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On the transport capacity of wireless ad-hoc networks

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Abstract

We show how transport capacity in bit-meters/s/Hz scales as the node density increases in ad-hoc networks. This was considered by many including [2, 3, 4]. In this paper, we do not consider the scaling law of the transport capacity of the entire network, but focus on the transport capacity of one source-destination pair. We assume opportunistic routing [1] is used to select the best relay node that is closest to the destination among nodes who successfully decode the current packet. We assume the squared channel gain decays as $e^{-\alpha d}/d^{\gamma}$, where $\alpha \geq 0$ is the absorption constant, $\gamma \geq 0$ is the path exponent, and d is the distance between a transmitter-receiver pair. We assume uniformly distributed nodes in n dimensions.

We show the transport capacity is bounded from above if there is no fading. Assuming slow flat Rayleigh fading and $\alpha = 0$, we show the transport capacity scales as $\Theta((\ln \lambda)^{1/\gamma})$ as the node density λ tends to infinity for one-dimensional networks. For any dimensional network, the transport capacity scales as $\Theta(\sqrt{\ln \lambda})$ if $\alpha = 0$ and $\gamma = 2$.

1. Introduction

The transport capacity of wireless ad-hoc networks was analyzed by Gupta and Kumar [2] and in related papers [3, 4]. They show asymptotically how the capacity scales in the number of nodes. Such scaling depends on many factors including traffic model and node placements.

Since the per-node transport capacity of a pure wireless network scales as $\Theta(1/\sqrt{n})$ in the number of nodes n under a very general set of assumptions [2], it is obvious that letting all nodes transmit simultaneously becomes impractical as the node density increases. Therefore, such a wireless network can be more useful for intermittent traffic, e.g., emergency traffic. In this case, a more important metric is how fast we can convey such information from the source to the destination. For this reason, we consider there is only one source-destination pair in the entire network and consider the maximum possible transport capacity in bit-meters/sec/Hz for the pair. Our analysis can also be applied to the case when the bandwidth is abundant such that the actively used bandwidth scales linearly with the number of senders.

We show how the transport capacity scales as the node density increases. We consider two scenarios, with and without fading. As fading provides the multi-user diversity gain [5], we can expect similar effect in opportunistic routing [1, 6, 7], i.e., the capacity is expected to increase as the node density increases. In [4], the effect of fading on transport capacity was analyzed. However, only the cases when the absorption constant is positive or when the path loss exponent is greater than 3 were considered in [4]. In this paper, we consider more general cases when the path exponent has any value. Furthermore, our approach is different from that of [4] since we do not consider the scaling law of the transport capacity of the entire network, but focus on the transport capacity of one source-destination pair.

This paper is organized as follows. In Section 2 we define our network and channel models. In Section 3 we analyze the transport capacity.

2. Network Model

In this section we describe our network model. We first assume one- or two-dimensional distribution of nodes, where nodes are uniformly distributed in infinitely big one- or two-dimensional line or plane, respectively. Let λ and μ denote the node densities for the one- and two-dimensional networks, i.e., λ is the node density (number of nodes per meter) for the one-dimensional network and μ is the node density (number of nodes per square meter) for the two-dimensional network.

We assume multi-hop routing. When there are one or more receivers correctly decoding the packet, the one that is closest to the destination relays the

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packet. Finding the node closest to the destination among nodes successfully receiving the packet can be done by using timers whose values depend on the distance to the destination [1]. Each node is programmed to broadcast a short signaling message upon expiration of the timer to indicate that it will be responsible for relaying the message. Upon receiving such a message, all other nodes discard the current message. When no receiver correctly decodes the packet, no node will transmit the signaling message and the current transmitter retransmits. Such a signaling message should be broadcasted with a high power to have enough coverage. If the signaling message size is much shorter than the data packet size, the overhead due to the signaling can be ignored.

This is repeated until the packet reaches the destination. We assume the destination is infinitely far away. Each hop is independent in this case if we assume independent fading and Poisson distribution of nodes that is randomized for each transmission, e.g., due to mobility. Even if there is no mobility, our asymptotic analysis still holds at high node density regime as long as the fading is independent. Note that it is enough to analyze the maximum traverse distance per hop under these assumptions. Therefore, from now on we focus on a single hop, where there is one transmitter and multiple receivers, where one of them will be relaying the packet in the next hop.

We assume no other cooperation between relay nodes. We assume no other signaling messages and thus no channel state information (CSI) is available at the transmitter. Therefore, we assume the transmitter transmits at a fixed power and at a fixed rate.

Let (x_k, y_k) denote the coordinate of the k-th receiving node, where $k = 1, 2, \ldots$. In the one-dimensional network, $y_k = 0$ for all k. We assume the transmitting node is at the origin. Let $d_k = \sqrt{x_k^2 + y_k^2}$ denote the distance in meters between the transmitter and the k-th receiver. Note that the final destination is infinitely far away on the x axis, i.e., at $(\infty, 0)$. The received SNR at the k-th receiver is given by

$$\frac{e^{-\alpha d_k}}{d_k^\gamma}\frac{P}{N_0 W}|h_k|^2,$$

where h_k is the fading coefficient from the transmitter to the receiver, P is the transmission power, W is the bandwidth and N_0 is the noise spectral density assuming the additive-white Gaussian noise (AWGN) channel. We assume $h_k = 1$ if there is no fading, i.e., line of sight, and assume $|h_k|^2$ is exponentially distributed and $E[|h_k|^2] = 1$ for Rayleigh fading. We assume h_k's are independent and they do not change during the transmission of a packet but change randomly for the next transmission, i.e., block fading. Note that the above propagation model is not very accurate when d_k is very small, but it will be asymptotically accurate since as we will show later d_k for the node closest to the destination tends to infinity as the node density tends to infinity if there is fading.

3. Transport Capacity

In this section, we analyze the transport capacity of our network. We define the transport capacity $C(R)$ in [bit-meters/s/Hz] given the transmission rate R [b/s/Hz] as follows

$$C(R) = E\left[\sup_k (R_k x_k)^+\right],$$

where $x^+ = x$ if $x > 0$ and $x^+ = 0$ otherwise and $R_k = R$ if $C_k \geq R$ and $R_k = 0$ otherwise, where $C_k = \log_2\left(1 + \frac{e^{-\alpha d_k}}{d_k^\gamma}|h_k|^2\mathrm{SNR}\right)$ [b/s/Hz] and $\mathrm{SNR} = P/N_0W$. The expectation is over all instances of node locations and over all fading instances. Note that $\sup_k(R_k x_k)^+ = 0$ if there is no successful reception, in which case the current transmitter needs to retransmit and the contribution to the transport capacity is zero for the failed transmission.

We define the transport capacity C^* as the supremum of $C(R)$ over all $R > 0$.

$$C^* = \sup_{R>0} C(R)$$

We define the maximum coverage radius L given the transmission rate R assuming no fading as follows

$$\log_2\left(1 + \frac{e^{-\alpha L}}{L^\gamma}\mathrm{SNR}\right) = R$$

We define the effective node density $\rho(x)$ at distance x (on the x axis) from the transmitter that includes only the nodes that satisfies $C_k \geq R$, i.e.,

$$\rho(x) = \lambda \Pr\left[\log_2\left(1 + \frac{e^{-\alpha x}}{x^\gamma}|h|^2\mathrm{SNR}\right) \geq R\right]$$

for the one-dimensional network and

$$\rho(x) = \int_{-\infty}^{\infty} \mu \Pr\left[\log_2\left(1 + \frac{e^{-\alpha\sqrt{x^2+y^2}}}{(x^2+y^2)^{\gamma/2}}|h|^2\mathrm{SNR}\right) \geq R\right] dy$$

for the two-dimensional network.

When there is fading, we get

$$\rho(x) = \lambda \exp\left[-e^{-\alpha(L-x)}(x/L)^\gamma\right] \qquad (1)$$

for the one-dimensional network and

$$\rho(x) = \int_{-\infty}^{\infty} \mu \exp\left[-e^{-\alpha(L-\sqrt{x^2+y^2})} \left(\frac{x^2+y^2}{L^2} \right)^{\gamma/2} \right] dy \quad (2)$$

for the two-dimensional network.

When there is no fading, we get

$$\rho(x) = \begin{cases} \lambda & x \le L \\ 0 & x > L \end{cases}$$

for the one-dimensional network and

$$\rho(x) = \begin{cases} 2\mu\sqrt{L^2 - x^2} & x \le L \\ 0 & x > L \end{cases}$$

for the two-dimensional network. Note that these are equivalent to the fading case if we set $\gamma = 0$ and $\alpha \to \infty$ or if we set $\alpha = 0$ and $\gamma \to \infty$ in (1) and (2).

The following shows how $C(R)$ can be computed.

Proposition 1 *The transport capacity $C(R)$ given the transmission rate R is given by the following*

$$C(R) = R \int_0^{\infty} e^{-\int_s^{\infty} \rho(t)dt} s\rho(s) ds$$

Proof: Define $\varphi(x)$ as follows

$$\varphi(x) = E\left[\sup_k \frac{(R_k x_k)^+}{R} \middle| \sup_k \frac{(R_k x_k)^+}{R} \le x \right]$$

Using $\varphi(x + \delta x) \sim x\rho(x)\delta x + (1 - \rho(x)\delta x)\varphi(x)$, we get

$$\varphi'(x) + \rho(x)\varphi(x) = x\rho(x)$$

Solving this differential equation for $\varphi(x)$ using the initialization condition $\varphi(0) = 0$, we get $C(R) = R\varphi(\infty) = R\int_0^{\infty} e^{-\int_s^{\infty} \rho(t)dt} s\rho(s)ds$ as shown above. ■

The transport capacity when there is no fading becomes

$$C(R) = RL\Phi_1(\bar{\lambda})$$

for the one-dimensional network, where $\Phi_1(\nu) = 1 - (1 - e^{-\nu})/\nu$ and $\bar{\lambda} = \lambda L$ and

$$C(R) = RL\Phi_2(\bar{\mu})$$

for the two-dimensional network, where

$$\Phi_2(\nu) = \int_0^1 e^{\nu(u\sqrt{1-u^2} + \sin^{-1} u - \pi/2)} 2\nu u \sqrt{1-u^2} du$$

and $\bar{\mu} = \mu L^2$.

Since $\Phi_1(\nu) \le 1$ and $\Phi_2(\nu) \le 1$, the transport capacity $C(R)$ is upper-bounded by RL for both one- and two-dimensional networks. The bound is tight as the node density tends to infinity. Therefore, the transport capacity C^* tends to R^*L^* as the node density tends to infinity, where

$$L^* = \underset{L}{\operatorname{argmax}}\, L \log_2 \left(1 + \frac{e^{-\alpha L}}{L^\gamma} \mathrm{SNR} \right) \quad (3)$$

and

$$R^* = \log_2 \left(1 + \frac{e^{-\alpha L^*}}{L^{*\gamma}} \mathrm{SNR} \right). \quad (4)$$

If $\alpha = 0$ and $\gamma = 2$, we get $R^* \sim 2.30$ and $L^* \sim 0.505$ assuming SNR $= 1$. From now on we assume SNR $= 1$ without loss of generality when $\alpha = 0$ since we can scale L accordingly.

If there is fading, the transport capacity $C(R)$ for one dimensional network is given by

$$C(R) = RL\Phi_{1f}(\bar{\lambda}, \bar{\alpha}, \gamma),$$

where $\bar{\alpha} = \alpha L$ and

$$\Phi_{1f}(\bar{\lambda}, \bar{\alpha}, \gamma) = \int_0^{\infty} \bar{\lambda} v \exp\left[-e^{-\bar{\alpha}(1-v)} v^\gamma \right] \times \exp\left[-\int_v^{\infty} \bar{\lambda} \exp[-e^{-\bar{\alpha}(1-u)} u^\gamma] du \right] dv$$

If $\alpha = 0$ and $\gamma = 1$, we get

$$\Phi_{1f}(\bar{\lambda}, 0, 1) = \gamma_0 + \Gamma(0, \bar{\lambda}) + \ln \bar{\lambda}, \quad (5)$$

where $\gamma_0 \sim 0.577$ is the Euler's constant and $\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\ dt$ is the incomplete gamma function. Since $\Gamma(0, x) \to 0$ as $x \to \infty$, the transport capacity in this case increases logarithmically in the node density. When $\alpha = 0$ and $\gamma > 1$, when $\alpha > 0$ and $\gamma = 1$, or when $\alpha > 0$ and $\gamma > 1$, the asymptotic transport capacity increases slower than the logarithmic growth of (5) in the node density since the asymptotic behavior depends only on $\rho(x)$ when x is large and $\rho(x)$ is attenuated more under one of these conditions. Figure 1 shows the transport capacity $C(R)$ for one-dimensional network with fading when $\alpha = 0$ and $\gamma = 2$. It also shows simulation results, where 10,000 transmissions are simulated with random node locations and fading. It matches our calculation very closely.

If $\alpha = 0$, we get the following scaling law for C^*.

Proposition 2 *The transport capacity C^* scales as $\Theta((\ln \lambda)^{1/\gamma})$ as λ tends to infinity for one-dimensional network with fading when $\alpha = 0$. Furthermore, the optimal rate R^* and the optimal L^* giving C^* are given by (4) and (3), respectively asymptotically as λ tends to infinity, i.e., the optimal rate and the optimal L for the no fading case.*

Proof: We give a brief sketch of the proof. When $\alpha = 0$, we get

$$\Phi_{1f}(\bar{\lambda}, 0, \gamma) = \int_0^\infty \bar{\lambda} v \exp\left[-v^\gamma - \frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{1}{\gamma}, v^\gamma\right)\right] dv$$

Since $\exp\left[-\frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{1}{\gamma}, v^\gamma\right)\right]$ exhibits a sharp transition from 0 to 1 at v^* as λ tends to infinity, we can use the following approximation $\Phi_{1f} \sim \int_{v^*}^\infty \bar{\lambda} v e^{-v^\gamma} dv = \frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{2}{\gamma}, v^{*\gamma}\right)$, where v^* is the solution to $\exp\left[-\frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{1}{\gamma}, v^{*\gamma}\right)\right] = \frac{1}{e}$, where v^* tends to infinity as $\bar{\lambda}$ tends to infinity. We get

$$\Phi_{1f} \sim v^* \tag{6}$$

and $v^* \sim (\ln \lambda)^{1/\gamma}$ asymptotically as the node density tends to infinity since $\Phi_{1f} \sim \frac{\bar{\lambda}}{\gamma} e^{-v^{*\gamma}} v^{*\gamma(2/\gamma-1)}$ and $\frac{\bar{\lambda}}{\gamma} e^{-v^{*\gamma}} v^{*\gamma(1/\gamma-1)} \sim 1$ using $\Gamma(a, x) = e^{-x} x^{a-1}(1 + O(1/x))$ and the fact that v^* is the solution to $\exp\left[-\frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{1}{\gamma}, v^{*\gamma}\right)\right] = \frac{1}{e}$. Note that the optimal rate R^* and the optimal L^* resulting in the optimal transport capacity C^* are the same as that of the no fading case (4) asymptotically as the node density tends to infinity since asymptotically $\Phi_{1f} \sim (\ln \lambda)^{1/\gamma}$ is a function of λ only and therefore C^* is maximized by (3) and (4). ■

The accuracy of (6) is demonstrated in Figure 2, which compares the transport capacity C^* with and without fading and also shows the asymptote (6) for the fading case. Note that v^* used in Figure 2 was obtained by numerically solving $\exp\left[-\frac{\bar{\lambda}}{\gamma}\Gamma\left(\frac{1}{\gamma}, v^{*\gamma}\right)\right] = \frac{1}{e}$ for v^* not by using $v^* \sim (\ln \lambda)^{1/\gamma}$. Furthermore, the values of L^* and R^* used in Figure 2 were calculated for each value of λ instead of using (3) and (4). These two should give us a better fit even for small λ's as can be seen from the figure. Note that the transport capacity for the no fading case is upper-bounded, but it is unbounded and scales as $\Theta(\sqrt{\ln \lambda})$ for the fading case with $\gamma = 2$.

For the two-dimensional network with fading and $\alpha = 0$, $\gamma = 2$, we get

$$\begin{aligned}\rho(x) &= \int_{-\infty}^\infty \mu \exp[-(x^2 + y^2)/L^2] dy \\ &= e^{-x^2/L^2} \sqrt{\pi} L \mu,\end{aligned}$$

i.e., the result is identical to the one-dimensional case if we use $\lambda = \mu L \sqrt{\pi}$. Similar is also true for dimensions higher than two.

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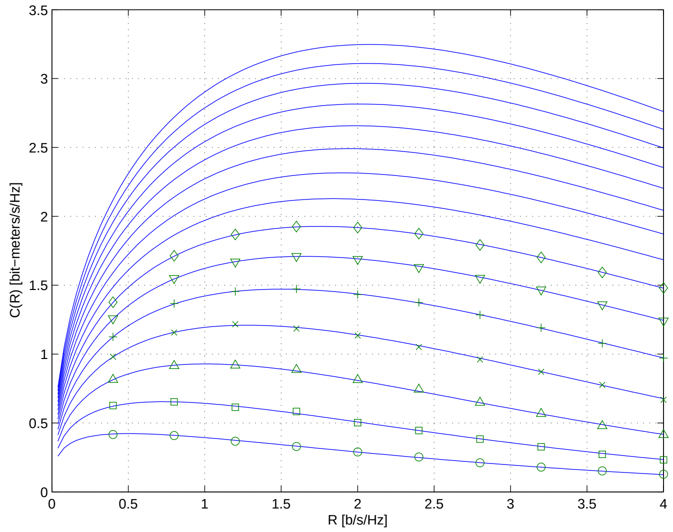

Figure 1: Transport capacity $C(R)$ computed for one-dimensional network with fading when $\alpha = 0$ and $\gamma = 2$. Curves from bottom to top correspond to $\lambda = 2^k$, where $k = 0, 1, \ldots, 14$. Also shown are simulation results for $\lambda = 2^k$, where $k = 0, 1, \ldots, 6$ ($\bigcirc$, $\Box$, $\triangle$, $\times$, $+$, $\bigtriangledown$, $\Diamond$, respectively).

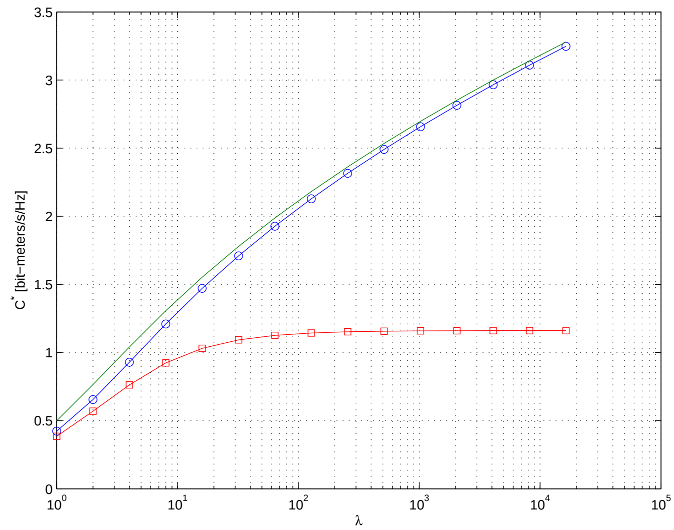

Figure 2: Transport capacity C^* for fading ($\bigcirc$) and no fading ($\Box$) cases for one-dimensional network with $\alpha = 0$ and $\gamma = 2$. Asymptote (6) (solid curve without markers) is also plotted for the fading case.