Comments

Comments on "Hierarchical Cubic Networks"

Sang Kyun Yun, *Member*, *IEEE* and Kyu Ho Park, *Member*, *IEEE*

Abstract—Ghose and Desai [1] introduced a new interconnection for large-scale distributed memory multiprocessors called the Hierarchical Cubic Network (HCN). The HCN is topologically superior to a comparable hypercube. They also proposed optimal routing algorithms for the HCN and obtained its diameter, which is about three-fourths the diameter of a comparable hypercube. However, their routing algorithm is not distance-optimal. In this paper, we propose an optimal routing algorithm for the HCN and show that HCN has about two-thirds the diameter of a comparable hypercube.

Index Terms—Interconnection networks, hypercubes, hierarchical cubic network(HCN), routing algorithm.

1 Introduction

The hypercube networks have been used as the interconnection network in a number of commercial and experimental parallel computers. A variety of interconnection networks based on the hypercube have also been proposed [2], [3], [4], [5], [6], [7], [8], [9]. For large-scale systems, the number of links for the hypercube may become prohibitively large. Ghose and Desai presented a new interconnection network called the Hierarchical Cubic Network (HCN) [1], [10], [11]. The HCN uses almost half as many links as a comparable hypercube and yet has a smaller diameter than a comparable hypercube and emulates desirable properties of a hypercube very efficiently.

They presented the optimal routing (OPT) algorithms in the HCN and showed that the HCN has three-fourths the diameter of a comparable hypercube [1], [11]. However, we found that their OPT algorithm is not distance-optimal. Thus, the diameter of the HCN obtained under the OPT algorithm is not correct. In this paper, we propose the distance-optimal routing algorithm in the HCN and show that the diameter of the HCN is about two-thirds the diameter of a comparable hypercube.

An HCN(n, n) is a hierarchical network consisting of 2^n clusters, each of which is an n-dimensional hypercube. Each node of the HCN(n, n) is addressed by a pair of numbers (I, J), where I is an n-bit cluster number and J is an n-bit address of the node within a cluster. Each node in the HCN(n, n) has $(n + 1)$ links connected to it. The links within a hypercube cluster are referred to as *local links* and the links between two clusters are referred to as *external links*. Clusters are interconnected by using external links to construct the HCN(n, n) using the following rule:

- If $I \neq J$, a node (I, J) is connected to the node (J, I) using its external link, which is called *nondiameter link*.
- A node (I, I) is connected to the node (I', I'), where I' is the bitwise complement of I, using its external link, which is called *diameter link*.

An HCN(3, 3) network is shown in Fig. 1.

2 An Optimal Routing Algorithm

2.1 Routing Algorithms for the HCN

Let us consider routing algorithms from a source node (I, J) to a destination node (K, L) in the HCN(n, n). When two nodes are in the same cluster, i.e., $I = K$, the shortest routing path is determined by conventional algorithms in hypercube and denoted by $(I, J) \Rightarrow (I, L)$.

For two nodes (I, J) and (K, L), such that $I \neq K$, Ghose and Desai proposed three routing algorithms as follows:

1) *Routing Algorithm A* :

$$(I, J) \Rightarrow (I, K) \rightarrow (K, I) \Rightarrow (K, L)$$

2) *Routing Algorithm B* :

$$(I, J) \Rightarrow (I, I) \rightarrow (I', I') \Rightarrow (I', K) \rightarrow (K, I') \Rightarrow (K, L)$$

3) *Routing Algorithm C* : for $I \neq L$ and $K \neq L$

$$(I, J) \Rightarrow (I, L) \rightarrow (L, I) \Rightarrow (L, K) \rightarrow (K, L)$$

Let $H(A, B)$ be the number of bits in which two binary numbers A and B differ. For two n-bit numbers $A = A_{n-1} \cdots A_0$ and $B_{n-1} \cdots B_0$, we have

$$H(A, B) = \sum_{i=0}^{n-1} A_i \oplus B_i,$$

where $\oplus$ is an exclusive-or operator. The distances R_A, R_B, and R_C following these three routing algorithms are thus

$$R_A = H(J, K) + H(I, L) + 1 \quad (1)$$

$$R_B = H(I, J) + H(I', K) + H(I', L) + 2 \quad (2)$$

$$R_C = H(J, L) + H(I, K) + 2 \quad (3)$$

Routing algorithms A and C provide the paths which do not traverse any diameter link. Of these two paths, the shorter one is the shortest among paths not traversing any diameter link, which will be shown in Section 2.3. Routing algorithm B provides the path traversing a diameter link. However, the path following routing algorithm B is not always the shortest among paths traversing a diameter link. Their OPT routing algorithm, which uses the one that provides the shortest routing distances among three algorithms, is not distance-optimal.

2.2 Routing Algorithm *B**

In this subsection, we describe the routing algorithm B^* which provides the shortest path among paths traversing a diameter link. Consider the routing algorithm $B^*(M)$, which provides the path traversing the diameter link of a cluster M as follows:

Routing Algorithm B(M)* :

$$(I, J) \Rightarrow \underbrace{(I, M) \rightarrow (M, I) \Rightarrow (M, M)}_{\text{part 1}} \rightarrow \underbrace{(M', M') \Rightarrow (M', K) \rightarrow (K, M')}_{\text{part 2}} \Rightarrow (K, L).$$

In the path of the routing algorithm $B^*(M)$, when $M = I$, part 1 degenerates to the node (I, I) and, when $M = K'$, part 2 degenerates to the node (K, K). The routing algorithm B is the routing algorithm $B^*(I)$. The routing distance $R_{B^*(M)}$ following this algorithm is thus

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- *S.K. Yun is with the Department of Computer Science, Seowon University, 231 Mochung-dong Heungdeok-gu Chongju-si, Chungbuk 361-742, Korea. E-mail: skyun@dragon.seowon.ac.kr.*
- *K.H. Park is with the Department of Electrical Engineering, Korea Advanced Institute of Science and Technology, 373-1 Kusong-dong Yusong-gu, Taejeon 305-701, Korea. E-mail: kpark@eekaist.kaist.ac.kr.*

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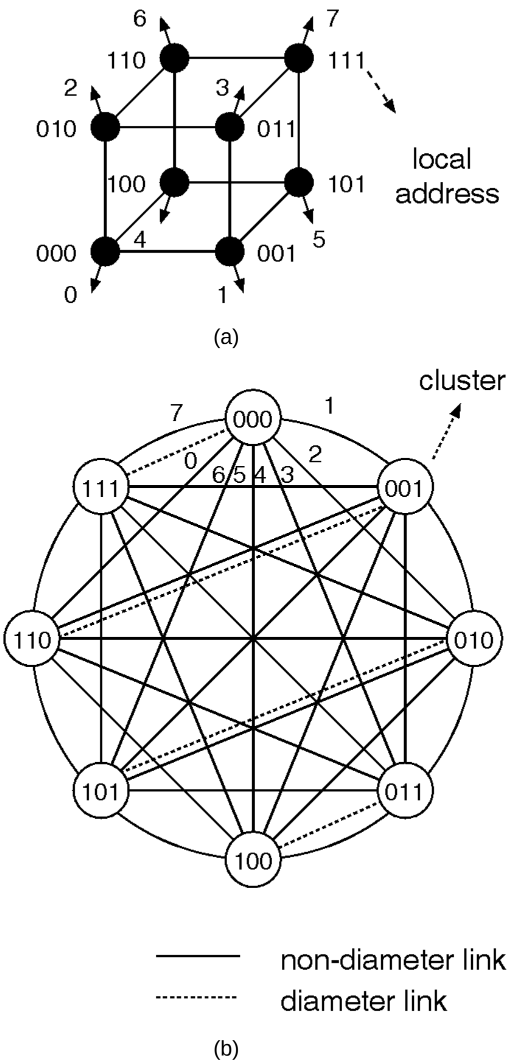

Fig. 1: HCN(3, 3) network. (a) A cluster, (b) HCN (3, 3).

$$R_{B^*(M)} = H(M, I) + H(M, J) + H(M', K) + H(M', L) + \delta, \quad (4)$$

where δ is 1 if $M = I = K'$, 2 if $M = I$ or $M = K'$ ($I \neq K'$), and 3, otherwise.

Such a cluster M that $R_{B^*(M)}$ is minimized is called a *minimizing* cluster. We can find a minimizing cluster M as follows. Let $Q = H(M, I) + H(M, J) + H(M', K) + H(M', L)$ and find the cluster M such that Q is minimized, which is called the *Q-minimizing cluster*. For any n-bit number P, let P_i denote the ith bit of P. Since $H(M', K) = H(M, K')$, $H(M', L) = H(M, L')$, we have

$$\begin{aligned} Q &= H(M, I) + H(M, J) + H(M, K') + H(M, L') \\ &= \sum_{i=0}^{n-1} \left\{ (M_i \oplus I_i) + (M_i \oplus J_i) + (M_i \oplus K_i') + (M_i \oplus L_i') \right\}. \quad (5) \end{aligned}$$

The set of Q-minimizing clusters can be obtained by finding in bitwise M_i such that each summation term $(M_i \oplus I_i) + (M_i \oplus J_i) + (M_i \oplus K_i') + (M_i \oplus L_i')$ is minimized as follows:

- If $I_i J_i K_i' L_i' \in \{0111, 1011, 1101, 1110, 1111\}$, then $M_i = 1$.
- If $I_i J_i K_i' L_i' \in \{0000, 0001, 0010, 0100, 1000\}$, then $M_i = 0$.
- If $I_i J_i K_i' L_i' \in \{0011, 0101, 0110, 1001, 1010, 1100\}$, then $M_i = X$, where X means *don't care* (0 or 1).

For example, for two nodes $(I, J) = (001, 010)$ and $(K, L) = (101, 111)$ in HCN(3, 3), three four-bit values $I_i J_i K_i' L_i'$s are 0000, 0110, and 1000 and, so, we obtain $M = 0X0$ and the set of Q-minimizing clusters is {000, 010}.

THEOREM 1. *For two nodes (I, J) and (K, L), let Q_{min} be the set of Q-minimizing clusters. If I (or K') belongs to Q_{min}, only I (or K') is a minimizing cluster and, otherwise, all the clusters in Q_{min} are minimizing clusters.*

PROOF. The distance of the path following the routing algorithm $B^*(M)$ is $R_{B^*(M)} = Q + \delta$. At first, let us consider the case that $I \neq K'$. If I (or K') belongs to Q_{min}, when $M = I$ (or K'), $R_{B^*(M)}$ is minimized, since both Q and δ are minimized. Therefore, I (or K') is a minimizing cluster. Otherwise, when $\delta = 2$, Q is not minimized and, therefore, $R_{B^*(M)}$ may not be minimized, but for any cluster M in Q_{min}, $R_{B^*(M)}$ is minimized although $\delta = 3$. Therefore, all the clusters in Q_{min} are minimizing clusters.

When $I = K'$, we obtain $M_i = I_i$ or $M_i = X$ for each i and, thus, I always belongs to Q_{min}. If $M = I$, then δ is 1. Therefore, I is a minimizing cluster. Thus, the theorem is proven. □

In the above example, since $K' = 010$ belongs to Q_{min}, the minimizing cluster M is K'. Note that both I and K' may be not minimizing clusters if $I \neq K'$. Thus, the path following routing algorithm B (routing algorithm $B^*(I)$) may not be the shortest among paths traversing a diameter link. For a minimizing cluster M, the routing algorithm $B^*(M)$ is called the *routing algorithm B^**.

EXAMPLE 1. Find the minimizing cluster M for two nodes $(I, J) = (000000, 000110)$ and $(K, L) = (011001, 101101)$ in HCN(6, 6). Six four-bit values $I_i J_i K_i' L_i'$s are 0010, 0001, 0000, 0110, 0111, 0000 and we obtain $M = 000X10$ and $Q_{min} = \{000010, 000110\}$. Since $I(= 000000)$ and $K'(= 100110)$ do not belong to Q_{min}, I and K' are not minimizing cluster and all the clusters in Q_{min} are minimizing clusters.

2.3 An Optimal Routing Algorithm

In this section, we establish the optimal routing algorithm for HCN(n, n) and prove its optimality.

THEOREM 2. *Suppose P is an arbitrary path from a source node (I, J) to a destination node (K, L) in the HCN(n, n), where $I \neq K$. The following is satisfied.*

1) *If path P contains three or more nondiameter links, then P is not the shortest.*
2) *If path P contains two or more diameter links, then P is not the shortest.*
3) *Routing algorithm A provides the shortest path that contains one nondiameter link.*
4) *Routing algorithm C provides the shortest path that contains two nondiameter links.*
5) *Routing algorithm B^* provides the shortest path that contains one diameter link.*

PROOF. The proof is given in the Appendix.

With Theorem 2, we can design the optimal routing algorithm from a source node (I, J) to a destination node (K, L) in the HCN(n, n), where $I \neq K$. The optimal routing algorithm is the algorithm which provides the shortest path among routing algorithms A, B^*, and C. The distance d between two nodes (I, J) and (K, L) is the minimum value among three distances. In other words,

$$d = \min(R_A, R_{B^*}, R_C) \quad (6)$$

EXAMPLE 2. We find the shortest routing path between two nodes (000000, 000110) and (011001, 101101). From Example 1, we select minimizing cluster address $M = 000010$. Three distances are as follows: $R_A = 5 + 4 + 1 = 10$, $R_C = 4 + 3 + 2 = 9$,

TABLE 1
VALUES OF R_{Ai}, R_{Ci}, R_{B*i}, AND M_i

group	$I_iJ_iK_iL_i$	R_{Ai}	R_{Ci}	R_{B*i}	M_i
1	0000, 1111	0	0	2	$M_i = X$
2	0110, 1001	0	2	2	$M_i = X$
3	0101, 1010	2	0	2	$M_i = X$
4	0011, 1100	2	2	0	$M_i = I_i = K_i'$
5	0010, 1101, 1000, 0111	1	1	1	$M_i = I_i = K_i'$
6	0001, 1110	1	1	1	$M_i = I_i \neq K_i'$
7	0100, 1011	1	1	1	$M_i = K_i \neq I_i$

and $R_{B*} = 1 + 1 + 2 + 1 + 3 = 8$. R_{B*} is the smallest. Thus, the path following routing algorithm $B*$ is the shortest path.

When there are any faulty components in an HCN(n, n), if the path following the optimal routing algorithm goes through a faulty component, the message routing may not be done along the path. Routing algorithm A has no alternate path with the same distance. Routing algorithm $B*$ can have an alternate path traversing another minimizing cluster. Routing algorithm C can also have an alternate path, which will be noted in the Appendix. Thus, if the optimal routing path is determined by routing algorithm $B*$ or C, it can have an alternate optimal routing path. If there is no alternate optimal path, the message must be routed through any nonoptimal path.

3 THE DIAMETER OF THE HCN(n, n)

In this section, we derive the exact expression for the diameter of a HCN(n, n). Ghose and Desai showed that the diameter of the HCN(n, n) under their optimal routing algorithm is $n + \lfloor n/2 \rfloor + 1$ [1]. However, since their routing algorithm is not optimal, the diameter obtained by them is incorrect. We can use the optimal routing algorithm developed in the previous section to derive the diameter of the HCN(n, n) and the next theorem gives its diameter.

THEOREM 3. *The diameter of the HCN(n, n) is* $n + \lfloor (n+1)/3 \rfloor + 1$.

PROOF. Let us consider the distance between a source node (I, J) and a destination node (K, L). If $I = K$, the maximum distance is n since two nodes are in the same hypercube. If $I \neq K$, the distance of two nodes is the minimum value among three distances R_A, R_{B*}, and R_C. The three distances can be represented as follows:

$$R_A = H(J,K) + H(I,L) + 1 = \sum_{i=1}^{n} R_{Ai} + 1 \quad (7)$$

$$R_C = H(J,L) + H(I,K) + 2 = \sum_{i=1}^{n} R_{Ci} + 2 \quad (8)$$

$$R_{B*} = H(M,I) + H(M,J) + H(M,K') + H(M,L') + \delta = \sum_{i=1}^{n} R_{B*i} + \delta, \quad (9)$$

where M is a minimizing cluster, $\delta \leq 2$ if $M = I$ or $M = K'$; otherwise, $\delta = 3$, and R_{Ai}, R_{Ci}, R_{B*i} are as follows: $R_{Ai} = (J_i \oplus K_i) + (I_i \oplus L_i)$, $R_{Ci} = (J_i \oplus L_i) + (I_i \oplus K_i)$, $R_{B*i} = (M_i \oplus I_i) + (M_i \oplus J_i) + (M_i \oplus K_i') + (M_i \oplus L_i')$.

The diameter of the HCN(n, n) is

$$D = \max_{(I,J),(K,L)} \{d\} = \max_{(I,J),(K,L)} \left\{ \min\left(R_A, R_{B*}, R_C\right) \right\}. \quad (10)$$

It is difficult to manipulate the equations for R_A, R_C, R_{B*} expressed in exclusive-or operator ($\oplus$). We can express the equations for R_A, R_C, R_{B*} in plus operator (+) as follows:

For each four-bit value $I_iJ_iK_iL_i$ ($0 \leq i \leq n - 1$), the values of R_{Ai}, R_{Ci}, R_{B*i}, and M_i are listed in Table 1. n four-bit values $I_iJ_iK_iL_i$ s are divided into seven groups according to their values of R_{Ai}, R_{Ci}, R_{B*i} and M_i. Let a_k be the number of four-bit values in group k. For example, for two nodes $(I, J) = (01001, 10110)$ and $(K, L) = (00001, 11100)$, five four-bit values $I_iJ_iK_iL_i$s are 0101, 1001, 0101, 0100, 1010 and we have $a_2 = 1$, $a_3 = 3$, $a_7 = 1$, and, for other k, $a_k = 0$. The sum of all a_ks is $a_1 + a_2 + \cdots + a_7 = n$, where all a_ks are nonnegative integers. If $a_6 = 0$, then $M = K'$, since $M_i = K_i'$ or $M_i = X$ for all i and if $a_7 = 0$, then $M = I$, since $M_i = I_i$ or $M_i = X$ for all i. Thus, if $a_6 = 0$ or $a_7 = 0$, we have $\delta \leq 2$.

Three distances can be expressed in terms of a_ks as follows:

$$R_A = 2a_3 + 2a_4 + A + 1 \quad (11)$$

$$R_C = 2a_2 + 2a_4 + A + 2 \quad (12)$$

$$R_{B*} = 2a_1 + 2a_2 + 2a_3 + A + \delta, \quad (13)$$

where $A = a_5 + a_6 + a_7$ and, if $a_6 = 0$ or $a_7 = 0$, $\delta \leq 2$; otherwise, $\delta = 3$. Since the sum of all a_ks is n, we have

$$a_1 + a_2 + a_3 + a_4 + A = n. \quad (14)$$

In order to find the diameter, we consider the upper bounds of distances for the four cases.

1) If $a_3 \leq a_2$ and $a_4 \leq a_1 + a_2$, the routing algorithm A is an optimal algorithm and the distance is

$$\min(R_A, R_C, R_{B*}) = R_A = 2a_3 + 2a_4 + A + 1 \leq 2a_1 + 4a_2 + A + 1,$$

where equality holds when $a_3 = a_2$ and $a_4 = a_1 + a_2$. When equality holds, we have $n = a_1 + a_2 + a_3 + a_4 + A = 2a_1 + 3a_2 + A$ and, so, $a_2 = (n - 2a_1 - A)/3 \leq \lfloor n/3 \rfloor$. Thus, we have

$$R_A \leq 2a_1 + 4a_2 + A + 1 = n + a_2 + 1 \leq n + \lfloor n/3 \rfloor + 1. \quad (15)$$

2) If $a_3 \leq a_2$ and $a_4 \geq a_1 + a_2 + 1$, the routing algorithm $B*$ is an optimal algorithm and the distance is

$$\min(R_A, R_C, R_{B*}) = R_{B*} = 2a_1 + 2a_2 + 2a_3 + A + \delta \leq 2a_1 + 4a_2 + A + \delta,$$

where equality holds when $a_3 = a_2$ and $a_4 = a_1 + a_2 + 1$. When equality holds, we have $n = a_1 + a_2 + a_3 + a_4 + A = 2a_1 + 3a_2 + A + 1$ and $a_2 = (n - 1 - 2a_1 - A)/3$. If $a_6 = 0$ or $a_7 = 0$, then $\delta \leq 2$, $A \geq 0$ and, so, $a_2 \leq \lfloor (n-1)/3 \rfloor$; otherwise, $\delta = 3$, $A \geq 2$ and, so, $a_2 \leq \lfloor (n-3)/3 \rfloor = \lfloor n/3 \rfloor - 1$. Thus, we have

$$R_{B*} \leq 2a_1 + 4a_2 + A + \delta = n + a_2 + (\delta - 1) \leq \max(n + \lfloor (n-1)/3 \rfloor + 1, n + \lfloor n/3 \rfloor + 1) = n + \lfloor n/3 \rfloor + 1. \quad (16)$$

3) If $a_3 \geq a_2 + 1$ and $a_4 \leq a_1 + a_3$, the routing algorithm C is an optimal algorithm and the distance is

$$\min(R_A, R_C, R_{B^*}) = R_C = 2a_2 + 2a_4 + A + 2 \leq 2a_1 + 4a_3 + A,$$

where equality holds when $a_3 = a_2 + 1$ and $a_4 = a_1 + a_3$. When equality holds, we have $n = a_1 + a_2 + a_3 + a_4 + A = 2a_1 + 3a_3 + A - 1$ and $a_3 \leq \lfloor (n+1)/3 \rfloor$. Thus, we have

$$R_C \leq 2a_1 + 4a_3 + A = n + a_3 + 1 \leq n + \lfloor (n+1)/3 \rfloor + 1. \quad (17)$$

4) If $a_3 \geq a_2 + 1$ and $a_4 \geq a_1 + a_3 + 1$, the routing algorithm B^* is an optimal algorithm and the distance is

$$\min(R_A, R_C, R_{B^*}) = R_{B^*} = 2a_1 + 2a_2 + 2a_3 + A + \delta \leq 2a_1 + 4a_3 + A + \delta - 2,$$

where equality holds when $a_3 = a_2 + 1$ and $a_4 = a_1 + a_3 + 1$. When equality holds, $n = a_1 + a_2 + a_3 + a_4 + A = 2a_1 + 3a_3 + A$ and $a_3 = (n - 2a_1 - A)/3$. If $a_6 = 0$ or $a_7 = 0$, $\delta \leq 2$, $A \geq 0$ and, so, $a_3 \leq \lfloor n/3 \rfloor$; otherwise, $\delta = 3$, $A \geq 2$ and, so, $a_3 \leq \lfloor (n-2)/3 \rfloor = \lfloor (n+1)/3 \rfloor - 1$. Thus, we have

$$\begin{aligned} R_{B^*} &\leq 2a_1 + 4a_3 + A + \delta - 2 = n + a_3 + \delta - 2 \\ &\leq \max(n + \lfloor n/3 \rfloor, n + \lfloor (n+1)/3 \rfloor) = n + \lfloor (n+1)/3 \rfloor. \quad (18) \end{aligned}$$

From (15)-(18), the maximum of distances is $n + \lfloor (n+1)/3 \rfloor + 1$, which is the upper bound in (17). Therefore, the diameter of HCN(n, n) is $n + \lfloor (n+1)/3 \rfloor + 1$. □

Thus, the HCN(n, n) has about two thirds the diameter of a comparable hypercube, $2n$-cube.

4 Conclusion

In this paper, we have presented an optimal routing algorithm for the HCN(n, n). Based on this optimal algorithm, we have derived an exact expression for the diameter of the HCN(n, n). It has been shown that the diameter derived in this paper is smaller than the previously known bound.

Appendix

Proof of Theorem 2.

1) Suppose (I, J) and (K, L) are the source node and the destination node. Path P containing k nondiameter links ($k \geq 3$) is constructed as follows: $(M_0, M_{-1}) \Rightarrow (M_0, M_1) \to (M_1, M_0) \Rightarrow (M_1, M_2) \to (M_2, M_1) \Rightarrow \cdots \to (M_k, M_{k-1}) \Rightarrow (M_k, M_{k+1})$, where $M_0 = I$, $M_{-1} = J$, $M_k = K$, $M_{k+1} = L$. The routing distance of P is $R_P = \sum_{i=1}^{k+1} H(M_{i-2}, M_i) + k$. Let us consider the two cases.

- When k is odd: We now construct another path Q as follows: $(M_0, M_{-1}) \Rightarrow (M_0, M_1) \Rightarrow (M_0, M_3) \Rightarrow \cdots \Rightarrow (M_0, M_k) \to (M_k, M_0) \Rightarrow (M_k, M_2) \Rightarrow (M_k, M_4) \Rightarrow \cdots \Rightarrow (M_k, M_{k+1})$. Path Q contains one nondiameter link. The routing distance of Q is $R_Q = \sum_{i=1}^{k+1} H(M_{i-2}, M_i) + 1$. Thus, we have $R_Q < R_P$ and path Q is shorter than path P.
- When k is even: We now construct another path Q as follows: $(M_0, M_{-1}) \Rightarrow (M_0, M_1) \Rightarrow (M_0, M_3) \Rightarrow \cdots \Rightarrow (M_0, M_{k+1}) \to (M_{k+1}, M_0) \Rightarrow (M_{k+1}, M_2) \Rightarrow (M_{k+1}, M_4) \cdots \Rightarrow (M_{k+1}, M_k) \to (M_k, M_{k+1})$. Path Q contains two nondiameter link. The routing distance of Q is $R_Q = \sum_{i=1}^{k+1} H(M_{i-2}, M_i) + 2$.

  Thus, we have $R_Q < R_P$ and path Q is shorter than path P.

Therefore, if path P contains three or more nondiameter links, then P is not the shortest.

2) Suppose (I, J) and (K, L) are the source node and the destination node. For any two nodes u and v, the path $u \stackrel{*}{\Rightarrow} v$ represents the routing path within the hypercube if two nodes are in the same cluster and the path following the routing algorithm A, otherwise. Path P containing k diameter links ($k \geq 2$) between two nodes is constructed as follows:

$$(I,J) \stackrel{*}{\Rightarrow} (M_1, M_1) \to (M_1', M_1') \stackrel{*}{\Rightarrow} (M_2, M_2) \to (M_2', M_2') \stackrel{*}{\Rightarrow} \cdots \to \cdots \stackrel{*}{\Rightarrow} (M_k, M_k) \to (M_k', M_k') \stackrel{*}{\Rightarrow} (K, L).$$

The routing distance of P is

$$\begin{aligned} R_P = {} & H(J, M_1) + H(I, M_1) + \sum_{i=1}^{k-1} 2H(M_i', M_{i+1}) + \\ & H(M_i', K) + H(M_k', L) + \delta, \end{aligned}$$

where $\delta \geq k$, since at least k diameter links are contained. Let us consider the two cases.

- When k is even: We now construct another path Q as follows: $(I, J) \Rightarrow (I, M_1) \Rightarrow \cdots \Rightarrow (I, M_{2j-1}) \Rightarrow (I, M_{2j}') \Rightarrow \cdots \Rightarrow (I, M_k') \Rightarrow (I, K) \to (K, I) \Rightarrow (K, M_1) \Rightarrow \cdots \Rightarrow (K, M_{2j-1}) \Rightarrow (K, M_{2j}') \Rightarrow \cdots \Rightarrow (K, M_k') \Rightarrow (K, L)$. Path Q does not contain any diameter link. The routing distance of Q is

$$\begin{aligned} R_Q = {} & H(J, M_1) + H(I, M_1) + \sum_{i=1}^{k-1} 2H(M_i', M_{i+1}) + \\ & H(M_k', K) + H(M_k', L) + 1. \end{aligned}$$

  Obviously, we have $R_Q < R_P$. Thus, path Q is the shorter than path P.
- When k is odd: We now construct another path Q as follows: $(I, J) \stackrel{*}{\Rightarrow} (M_1, M_1) \to (M_1', M_1') \Rightarrow (M_1', M_2) \cdots \Rightarrow (M_1', M_{2j}) \Rightarrow (M_1', M_{2j+1}') \Rightarrow \cdots \Rightarrow (M_1', M_k') \Rightarrow (M_1', K) \to (K, M_1') \Rightarrow (K, M_2) \Rightarrow \cdots \Rightarrow (K, M_{2j}) \Rightarrow (K, M_{2j+1}) \cdots \Rightarrow (K, M_k') \Rightarrow (K, L)$. Path Q contains exactly one diameter link. The routing distance of Q is

$$\begin{aligned} R_Q = {} & H(J, M_1) + H(I, M_1) + \sum_{i=1}^{k-1} 2H(M_i', M_{i+1}) + \\ & H(M_k', K) + H(M_k', L) + \delta^*, \end{aligned}$$

  where δ^* is 1 if $M_1 = I$ and 2, otherwise. Since $k \geq 3$, we have $R_Q < R_P$. Thus, path Q is shorter than path P.

Therefore, if path P contains two or more diameter links, then P is not the shortest.

3) Since the nondiameter link that connects between (I, K) and (K, I) is a unique link that connects two clusters I and K, the path that contains one nondiameter link must go through this link. Thus, the path following routing algorithm A is a unique path containing one nondiameter link. Therefore, routing algorithm A provides the shortest path that contains one nondiameter link.

4) Path P containing two nondiameter links is constructed as follows: $(I, J) \Rightarrow (I, M) \rightarrow (M, I) \Rightarrow (M, K) \rightarrow (K, M) \Rightarrow (K, L)$, where $M \neq I$ and $M \neq K$. The routing distance of P is $R_P = H(J, M) + H(I, K) + H(M, L) + 2$. From (3), $R_C = H(J, L) + H(I, K) + 2$. Since $H(J, M) + H(M, L) \geq H(J, L)$, we have $R_P \geq R_C$. Therefore, routing algorithm C provides the shortest path that contains two nondiameter links. Note that if M satisfies $H(J, M) + H(M, L) = H(J, L)$, path P has the same distance as the path by routing algorithm C and can be used as its alternate path.
5) It is obvious from the definition of routing algorithm B^*. □

REFERENCES

[1] K. Ghose and K.R. Desai, "Hierarchical Cubic Network," *IEEE Trans. Parallel and Distributed Systems*, vol. 6, no. 4, pp. 427–435, Apr. 1995.
[2] A. El-Amawy and S. Latifi, "Properties and Performance of Folded Hypercubes," *IEEE Trans. Parallel and Distributed Systems*, vol. 2, no. 1, pp. 31–42, Jan. 1991.
[3] A. Esfahanian, L.M. Ni, and B.E. Sagan, "The Twisted n-Cube with Application to Multiprocessing," *IEEE Trans. Computers*, vol. 40, no. 1, pp. 88–93, Jan. 1991.
[4] K. Hwang and J. Ghosh, "Hypernet: A Communication Efficient Architecture for Constructing Massively Parallel Computers," *IEEE Trans. Computers*, vol. 36, no. 12, pp. 1,450–1,466, Dec. 1987.
[5] J.M. Kumar and L.M. Patnaik, "Extended Hypercube: A Hierarchical Interconnection Network of Hypercube," *IEEE Trans. Parallel and Distributed Systems*, vol. 3, no. 1, pp. 45–57, Jan. 1992.
[6] N.F. Tzeng and S. Wei, "Enhanced Hypercube," *IEEE Trans. Computers*, vol. 40, no. 3, pp. 284–294, Mar. 1991.
[7] S.G. Ziavras, "A Versatile Family of Reduced Hypercube Interconnection Network," *IEEE Trans. Parallel and Distributed Systems*, vol. 5, no. 11, pp. 1,210–1,220, Nov. 1994.
[8] K. Efe, "A Variation on the Hypercube with Lower Diameter," *IEEE Trans. Computers*, vol. 40, no. 11, pp. 1,312–1,316, Nov. 1991.
[9] D.R Duh, G.H. Chen, and J. F. Fang, "Algorithms and Properties of a New Two-Level Network with Folded Hypercubes as Basic Modules," *IEEE Trans. Parallel and Distributed Systems*, vol. 6, no. 7, pp. 714–723, July 1995.
[10] K. Ghose and K.R. Desai, "The HCN: A Versatile Interconnection Network Based on Cubes," *Proc. Supercomputing*, pp. 426–435, 1989.
[11] K. Ghose and K.R. Desai, "The Design and Evaluation of the Hierarchical Cubic Network," *Proc. 19th Int'l Conf. Parallel Processing*, vol. I, pp. 355–362, 1990.