Function Augmented Sliding Mode Controller for Robot Manipulators

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Abstract— **In this paper, a sliding mode controller with function augmented sliding surfaces is proposed for the control of robot manipulators. By augmenting a function which satisfies some conditions, a sliding surface can be designed more generally than conventional method. The proposed controller always guarantees the existence of the sliding mode. Therefore, the overall system has no reaching phase and shows the robust performance all the time in the presence of parameter variations and external disturbances. Moreover, the global exponential stability of the overall system is guaranteed and the tracking error is uniformly continuous.**

The efficiency of the proposed method has been demonstrated by simulations for a trajectory tracking control of a two-link robot subject to parameter uncertainties and external disturbances.

I. Introduction

The robustness to variable payloads, torque disturbances, and parameter variations has been increasingly demanded on the development of modern industrial manipulators. As one of the control method for a trajectory tracking control of robot manipulators, the sliding mode control scheme has been received an increasing attention [1].

The design of a sliding mode control system, one of the robust control techniques, is based on the bounds of unknown parameters and uncertainties, and the control law is constructed in order to force the system state to stay on the predetermined sliding surface. Once the system is in the sliding mode, the system response is thereafter independent of parameter variations and disturbances. Therefore, the prescribed transient response can be obtained in the sliding mode regardless of system uncertainties. However, since the initial state may not be on the sliding surface, there exists the period called a "reaching phase" until the system state reaches the sliding surface. During this reaching phase, the system response is sensitive to parameter uncertainties and external disturbances.

In order to overcome this reaching phase problem, Slotine and Sastry suggested a time-varying sliding surface in the state space by imposing the constraint that the initial errors be zero in tracking control [2]. But, this situation is not general because the initial states of the actual system can be located arbitrarily in the state space.

A rotating sliding hyperplane was introduced by Harashima, *et al.* to guarantee the sliding mode occurrence throughout the entire response [3]. It deals mainly with the case that the initial state is in II or IV quadrant. However, it is difficult for this scheme to apply if the initial state is in I or III quadrant.

Choi, *et al.* proposed a moving sliding surface [4]. The proposed sliding surface was shifted to the initial condition and then moved/shifted until the surface cross the origin. Thereafter, it rotated to increase the error convergence speed. However, these mechanisms were performed in the discrete way. Thus, the algorithm is not rigorous from the mathematical viewpoint.

Therefore, a sliding mode control system with function augmented sliding surfaces is proposed in this paper to remove the reaching phase wherever the initial state is. By augmenting a function which satisfies some conditions, the reaching phase is successfully removed and the sliding surface can be designed more generally than the case that an exponentially decaying function is augmented [5]. Therefore, the overall system is always in the sliding mode and shows the robust performance at all times in the presence of parameter variations and external disturbances. Furthermore, the global exponential stability of the overall system is guaranteed and the tracking error is shown to be predetermined.

The validity of the proposed scheme has been shown through the simulation results.

II. Preliminaries

Assumption 1 *$\hat{h}(s) \in \Re(s)$ is exponentially stable and strictly proper, where $\hat{h}(s)$ represents the Laplace transformed form of the corresponding impulse response $h(t)$.*

Assumption 2 *$g : \Re_{+} \to \Re$, $g \in C^1[0,\infty)$, $\dot{g} \in L^{\infty}$ and $g \in L^p \cap L^{\infty}$ for some $p \in [1,\infty)$, where $C^1[0,\infty)$ represents the set of all first differentiable continuous functions defined on $[0,\infty)$.*

Theorem 1 *Let $g, h : \Re_{+} \to \Re$. If $g \in C^1[0,\infty)$, $g \in L^p \cap L^{\infty}$ and $\dot{g} \in L^{\infty}$ for some $p \in [1,\infty]$, then $y = h * g \in L^p \cap L^{\infty}$, y is uniformly continuous and $y \to 0$ as $t \to \infty$.*

Proof $g \in L^p \cap L^{\infty}$ implies that $\| g \|_{\infty} < \infty$ and $\| g \|_p < \infty$. And cleary, $h \in L^1$. Therefore, the following two facts can be derived.

$$\begin{aligned} \| y \|_{\infty} &= \| g * h \|_{\infty} \le \| g \|_{\infty} \| h \|_1 < \infty, \\ \| y \|_p &= \| g * h \|_p \le \| g \|_p \| h \|_1 < \infty. \end{aligned}$$

From the above two facts, $y = g * h \in L^p \cap L^\infty$.

Since $\dot{g} \in L^\infty$, $\| \dot{g} \|_\infty < \infty$. And clearly $h \in L^\infty$. Therefore, the following inequality can be derived.

$$\begin{aligned} \| \dot{y} \|_\infty &= \| \dot{g} * h + g(0)h(t) \|_\infty \\ &\leq \| \dot{g} * h \|_\infty + \| g(0)h(t) \|_\infty \\ &\leq \| \dot{g} \|_\infty \| h \|_1 + \| g(0)h(t) \|_\infty \\ &< \infty. \end{aligned}$$

This implies that $\dot{y} \in L^\infty$. Hence, y is uniformly continuous.

From the above results, $y = g * h \in L^p \cap L^\infty$ and y is uniformly continuous, one can conclude the following result.

$$y \to 0 \quad \text{as} \quad t \to \infty.$$

■

III. Modeling of Robot Manipulator

The dynamic equation of an n degree-of-freedom robot manipulator can be derived using Lagrangian formulation as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u + d, \qquad (1)$$

where $M(q)$ is an $n \times n$ inertia matrix, $C(q, \dot{q})$ is an $n \times n$ matrix corresponding to Coriolis and centrifugal factors, $G(q)$ is an $n \times 1$ vector caused by gravitational force, d is an $n \times 1$ bounded disturbance vector, q is an $n \times 1$ joint variable vector, and u is an $n \times 1$ input torque vector.

Let us define each matrices as $M = M^0 + \Delta M$, $C = C^0 + \Delta C$, and $G = G^0 + \Delta G$, where "^{0}" denotes the mean value and "Δ" denotes the estimation error. Assume that the $\Delta M_{ij}, \Delta C_{ij}$, and ΔG_i are bounded by M^m_{ij}, C^m_{ij}, and G^m_i respectively as $|\Delta M_{ij}| \leq M^m_{ij}$, $|\Delta C_{ij}| \leq C^m_{ij}$, and $|\Delta G_i| \leq G^m_i$, where "m" denotes the maximal absolute estimation error of each element, and $i, j = 1, 2, \cdots, n$. It is also assumed that $|d_i| \leq d^m_i$, where $i = 1, 2, \cdots, n$.

IV. Design of Control System

Let us define the trajectory tracking error as

$$e(t) = q(t) - q_d(t),$$

where $q_d(t)$ represents the desired trajectory. And we choose a function augmented sliding surface as

$$s(t) = \dot{e}(t) + \Lambda e(t) - g(t), \qquad (2)$$

where $s, g \in \Re^n$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Re^{n \times n}$, $g_i(0) = \dot{e}_i(0) + \lambda_i e_i(0)$, each $g_i(t)$ is chosen such that Assumption 2 in the section 2 holds, and $i = 1, 2, \cdots, n$. From the above definition, it is clear that $s(0) = 0$.

Let us define the following positive-definite function as a Lyapunov function candidate:

$$V = \frac{1}{2} s^T M s. \qquad (3)$$

Differentiating (3) with respect to time and adopting the skew-symmetricity of $\dot{M}(q) - 2C(q, \dot{q})$, we have

$$\begin{aligned} \dot{V} &= s^T M \dot{s} + s^T C s \\ &= s^T (M\ddot{q} - M\ddot{q}_d + M\Lambda\dot{e} - M\dot{g} + Cs) \\ &= s^T (u + d - G + M(\Lambda\dot{e} - \dot{g} - \ddot{q}_d) + C(s - \dot{q})) \quad (4) \end{aligned}$$

Therefore, the equivalent control law is

$$u_{eq} = -M^0 (\Lambda\dot{e} - \dot{g} - \ddot{q}_d) - C^0 (s - \dot{q}) + G^0. \qquad (5)$$

Now, we introduce the control input such as

$$u = u_{eq} - K \bullet sgn(s), \qquad (6)$$

where "$\bullet$" means the element-by-element multiplication of two vectors, and

$$\begin{aligned} K &:= M^m |\Lambda\dot{e} - \dot{g} - \ddot{q}_d| + C^m |s - \dot{q}| + G^m + d^m + \eta, \\ \eta &= [\, \eta_1, \eta_2, \cdots, \eta_n \,]^T, \qquad \eta_i > 0, \\ sgn(s) &= [\, sgn(s_1), sgn(s_2), \cdots, sgn(s_n) \,]^T, \\ sgn(s_i) &= \begin{cases} 1 & \text{if } s_i > 0 \\ 0 & \text{if } s_i = 0, \quad i = 1, 2, \cdots, n, \\ -1 & \text{if } s_i < 0 \end{cases} \end{aligned}$$

and the absolute of a vector denotes the vector whose element has its absolute value, i.e. $|x| = [\, |x_1|, |x_2|, \cdots, |x_n| \,]^T$.

Using the above control law, we can derive a following theorem about the sliding mode existence.

Theorem 2 *For the robot manipulator (1) with the control law (6), the system is in the sliding mode at all times.*

Proof By inserting (6) in (4), we can obtain the following inequality:

$$\begin{aligned} \dot{V} &= s^T \Big\{ -M^0 (\Lambda\dot{e} - \dot{g} - \ddot{q}_d) - C^0 (s - \dot{q}) + G^0 \\ &\quad - (M^m |\Lambda\dot{e} - \dot{g} - \ddot{q}_d| + C^m |s - \dot{q}| + G^m + d^m + \eta) \\ &\quad \bullet sgn(s) + d - G + M(\Lambda\dot{e} - \dot{g} - \ddot{q}_d) + C(s - \dot{q}) \Big\} \\ &= s^T \Big\{ (M - M^0)(\Lambda\dot{e} - \dot{g} - \ddot{q}_d) + \left(C - C^0\right)(s - \dot{q}) \\ &\quad - M^m |\Lambda\dot{e} - \dot{g} - \ddot{q}_d| \bullet sgn(s) - C^m |s - \dot{q}| \bullet sgn(s) \\ &\quad + (G^0 - G) - G^m \bullet sgn(s) + d - d^m \bullet sgn(s) \\ &\quad - \eta \bullet sgn(s) \Big\} \\ &\leq -\sum_{i=1}^{n} \eta_i |s_i|. \end{aligned}$$

Therefore, V is a Lyapunov function. From the above inequality, it is clear that $\dot{V} \leq 0$, and $\dot{V} = 0$ only for $s = 0$. Additionally, it is easily known from (2) that $s(0) = 0$. Therefore, the Lyapunov function $V(t)$ is equal to zero at all times. This also implies that

$$s = 0 \qquad \forall\ t \geq 0. \qquad (7)$$

Thus, the system is forced to stay in the sliding mode at all times. ■

From the above theorem, we know that $s_i(t) \equiv 0$ for all i. Hence, the sliding surface (2) can be rewritten as

$$s_i(t) = \left(\frac{d}{dt} + \lambda_i\right) e_i(t) - g_i(t) = 0 \qquad \forall\ t \geq 0. \quad (8)$$

Thus, the trajectory tracking error $e_i(t)$ can be thought as an output of the first-order low pass filter with an input signal $g_i(t)$. Since the above low pass filter is clearly exponentially stable and strictly proper, Assumption 1 is guaranteed. Hence, the following theorem can be derived for the stability.

Theorem 3 *For the robot manipulator (1) with the control law (6), the overall system is globally asymptotically stable.*

Proof It is sufficient to show the stability for some i.

Obviously, the low pass filter $\left(\frac{d}{dt}+\lambda_i\right)$ is exponentially stable and strictly proper. Hence, Assumption 1 is guaranteed because $\hat{h}(s)=\frac{1}{s+\lambda_i}$. And we have chosen the function, $g_i(t)$, such that Assumption 2 is guaranteed. In addition, the trajectory tracking error, $e_i(t)$, can be regarded as an output of the low pass filter with the input function $g_i(t)$ as mentioned above.

Therefore, the theorem can be proven directly from Theorem 1. ∎

Furthermore, the following theorem can be derived for more specific function $g(t)$.

Theorem 4 *For the robot manipulator (1) applied by the control input (6) with $g(t)$ bounded by some exponentially decaying function, the overall system is globally exponentially stable.*

Proof The proof is so obvious. So, we omit the proof. ∎

V. Design of an Augmented Function

In the previous section, a function augmented sliding surface has been proposed in order to remove the reaching phase problem. Including the proposed method in the previous section, almost all of the previous works proposed to overcome the reaching phase problem have used an initial condition of the state vector. The assumption that the initial condition is available is not an restrictive condition because the measured and/or estimated data for the state vector can be obtained at each sampling time. Thus, the *initial* condition is the data obtained when the control system starts to operate. Especially, for the robot manipulators, one can get the position and the velocity data at each sampling time.

The augmented function, $g(t)$, can be arbitrarily designed if Assumption 3 is guaranteed. One of the simple choice of $g(t)$ is an exponentially decaying function [5].

Let us rewrite the proposed function augmented sliding surface (2).

$$s(t)=\dot{e}(t)+\Lambda e(t)-g(t).$$

Let us define the desired decaying trajectory for the tracking error as d_{tr}. Then, one can arbitrarily design/plan the trajectory so that the trajectory of the tracking error decreases to zero in finite time. Then, the augmenting function $g(t)$ can be designed as following.

$$g(t)=\dot{d}_{tr}(t)+\Lambda d_{tr}(t). \tag{9}$$

Generally, the desired decaying trajectory for the trajectory error d_{tr} is designed as a continuous piecewise-differentiable function with a compact support. Thus, the following assumption is followed.

Assumption 3 *The desired decaying trajectory for the trajectory error $d_{tr}(t)$ is a continuous piecewise-differentiable function with a compact support satisfying following conditions.*

$$d_{tr}(0)=e(0), \quad \dot{d}_{tr}(0)=\dot{e}(0) \tag{10}$$

Remark 1 *If the augmented function $g(t)$ is designed as in (9) and the $d_{tr}(t)$ satisfies Assumption 3, then the overall system is globally exponentially stable.*

Proof Obviously, all the functions satisfying Assumption 3 is bounded by some exponentially decaying function. Therefore, the result comes directly from Theorem 4. ∎

Remark 2 *If the augmented function $g(t)$ is designed as in (9), then the overall system always shows invariance property to parameter uncertainties and external disturbances and the robot's joint trajectory can be described in advance.*

Proof From the definition of the augmented sliding surface (2) and the augmented function (9), $s(t)$ can be rewritten as follows:

$$\begin{aligned} s(t) &= \dot{e}(t)+\Lambda e(t)-g(t) \\ &= \dot{q}-\dot{q}_d+\Lambda(q-q_d)-\left(\dot{d}_{tr}+\Lambda d_{tr}\right) \\ &= \dot{q}-\dot{q}_d-\dot{d}_{tr}+\Lambda(q-q_d-d_{tr}) \\ &= \dot{e}_{tr}+\Lambda e_{tr} \end{aligned} \tag{11}$$

where, $e_{tr}=q-(q_d+d_{tr})$. Since $d_{tr}(t)$ satisfies Assumption 3, $e_{tr}(0)=\dot{e}_{tr}(0)=0$. Therefore, $s(t)\equiv 0\ \forall t\geq 0$ and $q(t)=q_d(t)+d_{tr}(t)\ \forall t\geq 0$. ∎

If the augmented function $g(t)$ is divided into two parts as in (9), the augmented function can be easily designed because the system trajectory is totally governed by $q(t)\equiv q_d(t)+d_{tr}(t)$.

VI. Simulation Results

The simulation has been carried out for a two degree-of-freedom robot manipulator model used by Yeung and Chen [6]. Parameter values are also the same as those of [6].

Figures 1∼4 show the results of the trajectory tracking control of the robot manipulator. Figure 1 shows the sliding surface function $s(t)$ for the joint 1 and 2 when the conventional control law is applied. Figure 2 shows the same function for the case that the proposed control system is employed. As can be shown in Figure 1, the sliding mode has occurred for the joint 1 after about 2 second. However, for the proposed control scheme, the system is always in the sliding mode (see Figure 2). Therefore, the overall system has the robust performance in the presence of model uncertainties and external disturbances.

The following two figures show the performance of the conventional control system and the proposed method in the presence of model uncertainties and external disturbances. Figure 3 shows the tracking error transients of joint 1 when the conventional control method is used, where one of two curves represents the error transient in the case of no disturbances and the other curve represents the same one when disturbances are added. As can be shown in this figure, the system performance is degraded by external disturbances when the system state is in reaching phase. On the other hand, since the system is always in the sliding mode when the proposed control scheme is employed, the tracking error does not be affected by external disturbances as can be shown in Figure 4. It implies that the curve shape of the tracking error can be predetermined from (8) regardless of the existence of parameter uncertainties and external disturbances.

VII. Conclusions

In this paper, a sliding mode control system with function augmented sliding surfaces was proposed for the tracking control of robot manipulators. By augmenting a function which satisfies Assumption 2, a sliding surface can be designed more generally than the conventional method. Since the proposed function augmented sliding control scheme always guarantees the occurrence of the sliding mode, the overall system has no reaching phase and shows the robust performance all the time in the presence of parameter variations and external disturbances. Global exponential stability was also guaranteed for the overall system.

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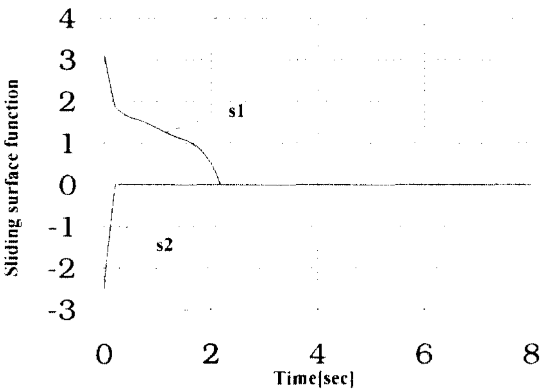

Fig. 1. Sliding surfaces (Conventional Case).

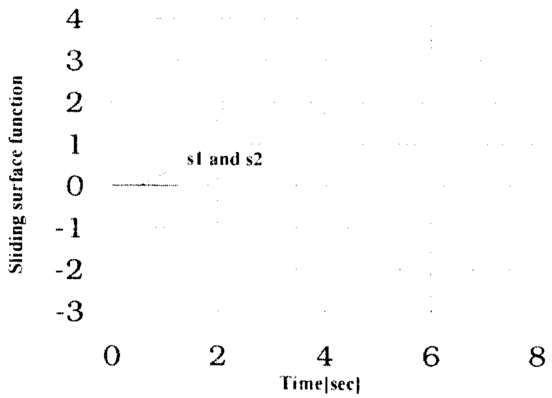

Fig. 2. Sliding surfaces (Proposed Case).

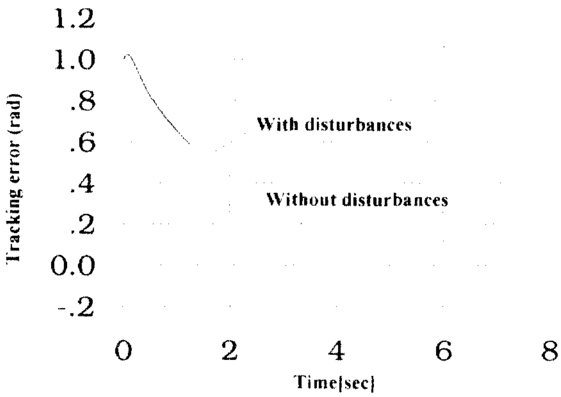

Fig. 3. Tracking error (Conventional Case).

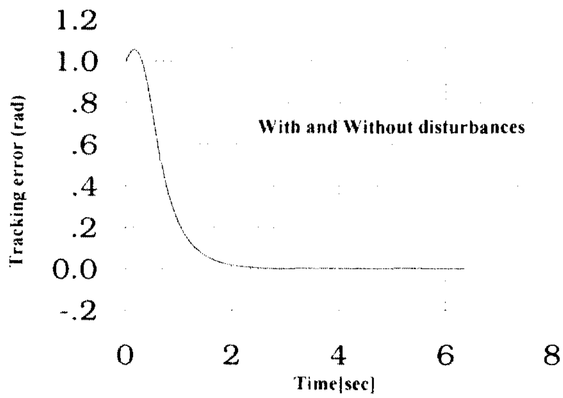

Fig. 4. Tracking error (Proposed Case).