# On Siegel invariants of certain CM-fields 

Ja Kyung Koo ${ }^{1}$. Gilles Robert ${ }^{2}$. Dong Hwa Shin ${ }^{3}$. Dong Sung Yoon ${ }^{4}$

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#### Abstract

We first construct Siegel invariants of some CM-fields in terms of special values of theta constants, which would be a generalization of Siegel-Ramachandra invariants of imaginary quadratic fields. We further describe Galois actions on these invariants and provide a numerical example to show that this invariant really generates the ray class field of a CM-field.


Keywords Abelian varieties • Class field theory • CM-fields • Shimura's reciprocity law • Theta constants

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[^0]
## 1 Introduction

Let $K$ be a number field and $\mathcal{O}_{K}$ be its ring of integers. For a proper nontrivial ideal $\mathfrak{f}$ of $\mathcal{O}_{K}$ we denote by $\mathrm{Cl}(\mathfrak{f})$ and $K_{\mathfrak{f}}$ the ray class group of $K$ modulo $\mathfrak{f}$ and its corresponding ray class field, respectively (see [4]). Suppose that there is a family $\left\{\Psi_{\mathfrak{f}}(C)\right\}_{C \in \mathrm{Cl}(\mathfrak{f})}$ of algebraic numbers, which we shall call a Siegel family, such that
(R1) each $\Psi_{\mathfrak{f}}(C)$ belongs to $K_{\mathfrak{f}}$,
(R2) $\Psi_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{f}}(D)}=\Psi_{\mathfrak{f}}(C D)$ for all $D \in \mathrm{Cl}(\mathfrak{f})$, where $\sigma_{\mathfrak{f}}: \mathrm{Cl}(\mathfrak{f}) \rightarrow \operatorname{Gal}\left(K_{\mathfrak{f}} / K\right)$ is the Artin reciprocity map for $\mathfrak{f}$.

Then, as is well known, every algebraic number $\Psi_{\mathfrak{f}}(C)$ becomes a primitive generator of $K_{\mathfrak{f}}$ over $K$ if

$$
\begin{equation*}
\sum_{C \in \mathrm{Cl}(\mathrm{f})} \chi(C) \ln \left|\Psi_{\mathfrak{f}}(C)\right| \neq 0 \tag{1}
\end{equation*}
$$

for any nontrivial character $\chi$ of $\mathrm{Cl}(\mathfrak{f})$ [11, Theorem 3 in Chapter 22], which motivates this paper. In particular, if $K$ is an imaginary quadratic field, then the SiegelRamachandra invariants form such a Siegel family having the properties (R1) and (R2) (see Sect. 2). Furthermore, they also satisfy (1) in most cases by the second Kronecker limit formula. These invariants are defined by the special values of certain modular units, Siegel functions, which can be expressed in terms of theta constants. However, before this work, relatively little was known for other types of number fields other than imaginary quadratic. For CM-fields, Shimura established the theory of complex multiplication and showed that abelian extensions over CM-fields are closely connected with complex abelian varieties. He also constructed class fields over CM-fields by using a Kummer variety, but it's too abstract to use in practice [18, Main Theorem 2 in §16].

In this paper, we shall construct a concrete example of a Siegel family over CMfields. Let $K$ be a CM-field and $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_{K}$ satisfying the conditions of Assumption 5.1 below. We shall first construct a meromorphic Siegel modular function of level $N(\geq 2)$, which would be a multi-variable generalization of the Siegel function, by making use of theta constants (Definition 4.2 and Proposition 4.5). Furthermore, we shall assign the special value $\Theta_{f}(C)$ of this function to each ray class $C$ in $\mathrm{Cl}(\mathfrak{f})$, and call it the Siegel invariant modulo $\mathfrak{f}$ at $C$ (Definition 5.3). This value depends only on $\mathfrak{f}$ and the class $C$ (Propositions 5.5 and 5.6 ), essentially by the fact that the Siegel modular variety is a moduli space for principally polarized abelian varieties. Finally, we are able to show by applying Shimura's reciprocity law that the Siegel invariant $\Theta_{\mathfrak{f}}(C)$, as a possible ray class invariant (Conjecture 6.4), satisfies the transformation formula (R2), that is,

$$
\Theta_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{f}}(D)}=\Theta_{\mathfrak{f}}(C D) \quad \text { for all } D \in \operatorname{Cl}(\mathfrak{f})
$$

(Theorem 6.3). By making use of the mathematical software Maple, we also present a numerical example with a non-imaginary quadratic CM-field $K$ for which $K_{\mathfrak{f}}$ is generated by $\Theta_{f}(C)$ over $K$ (Example 6.5).

## 2 Siegel-Ramachandra invariants

Let $\mathbf{v}=\left[\begin{array}{l}r \\ s\end{array}\right] \in \mathbb{Q}^{2} \backslash \mathbb{Z}^{2}$. The Siegel function $g_{\mathbf{v}}(\tau)$ on the complex upper half-plane $\mathbb{H}=\{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau)>0\}$ is given by the infinite product

$$
\begin{equation*}
g_{\mathbf{v}}(\tau)=-q^{\mathbf{B}_{2}(r) / 2} e^{\pi \mathrm{i} s(r-1)}\left(1-q^{r} e^{2 \pi \mathrm{i} s}\right) \prod_{n=1}^{\infty}\left(1-q^{n+r} e^{2 \pi \mathrm{i} s}\right)\left(1-q^{n-r} e^{-2 \pi \mathrm{i} s}\right), \tag{2}
\end{equation*}
$$

where $\mathbf{B}_{2}(x)=x^{2}-x+1 / 6$ is the second Bernoulli polynomial and $q=e^{2 \pi \mathrm{i} \tau}$. If $N(\geq 2)$ is a positive integer so that $N \mathbf{v} \in \mathbb{Z}^{2}$, then $g_{\mathbf{v}}(\tau)^{12 N}$ is a meromorphic modular function of level $N$ which has neither a zero nor a pole on $\mathbb{H}[6$, Theorem 1.2 in Chapter 2].

Let $K$ be an imaginary quadratic field. Let $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_{K}$, let $N$ be the smallest positive integer in $\mathfrak{f}$, and let $C \in \mathrm{Cl}(\mathfrak{f})$. Take any integral ideal $\mathfrak{c}$ in $C$, and let $\omega_{1}, \omega_{2} \in \mathbb{C}$ and $a, b \in \mathbb{Z}$ be such that

$$
\begin{aligned}
\mathfrak{f c}^{-1} & =\mathbb{Z} \omega_{1}+\mathbb{Z} \omega_{2} \quad \text { with } \omega=\omega_{1} / \omega_{2} \in \mathbb{H}, \\
N & =a \omega_{1}+b \omega_{2} .
\end{aligned}
$$

The Siegel-Ramachandra invariant $g_{f}(C)$ modulo $\mathfrak{f}$ at $C$ is defined as

$$
g_{f}(C)=g_{\left[\begin{array}{l}
a / N  \tag{3}\\
b / N
\end{array}\right]}(\omega)^{12 N} .
$$

This value depends only on $\mathfrak{f}$ and the class $C$, not on the choices of $\mathfrak{c}$ and $\omega_{1}, \omega_{2}[6$, Chapter 11, §1]. Furthermore, it lies in $K_{\mathfrak{f}}$ and satisfies

$$
g_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{f}}(D)}=g_{\mathfrak{f}}(C D) \quad(D \in \mathrm{Cl}(\mathfrak{f}))
$$

[6, Theorem 1.1 in Chapter 11]. In [14], Ramachandra constructed a primitive generator of $K_{\mathfrak{f}}$ over $K$ as a high power product of Siegel-Ramachandra invariants and singular values of the modular discriminant $\Delta$-function. In this direction, the first and the fourth author proved the following.

Proposition 2.1 Let $\mathfrak{f}=\prod_{\mathfrak{p} \mid \mathfrak{f}} \mathfrak{p}^{e_{\mathfrak{p}}}$ be the prime ideal factorization of $\mathfrak{f}$, and let

$$
G_{\mathfrak{p}}=\left(\mathcal{O}_{K} / \mathfrak{p}^{e_{\mathfrak{p}}}\right)^{\times} /\left\{\mu+\mathfrak{p}^{e_{\mathfrak{p}}} \mid \mu \in \mathcal{O}_{K}^{\times}\right\} .
$$

If $\left|G_{\mathfrak{p}}\right|>2$ for every $\mathfrak{p} \mid \mathfrak{f}$, then any nonzero power of $g_{\mathfrak{f}}(C)$ generates $K_{\mathfrak{f}}$ over $K$.
Proof See [9, Theorem 4.6].
Remark 2.2 This result is obtained by utilizing the second Kronecker limit formula [20, Theorem 9 in Chapter II] or [11, Theorem 2 in Chapter 22].

## 3 Actions on Siegel modular functions

In this section we shall briefly recall the action of an idele group on the field of meromorphic Siegel modular functions due to Shimura.

For a positive integer $g$ and a commutative ring $R$ with unity, we let

$$
\begin{aligned}
\mathrm{GSp}_{2 g}(R) & =\left\{\alpha \in \mathrm{GL}_{2 g}(R) \mid \alpha^{\mathrm{T}} J \alpha=v J \text { for some } v \in R^{\times}\right\} \text {where } J=\left[\begin{array}{cc}
O & -I_{g} \\
I_{g} & O
\end{array}\right] \\
\mathrm{Sp}_{2 g}(R) & =\left\{\alpha \in \mathrm{GL}_{2 g}(R) \mid \alpha^{\mathrm{T}} J \alpha=J\right\}
\end{aligned}
$$

Here, $\alpha^{\mathrm{T}}$ stands for the transpose of the matrix $\alpha$. Observe that the relation $\alpha^{\mathrm{T}} J \alpha=v J$ implies $\operatorname{det}(\alpha)=\nu^{g}[19,(1.11)]$. If $\alpha$ belongs to either $\operatorname{GSp}_{2 g}(R)$ or $\operatorname{Sp}_{2 g}(R)$, then $\alpha^{\mathrm{T}}$ also belongs to the same group [19, p. 17].

The symplectic group $\mathrm{Sp}_{2 g}(\mathbb{Z})$ acts on the Siegel upper half-space

$$
\mathbb{H}_{g}=\left\{Z \in M_{g}(\mathbb{C}) \mid Z^{\mathrm{T}}=Z, \operatorname{Im}(Z) \text { is positive definite }\right\}
$$

by

$$
\begin{equation*}
\gamma(Z)=(A Z+B)(C Z+D)^{-1} \quad\left(\gamma \in \operatorname{Sp}_{2 g}(\mathbb{Z}), Z \in \mathbb{H}_{g}\right), \tag{4}
\end{equation*}
$$

where $A, B, C, D$ are $g \times g$ block matrices of $\gamma=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right]$ (see e.g. [5, Proposition 1 in §1]). For a positive integer $N$ let

$$
\Gamma(N)=\left\{\gamma \in \operatorname{Sp}_{2 g}(\mathbb{Z}) \mid \gamma \equiv I_{2 g}\left(\bmod N \cdot M_{2 g}(\mathbb{Z})\right)\right\} .
$$

We call a holomorphic function $f: \mathbb{H}_{g} \rightarrow \mathbb{C}$ a Siegel modular form of weight $k$ and level $N$ if
(M1) $f(\gamma(Z))=\operatorname{det}(C Z+D)^{k} f(Z)$ for every $\gamma=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right] \in \Gamma(N)$,
(M2) $f$ is holomorphic at every cusp when $g=1$.
Every Siegel modular form $f$ can be expressed as

$$
f(Z)=\sum_{\beta} c(\beta) e(\operatorname{tr}(\beta Z) / N) \quad(c(\beta) \in \mathbb{C})
$$

where $\beta$ runs over all $g \times g$ positive semi-definite symmetric matrices with nondiagonal entries in $\frac{1}{2} \mathbb{Z}$ and diagonal entries in $\mathbb{Z}$, and $e(x)=e^{2 \pi \mathrm{i} x}(x \in \mathbb{R})$ [5, p. 44]. Here, we call $c(\beta)$ the Fourier coefficients of $f$. For a subfield $F$ of $\mathbb{C}$ we set
$\mathcal{M}_{k}(\Gamma(N), F)=$ the $F$-vector space of all Siegel modular forms of weight $k$ and level $N$ with Fourier coefficients in $F$,

$$
\begin{aligned}
\mathcal{M}_{k}(F)= & \bigcup_{N=1}^{\infty} \mathcal{M}_{k}(\Gamma(N), F), \\
\mathcal{A}_{0}(\Gamma(N), F)= & \text { the field of all meromorphic Siegel modular functions } \\
& \text { of the form } g / h, \text { with } g \in \mathcal{M}_{k}(F) \text { and } h \in \mathcal{M}_{k}(F) \backslash\{0\} \text { for some } k, \\
& \text { which are invariant under the group } \Gamma(N), \\
\mathcal{A}_{0}(F)= & \bigcup_{N=1}^{\infty} \mathcal{A}_{0}(\Gamma(N), F) .
\end{aligned}
$$

In particular, let

$$
\begin{aligned}
\mathcal{F}_{N} & =\mathcal{A}_{0}\left(\Gamma(N), \mathbb{Q}\left(\zeta_{N}\right)\right), \\
\mathcal{F} & =\bigcup_{N=1}^{\infty} \mathcal{F}_{N},
\end{aligned}
$$

where $\zeta_{N}=e(1 / N)$. For a number field $K$, let $K_{\mathrm{ab}}$ be the maximal abelian extension of $K$, and $K_{\mathbb{A}}^{\times}$be the idele group of $K$. By class field theory, every element $x$ of $K_{\mathbb{A}}^{\times}$ acts on $K_{\mathrm{ab}}$ as an automorphism. We denote this automorphism by $[x, K]$.

On the other hand, we let

$$
\begin{aligned}
G & =\mathrm{GSp}_{2 g}(\mathbb{Q}), \\
G_{+} & =\left\{\alpha \in G \mid \alpha^{\mathrm{T}} J \alpha=\nu J \text { for some } v>0\right\}, \\
G_{\mathbb{A}} & =\text { the adelization of } G \text { with } G_{0} \text { and } G_{\infty}
\end{aligned}
$$

the non-archimedean part and the archimedean part, respectively,
$G_{\mathbb{A}+}=G_{0} G_{\infty+}$, where $G_{\infty+}$ is the identity component of $G_{\infty}$.
Shimura presented in [19, Theorem 8.10] a group homomorphism

$$
\tau: G_{\mathbb{A}+} \rightarrow \operatorname{Aut}(\mathcal{F})
$$

satisfying the following properties: Let $f \in \mathcal{F}$.
(A1) $f^{\tau(\alpha)}=f \circ \alpha$ for all $\alpha \in G_{+}$, where $\alpha$ acts on $\mathbb{H}_{g}$ by the same way as in (4).
(A2) $f^{\tau(\iota(s))}=f^{[s, \mathbb{Q}]}$ for all $s \in \prod_{p} \mathbb{Z}_{p}^{\times}$, where $\iota(s)=\left[\begin{array}{cc}I_{g} & O \\ O & s^{-1} I_{g}\end{array}\right]$. Here, the action of $[s, \mathbb{Q}]$ on $f$ is understood as the action of it on the Fourier coefficients of $f$ (see also [16, Theorem 5]).

Note that the mapping $s \mapsto[s, \mathbb{Q}]$ yields an isomorphism of $\prod_{p} \mathbb{Z}_{p}^{\times}$onto $\operatorname{Gal}(\mathbb{Q} \mathrm{ab} / \mathbb{Q})$ [19, § 8.1]. Then, $\mathcal{F}_{N}$ coincides with the fixed field of $\mathcal{F}$ by the subgroup

$$
\mathbb{Q}^{\times} \cdot\left\{\alpha \in G_{\mathbb{A}+} \mid \alpha_{p} \in \mathrm{GL}_{2 g}\left(\mathbb{Z}_{p}\right), \alpha_{p} \equiv I_{2 g}\left(\bmod N \cdot M_{2 g}\left(\mathbb{Z}_{p}\right)\right) \text { for all primes } \mathrm{p}\right\}
$$

of $G_{\mathbb{A}+}[16$, Theorem 3] or $[19$, Theorem 8.10 (6)].

## 4 Siegel modular functions in terms of theta constants

Let $g$ and $N$ be positive integers, and let $\mathbf{r}, \mathbf{s} \in(1 / N) \mathbb{Z}^{g}$. The (classical) theta constant $\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ is defined by

$$
\theta\left(\left[\begin{array}{l}
\mathbf{r}  \tag{5}\\
\mathbf{s}
\end{array}\right], Z\right)=\sum_{\mathbf{n} \in \mathbb{Z}^{g}} e\left(\frac{1}{2}(\mathbf{n}+\mathbf{r})^{\mathrm{T}} Z(\mathbf{n}+\mathbf{r})+(\mathbf{n}+\mathbf{r})^{\mathrm{T}} \mathbf{s}\right) \quad\left(Z \in \mathbb{H}_{g}\right)
$$

For a matrix $E \in M_{g}(\mathbb{Z})$ we mean by $\{E\}$ the $g$-vector whose components are the diagonal entries of $E$.

Lemma 4.1 We have the following properties of theta constants.
(i) $\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ is identically zero if and only if $\mathbf{r}, \mathbf{s} \in(1 / 2) \mathbb{Z}^{g}$ and $e\left(2 \mathbf{r}^{\mathrm{T}} \mathbf{s}\right)=-1$.
(ii) $\theta\left(-\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)=\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$.
(iii) If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{g}$, then $\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right]+\left[\begin{array}{l}\mathbf{a} \\ \mathbf{b}\end{array}\right], Z\right)^{N}=\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)^{N}$.
(iv) If $\gamma=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right] \in \operatorname{Sp}_{2 g}(\mathbb{Z})$, then

$$
\begin{aligned}
\theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], \gamma(Z)\right)^{4 N}= & \zeta(\gamma)^{2 N} \operatorname{det}(C Z+D)^{2 N} \\
& \times e\left(2 N\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right) \theta\left(\left[\begin{array}{l}
\mathbf{r}^{\prime} \\
\mathbf{s}^{\prime}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{c}
\left\{A^{\mathrm{T}} C\right\} \\
\left\{B^{\mathrm{T}} D\right\}
\end{array}\right], Z\right)^{4 N},
\end{aligned}
$$

where $\left[\begin{array}{l}\mathbf{r}^{\prime} \\ \mathbf{s}^{\prime}\end{array}\right]=\gamma^{\mathrm{T}}\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right]$ and $\zeta(\gamma)$ is a 4-th root of unity which depends only on $\gamma$.
(v) The function $\theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right) / \theta\left(\left[\begin{array}{l}\mathbf{0} \\ \mathbf{0}\end{array}\right], Z\right)$ belongs to $\mathcal{F}_{2 N^{2}}$. Furthermore, if $\alpha \in$ $G_{\mathbb{A}+} \cap \prod_{p} \mathrm{GL}_{2 g}\left(\mathbb{Z}_{p}\right)$ is such that $\alpha_{p} \equiv\left[\begin{array}{cc}I_{g} & O \\ O & t I_{g}\end{array}\right]\left(\bmod 2 N^{2} \cdot M_{2 g}\left(\mathbb{Z}_{p}\right)\right)$ for all rational primes $p$ with a positive integer $t$, then

$$
\left(\frac{\theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)}{\theta\left(\left[\begin{array}{l}
\mathbf{0} \\
\mathbf{0}
\end{array}\right], Z\right)}\right)^{\tau(\alpha)}=\frac{\theta\left(\left[\begin{array}{c}
\mathbf{r} \\
t \mathbf{s}
\end{array}\right], Z\right)}{\theta\left(\left[\begin{array}{l}
\mathbf{0} \\
\mathbf{0}
\end{array}\right], Z\right)}
$$

Proof (i) See [3, Theorem 2].
(ii) This is immediate from the definition (5).
(iii) See [17, p. 676 (13)].
(iv) See [1, Theta Transformation Formula 8.6.1]. Here, one can show that

$$
\zeta(\gamma)=\kappa\left(\gamma^{-1}\right)^{2},
$$

where $\kappa\left(\gamma^{-1}\right)$ is a 8 -th root of unity [1, p. 229] given in [1, Theta Transformation Formula 8.6.1].
(v) See [17, Proposition 1.7].

Let

$$
\begin{aligned}
& S_{-}=\left\{\left.\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \right\rvert\, \mathbf{a}, \mathbf{b} \in\{0,1 / 2\}^{g} \text { such that } e\left(2 \mathbf{a}^{\mathrm{T}} \mathbf{b}\right)=-1\right\}, \\
& \left.S_{+}=\left\{\left.\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right] \right\rvert\, \mathbf{c}, \mathbf{d} \in\{0,1 / 2\}^{g} \text { such that } e\left(2 \mathbf{c}^{\mathrm{T}} \mathbf{d}\right)=1\right)\right\} .
\end{aligned}
$$

By Lemma 4.1 (i) and (iv), each element $\gamma=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right] \in \operatorname{Sp}_{2 g}(\mathbb{Z})$ induces a permutation of the set $S_{-}$(and $S_{+}$) by the map

$$
\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \mapsto \gamma^{\mathrm{T}}\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{l}
\left\{A^{\mathrm{T}} C\right\} \\
\left\{B^{\mathrm{T}} D\right\}
\end{array}\right]\left(\bmod \mathbb{Z}^{2 g}\right)
$$

Definition 4.2 We define a function

$$
\begin{aligned}
\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)= & 2^{4 N} e\left(-2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right) \mathbf{r}^{\mathrm{T}} \mathbf{s}\right) \\
& \times \frac{\prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]_{\in S_{-}} \theta\left(\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]-\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)^{4 N\left(2^{g}+1\right)}}^{\prod_{\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]_{\in S_{+}}} \theta\left(\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right], Z\right)^{4 N\left(2^{g}-1\right)}}\left(Z \in \mathbb{H}_{g}\right) .}{} \quad . .
\end{aligned}
$$

Remark 4.3 (i) One can easily check that

$$
\left|S_{-}\right|=2^{g-1}\left(2^{g}-1\right) \text { and }\left|S_{+}\right|=2^{g-1}\left(2^{g}+1\right)
$$

Hence we have

$$
\operatorname{lcm}\left(\left|S_{-}\right|,\left|S_{+}\right|\right)=2^{g-1}\left(2^{g}-1\right)\left(2^{g}+1\right)=\left|S_{-}\right|\left(2^{g}+1\right)=\left|S_{+}\right|\left(2^{g}-1\right)
$$

(ii) When $g=1$, let $N \geq 2$ and $\left[\begin{array}{l}r \\ s\end{array}\right] \in(1 / N) \mathbb{Z}^{2} \backslash \mathbb{Z}^{2}$. By using Jacobi’s triple product identity [2, (17.3)] which reads

$$
\sum_{n \in \mathbb{Z}} a^{n} q^{n^{2} / 2}=\prod_{n=1}^{\infty}\left(1-q^{n}\right)\left(1+a q^{n-1 / 2}\right)\left(1+a^{-1} q^{n-1 / 2}\right) \quad\left(a \in \mathbb{C}^{\times}\right)
$$

one can justify

$$
\Theta\left(\left[\begin{array}{l}
r \\
s
\end{array}\right], \tau\right)=g_{\left[\begin{array}{r}
r \\
s
\end{array}\right]}(\tau)^{12 N} \quad(\tau \in \mathbb{H}) .
$$

This shows that the function in Definition 4.2 would be a multi-variable generalization of the Siegel function described in (2).

Lemma 4.4 We have the following transformation formulas for $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$.
(i) $\Theta\left(\left[\begin{array}{l}\mathbf{a} \\ \mathbf{b}\end{array}\right]+\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)=\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right]\right.$, $\left.Z\right)$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{g}$.
(ii) $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], \gamma(Z)\right)=\Theta\left(\gamma^{\mathrm{T}}\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ for all $\gamma \in \mathrm{Sp}_{2 g}(\mathbb{Z})$.
(iii) If $\alpha \in G_{\mathbb{A}+} \cap \prod_{p} \mathrm{GL}_{2 g}\left(\mathbb{Z}_{p}\right)$ is such that $\alpha_{p} \equiv\left[\begin{array}{cc}I_{g} & O \\ O & t I_{g}\end{array}\right]\left(\bmod 2 N^{2} \cdot M_{2 g}\left(\mathbb{Z}_{p}\right)\right)$ for all rational primes $p$ with a positive integer $t$, then

$$
\Theta\left(\left[\begin{array}{c}
\mathrm{r} \\
\mathrm{~s}
\end{array}\right], Z\right)^{\tau(\alpha)}=\Theta\left(\left[\begin{array}{c}
\mathrm{r} \\
t \mathrm{~s}
\end{array}\right], Z\right) .
$$

Proof (i) This is immediate from Lemma 4.1 (iii) and Definition 4.2.
(ii) Let $\gamma=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right] \in \operatorname{Sp}_{2 g}(\mathbb{Z})$. For $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^{g}$, we set

$$
\left[\begin{array}{l}
\mathbf{x}^{\prime} \\
\mathbf{y}^{\prime}
\end{array}\right]=\gamma^{T}\left[\begin{array}{l}
\mathbf{x} \\
\mathbf{y}
\end{array}\right]=\left[\begin{array}{l}
A^{T} \mathbf{x}+C^{T} \mathbf{y} \\
B^{T} \mathbf{x}+D^{T} \mathbf{y}
\end{array}\right] .
$$

Here we observe that

$$
\begin{aligned}
& \prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \in S_{-}} e\left(2 N\left(2^{g}+1\right)\left((\mathbf{a}-\mathbf{r})^{\mathrm{T}}(\mathbf{b}-\mathbf{s})-\left(\mathbf{a}^{\prime}-\mathbf{r}^{\prime}\right)^{\mathrm{T}}\left(\mathbf{b}^{\prime}-\mathbf{s}^{\prime}\right)\right)\right) \\
& \prod_{\left[\begin{array}{c}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]_{\in S_{+}} e\left(2 N\left(2^{g}-1\right)\left(\mathbf{c}^{\mathrm{T}} \mathbf{d}-\left(\mathbf{c}^{\prime}\right)^{\mathrm{T}} \mathbf{d}^{\prime}\right)\right)} \\
& =e\left(2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right) \\
& \times \frac{\prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \in S_{-}} e\left(2 N\left(\mathbf{a}^{\mathrm{T}} \mathbf{b}-\left(\mathbf{a}^{\prime}\right)^{\mathrm{T}} \mathbf{b}^{\prime}\right)\right)}{\prod_{\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]_{\in S_{+}}} e\left(-2 N\left(\mathbf{c}^{\mathrm{T}} \mathbf{d}-\left(\mathbf{c}^{\prime}\right)^{\mathrm{T}} \mathbf{d}^{\prime}\right)\right)} \\
& =e\left(2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text { because } A D^{\mathrm{T}}+B C^{\mathrm{T}}=I_{g}+2 B C^{\mathrm{T}} \\
& =e\left(2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& \times \prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \in\{0,1 / 2\}^{2 g}} e\left(-2 N\left(\mathbf{a}^{\mathrm{T}} A B^{\mathrm{T}} \mathbf{a}+\mathbf{b}^{\mathrm{T}} C D^{\mathrm{T}} \mathbf{b}\right)\right) \\
& \\
& \text { because } S_{+}=\{0,1 / 2\}^{2 g} \backslash S_{-} \\
& =e\left(2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right) \\
& \times \prod_{\mathbf{a} \in\{0,1 / 2\}^{g}} e\left(-2^{g+1} N\left(\mathbf{a}^{\mathrm{T}}\left(A B^{\mathrm{T}}+C D^{\mathrm{T}}\right) \mathbf{a}\right)\right) \\
& = \\
& e\left(2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\mathbf{r}^{\mathrm{T}} \mathbf{s}-\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right)
\end{aligned}
$$

Hence we derive
$\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], \gamma(Z)\right)$

$$
=2^{4 N} e\left(-2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right)\left(\left(\mathbf{r}^{\prime}\right)^{\mathrm{T}} \mathbf{s}^{\prime}\right)\right)
$$

$$
\times \frac{\prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]_{\in S_{-}} \theta\left(\gamma^{\mathrm{T}}\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{l}
\left\{A^{\mathrm{T}} C\right\} \\
\left\{B^{\mathrm{T}} D\right\}
\end{array}\right]-\left[\begin{array}{l}
\mathbf{r}^{\prime} \\
\mathbf{s}^{\prime}
\end{array}\right], Z\right)^{4 N\left(2^{g}+1\right)}}^{\prod_{\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]_{\in S_{+}}} \theta\left(\gamma^{\mathrm{T}}\left[\begin{array}{l}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{l}
\left\{A^{\mathrm{T}} C\right\} \\
\left\{B^{\mathrm{T}} D\right\}
\end{array}\right], Z\right)^{4 N\left(2^{g}-1\right)}}}{\text { by Lemma } 4.1 \text { (iv) }} \text { ( }
$$

by Lemma 4.1 (iii) and the fact that $\gamma$ is a permutation of $S_{-}$(and $S_{+}$)
$=\Theta\left(\left[\begin{array}{l}\mathbf{r}^{\prime} \\ \mathbf{s}^{\prime}\end{array}\right], Z\right)$.
(iii) Since $t$ is odd, $\left[\begin{array}{l}\mathbf{a} \\ \mathbf{b}\end{array}\right] \mapsto\left[\begin{array}{c}\mathbf{a} \\ t \mathbf{b}\end{array}\right]\left(\bmod \mathbb{Z}^{2 g}\right)$ gives rise to a permutation of $S_{-}$(and $S_{+}$). Furthermore, it follows from $[19, \S 8.1]$ that

$$
e(1 / N)^{\tau(\alpha)}=e(t / N)
$$

Hence we see by Lemma 4.1 (v) that

$$
\Theta\left(\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)^{\tau(\alpha)}
$$

$$
\begin{aligned}
& =2^{4 N} e\left(-2^{g} t N\left(2^{g}-1\right)\left(2^{g}+1\right) \mathbf{r}^{\mathrm{T}} \mathbf{s}\right) \frac{\prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \in S_{-}} \theta\left(\left[\begin{array}{c}
\mathbf{a} \\
t \mathbf{b}
\end{array}\right]-\left[\begin{array}{c}
\mathbf{r} \\
t \mathbf{s}
\end{array}\right], Z\right)^{4 N\left(2^{g}+1\right)}}{\prod_{\left[\begin{array}{c}
\mathbf{c} \\
\mathbf{d}
\end{array}\right] \in S_{+}} \theta\left(\left[\begin{array}{c}
\mathbf{c} \\
t \mathbf{d}
\end{array}\right], Z\right)^{4 N\left(2^{g}-1\right)}} \\
& =2^{4 N} e\left(-2^{g} N\left(2^{g}-1\right)\left(2^{g}+1\right) \mathbf{r}^{\mathrm{T}}(t \mathbf{s})\right) \frac{\prod_{\left[\begin{array}{l}
\mathbf{a} \\
\mathbf{b}
\end{array}\right] \in S_{-}} \theta\left(\left[\begin{array}{c}
\mathbf{a} \\
\mathbf{b}
\end{array}\right]-\left[\begin{array}{c}
\mathbf{r} \\
t \mathbf{s}
\end{array}\right], Z\right)^{4 N\left(2^{g}+1\right)}}{\prod_{\left[\begin{array}{c}
\mathbf{c} \\
\mathbf{d}
\end{array}\right]_{\in S_{+}} \theta\left(\left[\begin{array}{c}
\mathbf{c} \\
\mathbf{d}
\end{array}\right], Z\right)^{4 N\left(2^{g}-1\right)}}}
\end{aligned}
$$

by Lemma 4.1 (iii)

$$
=\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
t \mathbf{s}
\end{array}\right], Z\right) .
$$

Proposition 4.5 The function $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ belongs to $\mathcal{F}_{N}$.
Proof Since $\left|S_{-}\right|\left(2^{g}+1\right)=\left|S_{+}\right|\left(2^{g}-1\right)$, we have $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right) \in \mathcal{F}_{2 N^{2}}$ by Lemma 4.1 (v).

For any $\gamma \in \Gamma(N)$ we see that

$$
\begin{aligned}
\Theta\left(\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], \gamma(Z)\right)= & \Theta\left(\gamma^{\mathrm{T}}\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right) \text { by Lemma } 4.4 \text { (ii) } \\
= & \Theta\left(\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right) \text { by the fact } \gamma^{\mathrm{T}}\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right] \equiv\left[\begin{array}{l}
\mathbf{r} \\
\mathbf{s}
\end{array}\right]\left(\bmod \mathbb{Z}^{2 g}\right) \\
& \text { and Lemma } 4.4 \text { (i). }
\end{aligned}
$$

This claims that $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ lies in $\mathcal{A}_{0}\left(\Gamma(N), \mathbb{Q}\left(\zeta_{2 N^{2}}\right)\right)$.
Let $s$ be an element of $\prod_{p} \mathbb{Z}_{p}^{\times}$such that $[s, \mathbb{Q}]$ is the identity on $\mathbb{Q}\left(\zeta_{N}\right)$. Take a positive integer $t$ for which

$$
\iota(s)=\left[\begin{array}{cc}
I_{g} & O \\
O & s^{-1} I_{g}
\end{array}\right] \equiv\left[\begin{array}{cc}
I_{g} & O \\
O & t I_{g}
\end{array}\right]\left(\bmod 2 N^{2} \cdot M_{2 g}\left(\mathbb{Z}_{p}\right)\right) \quad \text { for all rational primes } p .
$$

Since $s_{p} \equiv 1\left(\bmod N \cdot \mathbb{Z}_{p}\right)$ for all rational primes $p$, we have $t \equiv 1(\bmod N)$. We then obtain

$$
\begin{aligned}
\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)^{[s, \mathbb{Q}]} & =\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right)^{\tau(\iota(s))} \text { by (A2) } \\
& =\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
t \mathbf{s}
\end{array}\right], Z\right) \quad \text { by Lemma } 4.4 \text { (iii) } \\
& =\Theta\left(\left[\begin{array}{c}
\mathbf{r} \\
\mathbf{s}
\end{array}\right], Z\right) \quad \text { by the fact } t \equiv 1(\bmod N) \text { and Lemma } 4.4 \text { (i). }
\end{aligned}
$$

This implies that every Fourier coefficient of $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ lies in $\mathbb{Q}\left(\zeta_{N}\right)$.
Therefore, we conclude that $\Theta\left(\left[\begin{array}{l}\mathbf{r} \\ \mathbf{s}\end{array}\right], Z\right)$ belongs to $\mathcal{F}_{N}$.

## 5 Siegel invariants

Let $n$ be a positive integer and $K$ be a CM-field with $[K: \mathbb{Q}]=2 n$, that is, $K$ is a totally imaginary quadratic extension of a totally real number field of degree $n$. Fix a set $\left\{\varphi_{1}, \ldots, \varphi_{n}\right\}$ of embeddings of $K$ into $\mathbb{C}$ such that $\varphi_{1}, \ldots, \varphi_{n}, \bar{\varphi}_{1}, \ldots, \bar{\varphi}_{n}$ are all the embeddings of $K$ into $\mathbb{C}$, which is called a CM-type of $K$. Take a finite Galois extension $L$ of $\mathbb{Q}$ containing $K$ and set

$$
\begin{aligned}
S & =\left\{\sigma \in \operatorname{Gal}(L / \mathbb{Q})|\sigma|_{K}=\varphi_{i} \text { for some } 1 \leq i \leq n\right\}, \\
S^{*} & =\left\{\sigma^{-1} \mid \sigma \in S\right\}, \\
H^{*} & =\left\{\gamma \in \operatorname{Gal}(L / \mathbb{Q}) \mid \gamma S^{*}=S^{*}\right\} .
\end{aligned}
$$

Let $K^{*}$ be the subfield of $L$ corresponding to the subgroup $H^{*}$ of $\operatorname{Gal}(L / \mathbb{Q})$ and $\left\{\psi_{1}, \ldots, \psi_{g}\right\}$ be the set of all the embeddings of $K^{*}$ into $\mathbb{C}$ obtained from the elements of $S^{*}$. Then

$$
K^{*}=\mathbb{Q}\left(\sum_{i=1}^{n} a^{\varphi_{i}} \mid a \in K\right)
$$

and it is also a CM-field with a (primitive) CM-type $\left\{\psi_{1}, \ldots, \psi_{g}\right\}[18$, Proposition 28 in §8.3]. Here, $K^{*}$ and $\left\{\psi_{1}, \ldots, \psi_{g}\right\}$ are called the reflex field and the reflex type of ( $K ;\left\{\varphi_{i}\right\}_{i=1}^{n}$ ), respectively. We define an embedding

$$
\begin{aligned}
\Psi: K^{*} & \rightarrow \mathbb{C}^{g} \\
a & \mapsto\left[\begin{array}{c}
a^{\psi_{1}} \\
\vdots \\
a^{\psi_{g}}
\end{array}\right] .
\end{aligned}
$$

For an element $c$ of $K^{*}$ which is purely imaginary, define an $\mathbb{R}$-bilinear form $E_{c}$ : $\mathbb{C}^{g} \times \mathbb{C}^{g} \rightarrow \mathbb{R}$ as

$$
E_{C}(\mathbf{u}, \mathbf{v})=\sum_{j=1}^{g} c^{\psi_{j}}\left(u_{j} \bar{v}_{j}-\bar{u}_{j} v_{j}\right) \quad\left(\mathbf{u}=\left[\begin{array}{c}
u_{1} \\
\vdots \\
u_{g}
\end{array}\right], \mathbf{v}=\left[\begin{array}{c}
v_{1} \\
\vdots \\
v_{g}
\end{array}\right] \in \mathbb{C}^{g}\right) .
$$

Then we know that

$$
E_{c}(\Psi(a), \Psi(b))=\operatorname{Tr}_{K^{*} / \mathbb{Q}}(c a \bar{b}) \text { for all } a, b \in K^{*} .
$$

Assumption 5.1 For the remainder of the paper we assume that the following two conditions hold:
(C1) The complex torus $\mathbb{C}^{g} / \Psi\left(\mathcal{O}_{K^{*}}\right)$ can be given a structure of a principally polarized abelian variety.
(C2) $\left(K^{*}\right)^{*}=K$.
Remark 5.2 (i) It is well known that a complex torus can be equipped with a structure of an abelian variety if and only if there is a non-degenerate Riemann form on the torus in the sense of $[18, \S 3.1]$. See also [1, §4.2].
(ii) The assumption ( C 1 ) is equivalent to saying that there is an element $\xi$ of $K^{*}$ satisfying the following properties:
(P1) $\xi^{\psi_{i}}$ lies on the positive imaginary axis for every $1 \leq i \leq g$.
(P2) The map $E_{\xi}$ yields a Riemann form on $\mathbb{C}^{g} / \Psi\left(\mathcal{O}_{K^{*}}\right)$.
(P3) $\delta_{K^{*}}^{-1}=\xi \mathcal{O}_{K^{*}}$, where $\delta_{K^{*}}$ is the different ideal of $K^{*}$.
See [18, Theorem 4 in §6.2]. In this case, we call the pair $\left(\mathbb{C}^{g} / \Psi\left(\mathcal{O}_{K^{*}}\right), E_{\xi}\right)$ a principally polarized abelian variety. If the narrow class number of the maximal real subfield $F$ of $K^{*}$ is one, then the assumption (C1) is always true. Indeed, one can choose an element $\zeta$ of $K^{*}$ such that $K^{*}=F(\zeta)$ and $\zeta^{\psi_{j}}$ lies on the positive imaginary axis for every $1 \leq j \leq g$ [18, p. 43]. Note that the different ideal $\delta_{\underline{K}^{*} / F}$ of $K^{*}$ over $F$ is generated by the elements $\alpha-\bar{\alpha}$ for $\alpha \in \mathcal{O}_{K^{*}}$. Since $\bar{\zeta}=-\zeta$, we see that $\zeta(\alpha-\bar{\alpha}) \in F$ for every $\alpha \in \mathcal{O}_{K^{*}}$. Hence $\zeta^{-1} \delta_{K^{*}}^{-1}=\left(\zeta \delta_{K^{*} / F}\right)^{-1} \delta_{F}^{-1}$ is generated by an ideal of $F$. Since the narrow class number of $F$ is one, $\zeta^{-1} \delta_{K^{*}}^{-1}=x \mathcal{O}_{K^{*}}$ for some totally positive element $x$ of $F^{\times}$. Then $\xi=x \zeta$ satisfies (P1)-(P3).
(iii) The assumption (C2) holds if and only if ( $K ;\left\{\varphi_{i}\right\}_{i=1}^{n}$ ) is a primitive CM-type, that is, the abelian varieties of this CM-type are simple [18, §8.2, Proposition 26].
(iv) Throughout this paper, we fix an element $\xi$ of $K^{*}$ satisfying (P1)-(P3) so that $\left(\mathbb{C}^{g} / \Psi\left(\mathcal{O}_{K^{*}}\right), E_{\xi}\right)$ becomes a principally polarized abelian variety.

By Assumption 5.1 (C2), one can define a group homomorphism

$$
\begin{aligned}
\varphi: K^{\times} & \rightarrow\left(K^{*}\right)^{\times} \\
a & \mapsto \prod_{i=1}^{n} a^{\varphi_{i}},
\end{aligned}
$$

and extend it naturally to a homomorphism of idele groups $\varphi: K_{\mathbb{A}}^{\times} \rightarrow\left(K^{*}\right)_{\mathbb{A}}^{\times}$. It is also known that for a fractional ideal $\mathfrak{a}$ of $K$ there exists a fractional ideal $\varphi(\mathfrak{a})$ of $K^{*}$
such that

$$
\begin{equation*}
\varphi(\mathfrak{a}) \mathcal{O}_{L}=\prod_{i=1}^{n}\left(\mathfrak{a} \mathcal{O}_{L}\right)^{\varphi_{i}} \tag{6}
\end{equation*}
$$

[18, Proposition 29 in §8.3].
For a number field $F$ and a nonzero integral ideal $\mathfrak{a}$ of $F$ let $\mathcal{N}_{F}(\mathfrak{a})$ be the absolute norm of $\mathfrak{a}$, namely, $\mathcal{N}_{F}(\mathfrak{a})=\left|\mathcal{O}_{F} / \mathfrak{a}\right|\left(\right.$ so, $\left.N_{F / \mathbb{Q}}(\mathfrak{a})=\mathcal{N}_{F}(\mathfrak{a}) \mathbb{Z}\right)$. In general, for a fractional ideal $\mathfrak{b}$ of $F$ with prime ideal factorization $\mathfrak{b}=\prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$ we define $\mathcal{N}_{F}(\mathfrak{b})=$ $\prod_{\mathfrak{p}} \mathcal{N}_{F}(\mathfrak{p})^{e_{\mathfrak{p}}}$. Furthermore, let $D_{F / \mathbb{Q}}(\mathfrak{b})$ be the discriminant ideal of $\mathfrak{b}$ and $d_{F / \mathbb{Q}}(\mathfrak{b})$ be its positive generator in $\mathbb{Q}$. We then have the relation

$$
d_{F / \mathbb{Q}}(\mathfrak{b})=\mathcal{N}_{F}(\mathfrak{b})^{2} d_{F / \mathbb{Q}}\left(\mathcal{O}_{F}\right)
$$

[10, Proposition 13 in Chapter III].
Let $K_{0}$ be the fixed field of $L$ by the subgroup

$$
\left.\left.\langle\sigma \in \operatorname{Gal}(L / \mathbb{Q})| \sigma\right|_{K}=\varphi_{i} \text { for some } i\right\rangle
$$

of $\operatorname{Gal}(L / \mathbb{Q})$. One can readily check that $K_{0}$ becomes either an imaginary quadratic subfield of $K$ and $K^{*}$, or $\mathbb{Q}$. In particular, we see from the assumption (C2) that $K_{0}=\mathbb{Q}$ when $g \geq 2$ [13, Remark (1) in p. 213] or [18, Theorem 3 in §6.2].

From now on, we let $\mathfrak{f}=\mathfrak{f}_{0} \mathcal{O}_{K}$ for a proper nontrivial ideal $\mathfrak{f}_{0}$ of $\mathcal{O}_{K_{0}}$. Let $C$ be a given ray class in $\mathrm{Cl}(\mathfrak{f})$. For an integral ideal $\mathfrak{c}$ in $C$ we set

$$
m_{\mathfrak{c}}=\sqrt[g]{\mathcal{N}_{K^{*}}\left(\left(f^{*}\right)^{-1} \varphi(\mathfrak{c})\right)}
$$

where $f^{*}=\mathfrak{f}_{0} \mathcal{O}_{K^{*}}$. Let $d_{0}=2 /\left[K_{0}: \mathbb{Q}\right]$. Then we get

$$
\begin{align*}
m_{\mathfrak{c}} & =\sqrt[g]{\mathcal{N}_{K^{*}}\left(\mathfrak{f}^{*} \overline{\mathfrak{f}^{*}}\right)^{-1 / 2} \mathcal{N}_{K^{*}}(\varphi(\mathfrak{c}) \overline{\varphi(\mathfrak{c})})^{1 / 2}} \\
& =\sqrt[g]{\mathcal{N}_{K^{*}}\left(\mathcal{N}_{K_{0}}\left(\mathfrak{f}_{0}\right)^{d_{0}} \mathcal{O}_{K^{*}}\right)^{-1 / 2} \mathcal{N}_{K^{*}}\left(\mathcal{N}_{K}(\mathfrak{c})\right)^{1 / 2}}  \tag{7}\\
& =\mathcal{N}_{K_{0}}\left(\mathfrak{f}_{0}\right)^{-d_{0}} \mathcal{N}_{K}(\mathfrak{c})
\end{align*}
$$

Since $f^{*} \overline{\mathfrak{f}^{*}}=\mathcal{N}_{K_{0}}\left(\mathfrak{f}_{0}\right)^{d_{0}} \mathcal{O}_{K^{*}}$, one can deduce from [7, Lemma 5.3] that

$$
\mathcal{P}_{\mathfrak{c}}=\left(\mathbb{C}^{g} / \Psi\left(\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1}\right), E_{\xi m_{\mathfrak{c}}}\right)
$$

is also a principally polarized abelian variety. Let $\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{2 g}\right\}$ be a symplectic basis of $\mathcal{P}_{\mathfrak{c}}$, and let $y_{1}, \ldots, y_{2 g}$ be elements of $f^{*} \varphi(\mathfrak{c})^{-1}$ satisfying $\mathbf{b}_{j}=\Psi\left(y_{j}\right)(1 \leq j \leq 2 g)$. As is well known [1, Proposition 8.1.1], the $g \times g$ matrix

$$
Z_{0}^{*}=\left[\mathbf{b}_{g+1} \cdots \mathbf{b}_{2 g}\right]^{-1}\left[\mathbf{b}_{1} \cdots \mathbf{b}_{g}\right]
$$

belongs to $\mathbb{H}_{g}$ and we call it a CM-point. Since the smallest positive integer $N$ in $\mathfrak{f}=\mathfrak{f}_{0} \mathcal{O}_{K}$ also belongs to $\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1}=\mathfrak{f}_{0} \varphi(\mathfrak{c})^{-1}$, we can express $N$ as

$$
\begin{equation*}
N=\sum_{j=1}^{2 g} r_{j} y_{j} \quad \text { for some unique integers } r_{1}, \ldots, r_{2 g} . \tag{8}
\end{equation*}
$$

Definition 5.3 We define the Siegel invariant $\Theta_{\mathfrak{f}}(C)$ modulo $f$ at $C$ by

$$
\Theta_{f}(C)=\Theta\left(\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right], Z_{0}^{*}\right) .
$$

Remark 5.4 When $g=1$, Assumption 5.1 always holds and $\Theta_{f}(C)$ becomes the Siegel-Ramachandra invariant modulo $\mathfrak{f}$ at $C$ described in (3).

This invariant $\Theta_{\mathfrak{f}}(C)$ is well defined, independent of the choices of a symplectic basis of $\mathcal{P}_{\mathfrak{c}}$ and an integral ideal $\mathfrak{c}$ in $C$ as follows:

Proposition $5.5 \Theta_{f}(C)$ does not depend on the choice of a symplectic basis of $\mathcal{P}_{\mathfrak{c}}$.
Proof Let $\left\{\widetilde{\mathbf{b}}_{1}, \ldots, \widetilde{\mathbf{b}}_{2 g}\right\}$ be another symplectic basis of $\mathcal{P}_{c}$ and let $\widetilde{Z}_{0}^{*}$ be the associated CM-point in $\mathbb{H}_{g}$. Then we have

$$
\left[\widetilde{\mathbf{b}}_{1} \cdots \tilde{\mathbf{b}}_{2 g}\right]=\left[\begin{array}{lll}
\mathbf{b}_{1} & \cdots & \mathbf{b}_{2 g}
\end{array}\right] \beta \quad \text { for some } \beta=\left[\begin{array}{cc}
A & B  \tag{9}\\
C & D
\end{array}\right] \in \operatorname{Sp}_{2 g}(\mathbb{Z})
$$

and

$$
\begin{equation*}
\widetilde{Z}_{0}^{*}=\beta^{\mathrm{T}}\left(Z_{0}^{*}\right) \tag{10}
\end{equation*}
$$

[7, Proposition 6.1].
Let $\widetilde{y}_{1}, \ldots, \widetilde{y}_{2 g}$ be elements of $\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1}$ such that $\widetilde{\mathbf{b}}_{j}=\Psi\left(\widetilde{y}_{j}\right)(1 \leq j \leq 2 g)$. Together with (8) we can express $N$ as

$$
N=\sum_{j=1}^{2 g} r_{j} y_{j}=\sum_{j=1}^{2 g} \tilde{r}_{j} \tilde{y}_{j} \text { for some unique integers } \widetilde{r}_{1}, \ldots, \widetilde{r}_{2 g} .
$$

Applying the embedding $\Psi$ in the previous equation and using (9), we see that

$$
\Psi(N)=\left[\begin{array}{lll}
\mathbf{b}_{1} & \cdots & \mathbf{b}_{2 g}
\end{array}\right]\left[\begin{array}{c}
r_{1} \\
\vdots \\
r_{2 g}
\end{array}\right]=\left[\begin{array}{lll}
\widetilde{\mathbf{b}}_{1} & \cdots & \widetilde{\mathbf{b}}_{2 g}
\end{array}\right]\left[\begin{array}{c}
\widetilde{r}_{1} \\
\vdots \\
\widetilde{r}_{2 g}
\end{array}\right]=\left[\begin{array}{lll}
\mathbf{b}_{1} & \cdots & \mathbf{b}_{2 g}
\end{array}\right] \beta\left[\begin{array}{c}
\widetilde{r}_{1} \\
\vdots \\
\widetilde{r}_{2 g}
\end{array}\right],
$$

from which we have

$$
\left[\begin{array}{c}
\widetilde{r}_{1}  \tag{11}\\
\vdots \\
\widetilde{r}_{2 g}
\end{array}\right]=\beta^{-1}\left[\begin{array}{c}
r_{1} \\
\vdots \\
r_{2 g}
\end{array}\right]
$$

We then derive that

$$
\begin{aligned}
\Theta\left(\left[\begin{array}{c}
\tilde{r}_{1} / N \\
\vdots \\
\widetilde{r}_{2 g} / N
\end{array}\right], \widetilde{Z}_{0}^{*}\right)= & \Theta\left(\beta^{-1}\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right], \beta^{\mathrm{T}}\left(Z_{0}^{*}\right)\right) \text { by (10) and (11) } \\
= & \Theta\left(\beta \beta^{-1}\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right], Z_{0}^{*}\right) \text { by the fact } \beta^{\mathrm{T}} \in \mathrm{Sp}_{2 g}(\mathbb{Z}) \\
& \text { and Lemma } 4.4 \text { (ii) } \\
= & \Theta\left(\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right], Z_{0}^{*}\right)
\end{aligned}
$$

This completes the proof.
Proposition $5.6 \Theta_{f}(C)$ does not depend on the choice of an integral ideal $\mathfrak{c}$ in $C$.
Proof Let $\boldsymbol{c}^{\prime}$ be another integral ideal in the class $C$, and so

$$
\mathfrak{c}^{\prime}=v \mathfrak{c} \text { for some } v \in K^{\times} \text {such that } v \equiv^{*} 1(\bmod \mathfrak{f})
$$

([4, § IV.1]).
Then we may write $\nu$ as

$$
v=1+x \text { for some } x \in \mathfrak{f c}^{-1} .
$$

Let

$$
\mathbf{b}_{j}^{\prime}=\Psi\left(\varphi\left(v^{-1}\right) y_{j}\right)=\left[\begin{array}{cccc}
\varphi\left(v^{-1}\right)^{\psi_{1}} & 0 & \cdots & 0 \\
0 & \varphi\left(v^{-1}\right)^{\psi_{2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \varphi\left(v^{-1}\right)^{\psi_{g}}
\end{array}\right] \mathbf{b}_{j} \quad \text { for } 1 \leq j \leq 2 g .
$$

It then follows from the proof of [7, Proposition 6.3] that $\left\{\mathbf{b}_{1}^{\prime}, \ldots, \mathbf{b}_{2 g}^{\prime}\right\}$ is a symplectic basis of $\mathcal{P}_{\mathfrak{c}^{\prime}}$, and the associated CM-point is

$$
\left[\mathbf{b}_{g+1}^{\prime} \cdots \mathbf{b}_{2 g}^{\prime}\right]^{-1}\left[\mathbf{b}_{1}^{\prime} \cdots \mathbf{b}_{g}^{\prime}\right]=\left[\mathbf{b}_{g+1} \cdots \mathbf{b}_{2 g}\right]^{-1}\left[\mathbf{b}_{1} \cdots \mathbf{b}_{g}\right]=Z_{0}^{*}
$$

Since $K_{0}$ is the subfield of $K$ fixed by the subgroup $\left.\langle\sigma \in \operatorname{Gal}(L / \mathbb{Q})| \sigma\right|_{K}=$ $\varphi_{i}$ for some $\left.i\right\rangle$ of $\operatorname{Gal}(L / \mathbb{Q})$, we have $\left(\mathfrak{f} \mathcal{O}_{L}\right)^{\varphi_{i}}=\left(\mathfrak{f}_{0} \mathcal{O}_{L}\right)^{\varphi_{i}}=\mathfrak{f}_{0} \mathcal{O}_{L}=\mathfrak{f}^{*} \mathcal{O}_{L}$. Hence we see from the fact $x \in \mathfrak{f c}^{-1}$ that

$$
\begin{equation*}
\varphi(\nu)=\prod_{i=1}^{n}(1+x)^{\varphi_{i}} \in K^{*} \cap\left(1+\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1} \mathcal{O}_{L}\right)=1+\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1} \tag{12}
\end{equation*}
$$

Since $N \in \mathfrak{f}^{*} \varphi\left(\mathfrak{c}^{\prime}\right)^{-1}$ and $\left\{\varphi\left(v^{-1}\right) y_{1}, \ldots, \varphi\left(\nu^{-1}\right) y_{2 g}\right\}$ is a $\mathbb{Z}$-basis for $\mathfrak{f}^{*} \varphi\left(\mathfrak{c}^{\prime}\right)^{-1}$, one can express $N$ as

$$
N=\sum_{j=1}^{2 g} r_{j}^{\prime} \varphi\left(v^{-1}\right) y_{j} \quad \text { for some integers } r_{1}^{\prime}, \ldots, r_{2 g}^{\prime}
$$

Hence we have

$$
\varphi(v)=\sum_{j=1}^{2 g}\left(r_{j}^{\prime} / N\right) y_{j},
$$

which implies by (8) and (12)

$$
\left[\begin{array}{c}
r_{1}^{\prime} / N \\
\vdots \\
r_{2 g}^{\prime} / N
\end{array}\right] \in\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right]+\mathbb{Z}^{2 g}
$$

Therefore, the proposition follows from Lemma 4.1 (iii).

## 6 Galois conjugates of Siegel invariants

Finally, we shall show that under Assumption 5.1 the Siegel invariant $\Theta_{f}(C)$ lies in the ray class field $K_{\mathfrak{f}}$ and satisfies the natural transformation formula via the Artin reciprocity map for $\mathfrak{f}$.

Let $h: K^{*} \rightarrow M_{2 g}(\mathbb{Q})$ be the regular representation with respect to the $\mathbb{Q}$-basis $\left\{y_{1}, \ldots, y_{2 g}\right\}$ of $K^{*}$, that is, $h$ is the map given by the relation

$$
h(a)\left[\begin{array}{c}
y_{1}  \tag{13}\\
\vdots \\
y_{2 g}
\end{array}\right]=a\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{2 g}
\end{array}\right] \quad\left(a \in K^{*}\right) .
$$

We naturally extend $h$ to the map $\left(K^{*}\right)_{\mathbb{A}} \rightarrow M_{2 g}\left(\mathbb{Q}_{\mathbb{A}}\right)$, and also denote it by $h$.

Proposition 6.1 (Shimura's Reciprocity Law) Let $f \in \mathcal{F}$. If $f$ is finite at $Z_{0}^{*} \in \mathbb{H}_{g}$, then $f\left(Z_{0}^{*}\right)$ belongs to $K_{a b}$. Moreover, if $s \in K_{\mathbb{A}}^{\times}$, then we get $h(\varphi(s)) \in G_{\mathbb{A}+}$ and

$$
f\left(Z_{0}^{*}\right)^{[s, K]}=f^{\tau\left(h\left(\varphi(s)^{-1}\right)\right)}\left(Z_{0}^{*}\right)
$$

Proof See [19, Lemma 9.5 and Theorem 9.6].
Remark 6.2 Observe that we are assuming $\left(K^{*}\right)^{*}=K$.
Theorem 6.3 If $\Theta_{\mathfrak{f}}(C)$ is finite, then it lies in $K_{\mathfrak{f}}$. Furthermore, it satisfies

$$
\Theta_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{f}}(D)}=\Theta_{\mathfrak{f}}(C D) \text { for all } D \in \mathrm{Cl}(\mathfrak{f}) \text {, }
$$

where $\sigma_{\mathfrak{f}}$ is the Artin reciprocity map for $\mathfrak{f}$.
Proof Since $\Theta_{\mathfrak{f}}(C) \in K_{\text {ab }}$ by Propositions 4.5 and 6.1, there is a positive integer $M$ such that $2 N^{2} \mid M$ and $\Theta_{\mathfrak{f}}(C) \in K_{\mathfrak{g}}$, where $\mathfrak{g}=M \mathcal{O}_{K}$. We can take integral ideals $\mathfrak{c} \in C$ and $\mathfrak{d} \in D$ which are relatively prime to $\mathfrak{g}$ by using the surjectivity of the natural map $\mathrm{Cl}(\mathfrak{g}) \rightarrow \mathrm{Cl}(\mathfrak{f})$. Let $\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{2 g}\right\}$ and $\left\{\mathbf{d}_{1}, \ldots, \mathbf{d}_{2 g}\right\}$ be symplectic bases of $\mathcal{P}_{\mathfrak{c}}$ and $\mathcal{P}_{\mathfrak{c} \mathfrak{d}}$, respectively. Furthermore, let $y_{1}, \ldots, y_{2 g}$ and $z_{1}, \ldots, z_{2 g}$ be elements of $\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1}$ and $\mathfrak{f}^{*} \varphi(\mathfrak{c d})^{-1}$, respectively, such that $\mathbf{b}_{j}=\Psi\left(y_{j}\right)$ and $\mathbf{d}_{j}=\Psi\left(z_{j}\right)$ for $1 \leq j \leq 2 g$.

Since $f^{*} \varphi(\mathfrak{c})^{-1} \subseteq f^{*} \varphi(\mathfrak{c d})^{-1}=f^{*} \varphi(\mathfrak{c})^{-1} \varphi(\mathfrak{d})^{-1}$, we have

$$
\left[\begin{array}{c}
y_{1}  \tag{14}\\
\vdots \\
y_{2 g}
\end{array}\right]=\delta\left[\begin{array}{c}
z_{1} \\
\vdots \\
z_{2 g}
\end{array}\right] \quad \text { for some } \delta \in M_{2 g}(\mathbb{Z}) \cap \operatorname{GL}_{2 g}(\mathbb{Q})
$$

and hence

$$
\left[\mathbf{b}_{1} \cdots \mathbf{b}_{2 g}\right]=\left[\mathbf{d}_{1} \cdots \mathbf{d}_{2 g}\right] \delta^{\mathrm{T}}
$$

If we let $Z_{0}^{*}$ and $Z_{1}^{*}$ be the CM-points associated with $\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{2 g}\right\}$ and $\left\{\mathbf{d}_{1}, \ldots, \mathbf{d}_{2 g}\right\}$, respectively, then we obtain

$$
\begin{equation*}
Z_{1}^{*}=\delta^{-1}\left(Z_{0}^{*}\right) \tag{15}
\end{equation*}
$$

We also obtain

$$
\begin{aligned}
{\left[\begin{array}{cc}
O & -I_{g} \\
I_{g} & O
\end{array}\right] } & =\left[E_{\xi m_{\mathfrak{c}}}\left(\mathbf{b}_{i}, \mathbf{b}_{j}\right)\right]_{1 \leq i, j \leq 2 g} \\
& =\delta\left[E_{\xi m_{\mathfrak{c}}}\left(\mathbf{d}_{i}, \mathbf{d}_{j}\right)\right]_{1 \leq i, j \leq 2 g} \delta^{\mathrm{T}} \\
& =\delta\left[m_{\mathfrak{c}} m_{\mathfrak{c d}}^{-1} E_{\xi m_{\mathfrak{c d}}}\left(\mathbf{d}_{i}, \mathbf{d}_{j}\right)\right]_{1 \leq i, j \leq 2 g} \delta^{\mathrm{T}} \\
& =m_{\mathfrak{c}} m_{\mathfrak{c d}}^{-1} \delta\left[\begin{array}{cc}
O & -I_{g} \\
I_{g} & O
\end{array}\right] \delta^{\mathrm{T}}
\end{aligned}
$$

This shows that

$$
\begin{equation*}
\delta \in M_{2 g}(\mathbb{Z}) \cap G_{+} \text {with } \operatorname{det}(\delta)=\left(m_{\mathfrak{c}}^{-1} m_{\mathfrak{c d}}\right)^{g}=\mathcal{N}(\mathfrak{d})^{g} \quad \text { by }(7) . \tag{16}
\end{equation*}
$$

Let $s=\left(s_{p}\right)_{p}$ be an idele of $K$ such that

$$
\begin{cases}s_{p} & =1  \tag{17}\\ s_{p}\left(\mathcal{O}_{K}\right)_{p} & =\mathfrak{d}_{p} \\ \text { if } p \nmid M .\end{cases}
$$

If we denote by $\widetilde{D}$ the ray class in $\mathrm{Cl}(\mathfrak{g})$ containing $\mathfrak{d}$, then we get by (17) that

$$
\begin{align*}
{\left.[s, K]\right|_{K_{\mathfrak{g}}} } & =\sigma_{\mathfrak{g}}(\widetilde{D}),  \tag{18}\\
\varphi(s)_{p}^{-1}\left(\mathcal{O}_{K^{*}}\right)_{p} & =\varphi(\mathfrak{d})_{p}^{-1} \quad \text { for all rational primes } p
\end{align*}
$$

By (13)-(18), we deduce that for each rational prime $p$, the components of each of the vectors

$$
h\left(\varphi(s)^{-1}\right)_{p}\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{2 g}
\end{array}\right] \quad \text { and } \quad \delta^{-1}\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{2 g}
\end{array}\right]
$$

form a basis of $\mathfrak{f}^{*} \varphi(\mathfrak{c d})_{p}^{-1}=\mathfrak{f}^{*} \varphi(\mathfrak{c})^{-1} \varphi(\mathfrak{d})_{p}^{-1}$. Thus there is a matrix $u=\left(u_{p}\right)_{p} \in$ $\prod_{p} \mathrm{GL}_{2 g}\left(\mathbb{Z}_{p}\right)$ satisfying

$$
\begin{equation*}
h\left(\varphi(s)^{-1}\right)=u \delta^{-1} . \tag{19}
\end{equation*}
$$

On the other hand, there exists a matrix $\gamma \in \mathrm{Sp}_{2 g}(\mathbb{Z})$ such that

$$
\delta \equiv\left[\begin{array}{cc}
I_{g} & O  \tag{20}\\
O & \mathcal{N}(\mathfrak{d}) I_{g}
\end{array}\right] \gamma\left(\bmod M \cdot M_{2 g}(\mathbb{Z})\right)
$$

by (16) and the surjectivity of the reduction $\operatorname{Sp}_{2 g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / M \mathbb{Z})$ [15]. Since $h\left(\varphi(s)^{-1}\right)_{p}=I_{2 g}$ for all $p \mid M$ by (17), we achieve $u_{p}=\delta$ for all $p \mid M$ by (19). Hence we obtain by (20) that

$$
u_{p} \gamma^{-1} \equiv\left[\begin{array}{cc}
I_{g} & O  \tag{21}\\
O & \mathcal{N}(\mathfrak{d}) I_{g}
\end{array}\right]\left(\bmod M \cdot M_{2 g}\left(\mathbb{Z}_{p}\right)\right)
$$

for all rational primes $p$.
If we write

$$
N=\sum_{j=1}^{2 g} r_{j} y_{j} \quad \text { for some integers } r_{1}, \ldots, r_{2 g}
$$

then we see by (14) that

$$
\begin{align*}
N & =\left[r_{1} \cdots r_{2 g}\right]\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{2 g}
\end{array}\right]=\left(\left[r_{1} \cdots r_{2 g}\right] \delta\right)\left(\delta^{-1}\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{2 g}
\end{array}\right]\right) \\
& =\left(\left[r_{1} \cdots r_{2 g}\right] \delta\right)\left[\begin{array}{c}
z_{1} \\
\vdots \\
z_{2 g}
\end{array}\right] . \tag{22}
\end{align*}
$$

$$
\begin{aligned}
& \text { Letting } \mathbf{v}=\left[\begin{array}{c}
r_{1} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right] \text { we derive that } \\
& \Theta_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{g}}(\widetilde{D})}=\Theta_{\mathfrak{f}}(C)^{[s, K]} \text { by (18) } \\
& =\Theta\left(\mathbf{v}, Z_{0}^{*}\right)^{[s, K]} \text { by Definition } 5.3 \\
& =\left.\Theta(\mathbf{v}, Z)^{\tau\left(h\left(\varphi(s)^{-1}\right)\right)}\right|_{Z=Z_{0}^{*}} \quad \text { by Proposition } 6.1 \\
& =\left.\Theta(\mathbf{v}, Z)^{\tau\left(u \delta^{-1}\right)}\right|_{Z=Z_{0}^{*}} \text { by (19) } \\
& =\left.\Theta(\mathbf{v}, Z)^{\tau\left(u \gamma^{-1}\right) \tau(\gamma) \tau\left(\delta^{-1}\right)}\right|_{Z=Z_{0}^{*}} \\
& =\left.\Theta\left(\left[\begin{array}{cc}
I_{g} & O \\
O & \mathcal{N}(\mathfrak{d})) I_{g}
\end{array}\right] \mathbf{v}, Z\right)^{\tau(\gamma) \tau\left(\delta^{-1}\right)}\right|_{Z=Z_{0}^{*}} \quad \text { by (21) and Lemma } 4.4 \text { (iii) } \\
& =\left.\Theta\left(\gamma^{\mathrm{T}}\left[\begin{array}{cc}
I_{g} & O \\
O & \mathcal{N}(\mathfrak{d}) I_{g}
\end{array}\right] \mathbf{v}, Z\right)^{\tau\left(\delta^{-1}\right)}\right|_{Z=Z_{0}^{*}} \quad \text { by Lemma } 4.4 \text { (ii) } \\
& =\left.\Theta\left(\delta^{\mathrm{T}} \mathbf{v}, Z\right)^{\tau\left(\delta^{-1}\right)}\right|_{Z=Z_{0}^{*}} \quad \text { by (20) and Lemma } 4.4 \text { (i) } \\
& =\Theta\left(\delta^{\mathrm{T}} \mathbf{v}, \delta^{-1}\left(Z_{0}^{*}\right)\right) \text { owing to the fact } \delta \in G_{+} \text {and (A1) } \\
& =\Theta_{\mathfrak{f}}(C D) \text { by (15), (22) and Definition 5.3. }
\end{aligned}
$$

In particular, if $D$ is the identity class of $\mathrm{Cl}(\mathfrak{f})$ then $\sigma_{\mathfrak{g}}(\widetilde{D})$ leaves $\Theta_{\mathfrak{f}}(C)$ fixed. Therefore, we conclude that $\Theta_{\mathfrak{f}}(C)$ lies in $K_{\mathfrak{f}}$ as desired.

Lastly, we expect from Proposition 2.1 and [18, Main Theorem 2 in §16] that under the Assumption 5.1 the following conjecture will turn out to be affirmative.

Conjecture 6.4 The Siegel invariant $\Theta_{\mathfrak{f}}(C)$ discussed here is a primitive generator of the fixed field of $\operatorname{ker}(\widetilde{\varphi})$ in the ray class field $K_{\mathfrak{f}}$ of a CM-field $K$, where $\widetilde{\varphi}: \mathrm{Cl}(\mathfrak{f}) \rightarrow$ $\mathrm{Cl}\left(f^{*}\right)$ is the natural homomorphism induced from the map $\varphi$ defined in (6). Here, $\mathrm{Cl}\left(f^{*}\right)$ is the ray class group of $K^{*}$ modulo $\mathfrak{f}^{*}$.

Example 6.5 Let $\ell$ be an odd prime and $g=(\ell-1) / 2$. Let $K=\mathbb{Q}\left(\zeta_{\ell}\right)$ with $\zeta_{\ell}=$ $e^{2 \pi \mathrm{i} / \ell}$. Then $[K: \mathbb{Q}]=2 g$. For each $1 \leq i \leq g$, let $\varphi_{i}$ be the element of $\operatorname{Gal}(K / \mathbb{Q})$ determined by $\zeta_{\ell}^{\varphi_{i}}=\zeta_{\ell}^{i}$. Then $\left(K ;\left\{\varphi_{1}^{-1}, \varphi_{2}^{-1}, \ldots, \varphi_{g}^{-1}\right\}\right)$ is a primitive CM-type and its reflex is $\left(K ;\left\{\varphi_{1}, \varphi_{2}, \ldots, \varphi_{g}\right\}\right)\left[18\right.$, p. 64]. Furthermore, $\left(\mathbb{C}^{g} / \Psi\left(\mathcal{O}_{K}\right), E_{\xi}\right)$ with
$\xi=\left(\zeta_{\ell}-\zeta_{\ell}^{-1}\right) / \ell$ becomes a principally polarized abelian variety, where $\Psi(a)=$ $\left[\begin{array}{c}a^{\varphi_{1}} \\ \vdots \\ a^{\varphi_{g}}\end{array}\right]$ for $a \in K$ [8, pp. 817-818]. Hence it satisfies the Assumption 5.1.

Assume that the class number of $K$ is 1 . Let $\mathfrak{f}=N \mathcal{O}_{K}$ for a positive integer $N$, and let $C \in \mathrm{Cl}(\mathfrak{f})$. Take an integral ideal $\mathfrak{c}$ of $K$ in $C$. Then $\mathfrak{c}=\lambda \mathcal{O}_{K}$ for some $\lambda \in \mathcal{O}_{K}$. Let

$$
x_{j}= \begin{cases}\zeta_{\ell}^{2 j} & \text { for } 1 \leq j \leq g \\ \sum_{k=1}^{j-g} \zeta_{\ell}^{2 k-1} & \text { for } g+1 \leq j \leq 2 g\end{cases}
$$

and $\varphi(a)=\prod_{i=1}^{g} a^{\varphi_{i}^{-1}}$ for $a \in K$. Then $\left\{N \varphi(\lambda)^{-1} x_{j}\right\}_{j=1}^{2 g}$ is a free $\mathbb{Z}$-basis of $\mathfrak{f} \varphi(\mathfrak{c})^{-1}$ and

$$
\left[E_{\xi m_{\mathrm{c}}}\left(\Psi\left(N \varphi(\lambda)^{-1} x_{i}\right), \Psi\left(N \varphi(\lambda)^{-1} x_{j}\right)\right)\right]_{1 \leq i, j \leq 2 g}=\left[\begin{array}{cc}
O & -I_{g} \\
I_{g} & O
\end{array}\right]
$$

Thus $\left\{\Psi\left(N \varphi(\lambda)^{-1} x_{j}\right)\right\}_{j=1}^{2 g}$ is a symplectic basis of $\left(\mathbb{C}^{g} / \Psi\left(\mathfrak{f} \varphi(\mathfrak{c})^{-1}\right), E_{\xi m_{\mathfrak{c}}}\right)$ and the corresponding CM-point is

$$
\begin{align*}
Z_{\ell}^{*}= & {\left[\Psi\left(N \varphi(\lambda)^{-1} x_{g+1}\right) \cdots \Psi\left(N \varphi(\lambda)^{-1} x_{2 g}\right)\right]^{-1} } \\
& \times\left[\Psi\left(N \varphi(\lambda)^{-1} x_{1}\right) \cdots \Psi\left(N \varphi(\lambda)^{-1} x_{g}\right)\right] \\
= & {\left[\Psi\left(x_{g+1}\right) \cdots \Psi\left(x_{2 g}\right)\right]^{-1}\left[\Psi\left(x_{1}\right) \cdots \Psi\left(x_{g}\right)\right] . } \tag{23}
\end{align*}
$$

Note that $Z_{\ell}^{*}$ does not depend on a ray class $C$. On the other hand, there exist integers $r_{1}, r_{2}, \ldots, r_{2 g}$ such that

$$
N=\sum_{j=1}^{2 g} r_{j}\left(N \varphi(\lambda)^{-1}\right) x_{j}
$$

that is, $\varphi(\lambda)=\sum_{j=1}^{2 g} r_{j} x_{j}$. Then we obtain

$$
\Theta_{f}(C)=\Theta_{\left[\begin{array}{c}
r_{1} / N  \tag{24}\\
r_{2} / N \\
\vdots \\
r_{2 g} / N
\end{array}\right]}{ }^{\left(Z_{\ell}^{*}\right)}
$$

Now, consider the special case where $K=\mathbb{Q}\left(\zeta_{5}\right)$ and $\mathfrak{f}=7 \mathcal{O}_{K}$. One can readily show that $\left[K_{\mathfrak{f}}: K\right]=30$ and

$$
\mathrm{Cl}(\mathfrak{f})=\left\langle C_{1}\right\rangle \cong \mathbb{Z} / 30 \mathbb{Z},
$$

where $C_{1}$ denotes the ray class in $\mathrm{Cl}(\mathfrak{f})$ containing the ideal $\left(3+\zeta_{5}\right) \mathcal{O}_{K}$. Here we note that $K_{\mathfrak{f}}$ is not a Kummer extension of $K$. Let $C_{k}=C_{1}^{k}$ for an integer $k$. Then we have

$$
\begin{aligned}
\varphi\left(3+\zeta_{5}\right) & =\left(3+\zeta_{5}\right)\left(3+\zeta_{5}^{3}\right) \\
& =-6 \zeta_{5}-9 \zeta_{5}^{2}-6 \zeta_{5}^{3}-8 \zeta_{5}^{4} \text { since } \sum_{k=0}^{4} \zeta_{5}^{k}=0 \\
& \equiv 5 x_{1}+6 x_{2}+0 \cdot x_{3}+x_{4}(\bmod \mathfrak{f})
\end{aligned}
$$

where

$$
x_{1}=\zeta_{5}^{2}, \quad x_{2}=\zeta_{5}^{4}, \quad x_{3}=\zeta_{5}, \quad x_{4}=\zeta_{5}+\zeta_{5}^{3} .
$$

Hence we see from (23), (24) and Lemma 4.4 (i) that

$$
\Theta_{\mathfrak{f}}\left(C_{1}\right)=\Theta_{\left[\begin{array}{c}
5 / 7 \\
6 / 7 \\
0 \\
1 / 7
\end{array}\right]}^{\left[\begin{array}{l}
\text { ( }
\end{array}\right] \approx(3.63991-3.56536 \mathrm{i}) \times 10^{-139},, ~}
$$

where

$$
Z_{5}^{*}=\left[\begin{array}{ll}
\zeta_{5} & \zeta_{5}+\zeta_{5}^{3} \\
\zeta_{5}^{2} & \zeta_{5}^{2}+\zeta_{5}
\end{array}\right]^{-1}\left[\begin{array}{ll}
\zeta_{5}^{2} & \zeta_{5}^{4} \\
\zeta_{5}^{4} & \zeta_{5}^{3}
\end{array}\right] .
$$

In like manner, we obtain

$$
\begin{aligned}
& \Theta_{\mathfrak{f}}\left(C_{2}\right)=\Theta_{\left[\begin{array}{l}
5 / 7 \\
5 / 7 \\
1 / 7 \\
3 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(-8.08524+6.22260 \mathrm{i}) \times 10^{-89}, \\
& \Theta_{\mathrm{f}}\left(C_{3}\right)=\Theta_{\left[\begin{array}{l}
6 / 7 \\
4 / 7 \\
3 / 7 \\
6 / 7
\end{array}\right]}^{\left(Z_{5}^{*}\right) \approx(-2.33222+4.31812 \mathrm{i}) \times 10^{-62},} \\
& \Theta_{\mathfrak{f}}\left(C_{4}\right)=\Theta_{\left[\begin{array}{c}
4 / 7 \\
0 \\
1 / 7 \\
2 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(3.60113+1.64858 \mathrm{i}) \times 10^{-93}, \\
& \Theta_{\mathfrak{f}}\left(C_{5}\right)=\Theta_{\left[\begin{array}{c}
4 / 7 \\
0 \\
0 \\
1 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx 6.72312 \times 10^{-102}, ~}^{\text {, }} \\
& \Theta_{f}\left(C_{6}\right)=\Theta_{\left[\begin{array}{c}
6 / 7 \\
0 \\
2 / 7 \\
1 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(-1.10112-0.39890 \mathrm{i}) \times 10^{-107}, \\
& \Theta_{\mathrm{f}}\left(C_{7}\right)=\Theta_{\left[\begin{array}{l}
1 / 7 \\
5 / 7 \\
2 / 7 \\
6 / 7
\end{array}\right]}^{\left(Z_{5}^{*}\right) \approx(0.52715+3.21425 \mathrm{i}) \times 10^{-98}, ~, ~, ~}
\end{aligned}
$$

$$
\begin{aligned}
& \Theta_{f}\left(C_{8}\right)=\Theta_{\left[\begin{array}{l}
0 / 7 \\
1 / 7 \\
1 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(-8.08524-6.22260 \mathrm{i}) \times 10^{-89}, \\
& \Theta_{\mathfrak{f}}\left(C_{9}\right)=\Theta_{\left[\begin{array}{c}
5 / 7 \\
2 / 7 \\
0 \\
4 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(2.28063+4.81439 \mathrm{i}) \times 10^{-117}, ~, ~, ~}^{1 / 7}{ }^{2} \\
& \Theta_{f}\left(C_{10}\right)=\Theta^{[ }\left[\begin{array}{c}
0 \\
3 / 7 \\
0 \\
0
\end{array}\right]\left(Z_{5}^{*}\right) \approx 9.88496 \times 10^{-328}, \\
& \Theta_{f}\left(C_{11}\right)=\Theta_{\left[\begin{array}{l}
0 \\
4 / 7 \\
4 / 7 \\
1 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(3.68776+3.68367 \mathrm{i}) \times 10^{-155},
\end{aligned}
$$

$$
\begin{aligned}
& \Theta_{\mathfrak{f}}\left(C_{13}\right)=\Theta_{\left[\begin{array}{l}
5 / 7 \\
1 / 7 \\
6 / 7 \\
6 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(0.52715-3.21425 \mathrm{i}) \times 10^{-98}, ~}^{\text {, }} \\
& \Theta_{\mathfrak{f}}\left(C_{14}\right)=\Theta_{\left[\begin{array}{l}
4 / 7 \\
5 / 7 \\
5 / 7 \\
5 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(1.23513+0.77554 \mathrm{i}) \times 10^{-99}, ~}^{\text {, }} \\
& \Theta_{\mathrm{f}}\left(C_{15}\right)=\Theta_{\left[\begin{array}{c}
0 \\
4 / 7 \\
5 / 7 \\
4 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx 6.12704 \times 10^{-67},
\end{aligned}
$$

$$
\begin{aligned}
& \Theta_{f}\left(C_{18}\right)=\Theta_{\left[\begin{array}{l}
4 / 7 \\
1 / 7 \\
6 / 7 \\
2 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(-6.14912-0.77268 i) \times 10^{-86}, ~, ~, ~}^{\text {, }} \\
& \Theta_{\mathfrak{f}}\left(C_{19}\right)=\Theta_{\left[\begin{array}{c}
0 \\
2 / 7 \\
2 / 7 \\
1 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(3.63991+3.56536 \mathrm{i}) \times 10^{-139}, \\
& \left.\Theta_{\mathfrak{f}}\left(C_{20}\right)=\Theta^{[ } \begin{array}{c}
0 \\
0 \\
5 / 7 \\
2 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx 5.78483 \times 10^{-283},
\end{aligned}
$$

$$
\begin{aligned}
& \Theta_{\mathfrak{f}}\left(C_{21}\right)=\Theta_{\left[\begin{array}{l}
2 / 7 \\
6 / 7 \\
2 / 7 \\
4 / 7
\end{array}\right]}^{\left(Z_{5}^{*}\right) \approx(2.28063-4.81439 i) \times 10^{-117},} \\
& \Theta_{\mathfrak{f}}\left(C_{22}\right)=\Theta_{\left[\begin{array}{c}
0 \\
5 / 7 \\
6 / 7 \\
4 / 7
\end{array}\right]}\left(Z_{5}^{*}\right) \approx(-1.38020-2.93978 \mathrm{i}) \times 10^{-89}, \\
& \Theta_{f}\left(C_{23}\right)=\Theta_{\left[\begin{array}{l}
3 / 7 \\
6 / 7 \\
5 / 7 \\
2 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(-2.33675-8.05623 i) \times 10^{-90}, ~, ~, ~, ~}^{\text {, }} \\
& \Theta_{\mathfrak{f}}\left(C_{24}\right)=\Theta_{\left[\begin{array}{l}
6 / 7 \\
5 / 7 \\
4 / 7 \\
6 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(-1.10112+0.39890 \mathrm{i}) \times 10^{-107}, ~}^{\text {, }} \\
& \Theta_{f}\left(C_{25}\right)=\Theta_{\left[\begin{array}{l}
6 / 7 \\
1 / 7 \\
2 / 7 \\
6 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx 8.93647 \times 10^{-206},} \\
& \Theta_{\mathrm{f}}\left(C_{26}\right)=\Theta_{\left[\begin{array}{l}
2 / 7 \\
1 / 7 \\
2 / 7 \\
2 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(1.23513-0.77554 \mathrm{i}) \times 10^{-99}, ~}^{\text {, }} \text {, } \\
& \Theta_{\mathfrak{f}}\left(C_{27}\right)=\Theta_{\left[\begin{array}{c}
1 / 7 \\
0 \\
5 / 7 \\
5 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(-2.33222-4.31812 \mathrm{i}) \times 10^{-62}, ~, ~, ~}^{\text {, }} \\
& \left.\Theta_{\mathfrak{f}}\left(C_{28}\right)=\Theta_{\left[\begin{array}{c}
3 / 7 \\
4 / 7 \\
2 / 7 \\
0
\end{array}\right]}^{\left[\begin{array}{l}
\text { ( }
\end{array}\right]}{ }_{5}^{*}\right) \approx(-1.29271-0.48794 \mathrm{i}) \times 10^{-71}, \\
& \Theta_{\mathfrak{f}}\left(C_{29}\right)=\Theta_{\left[\begin{array}{l}
1 / 7 \\
5 / 7 \\
1 / 7 \\
4 / 7
\end{array}\right]\left(Z_{5}^{*}\right) \approx(3.68776-3.68367 \mathrm{i}) \times 10^{-155}, ~, ~, ~, ~}^{\text {, }} \\
& \Theta_{\mathfrak{f}}\left(C_{30}\right)=\Theta^{\left[\begin{array}{c}
6 / 7 \\
0 \\
0 \\
0
\end{array}\right]}\left(Z_{5}^{*}\right) \approx 4.96289 \times 10^{-453} .
\end{aligned}
$$

Here we estimate these values with the aid of Maple software [12]. Observe that

$$
\begin{aligned}
& \Theta_{\mathfrak{f}}\left(C_{1}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{19}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{2}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{8}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{3}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{27}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{4}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{16}\right)}, \\
& \Theta_{\mathfrak{f}}\left(C_{6}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{24}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{7}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{13}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{9}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{21}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{11}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{29}\right)}, \\
& \Theta_{\mathfrak{f}}\left(C_{12}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{18}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{14}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{26}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{17}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{23}\right)}, \quad \Theta_{\mathfrak{f}}\left(C_{22}\right)=\overline{\Theta_{\mathfrak{f}}\left(C_{28}\right)},
\end{aligned}
$$

and

$$
\Theta_{\mathfrak{f}}\left(C_{5 k}\right) \in \mathbb{R} \quad \text { for } 1 \leq k \leq 6 .
$$

Since all conjugates of $\Theta_{\mathfrak{f}}\left(C_{1}\right)$ are distinct, we get

$$
K_{\mathfrak{f}}=K\left(\Theta_{\mathfrak{f}}(C)\right) \text { for } C \in \mathrm{Cl}(\mathfrak{f}) .
$$

## References

1. Birkenhake, C., Lange, H.: Complex Abelian Varieties. Grundlehren der Mathematischen Wissenschaften, vol. 302. Springer, Berlin (2004)
2. Fine, N.J.: Basic Hypergeometric Series and Applications. Mathematical Surveys and Monographs, vol. 27. Amer. Math. Soc, Providence, RI (1988)
3. Igusa, J.: On the graded ring of theta-constants (II). Am. J. Math. 88(1), 221-236 (1966)
4. Janusz, G.J.: Algebraic Number Fields. Graduation Studies in Math, vol. 7, 2nd edn. Amer. Math. Soc., Providence, RI (1996)
5. Klingen, H.: Introductory Lectures on Siegel Modular Forms. Cambridge Studies in Advanced Mathematics, vol. 20. Cambridge Univ. Press, Cambridge (1990)
6. Kubert, D., Lang, S.: Modular Units. Grundlehren der Mathematischen Wissenschaften, vol. 244. Spinger, New York (1981)
7. Koo, J.K., Shin, D.H., Yoon, D.S.: Siegel families with application to class fields. Proc. R. Soc. Edinb. Sect. A 148(4), 751-771 (2018)
8. Koo, J.K., Yoon, D.S.: Construction of class fields over cyclotomic fields. Kyoto J. Math. 56(4), 803829 (2016)
9. Koo, J.K., Yoon, D.S.: Construction of ray class fields by smaller generators and applications. Proc. R. Soc. Edinb. Sect. A 147(4), 781-812 (2017)
10. Lang, S.: Algebraic Number Theory. Texts in Math, vol. 110, 2nd edn. Springer, New York (1986)
11. Lang, S.: Elliptic Functions. Grad. Texts in Math, vol. 112, 2nd edn. Spinger, New York (1987)
12. Maple: Maplesoft. A division of Waterloo Maple Inc., Waterloo, ON (2019)
13. Mumford, D.: Abelian Varieties. Tata Inst. Fundam. Res. Stud. Math, vol. 5. Hindustan Book Agency, New Delhi (2008)
14. Ramachandra, K.: Some applications of Kronecker's limit formula. Ann. Math. (2) 80, 104-148 (1964)
15. Rapinchuk, A.S.: Strong approximation for algebraic groups. Thin Groups and Superstrong Approximation. Math. Sci. Res. Inst. Publ., vol. 61, pp. 269-298. Cambridge Univ. Press, Cambridge (2014)
16. Shimura, G.: On the Fourier coefficients of modular forms of several variables. Göttingen, Nachr. Akad. Wiss., pp. 261-268 (1975)
17. Shimura, G.: Theta functions with complex multiplication. Duke Math. J. 43(4), 673-696 (1976)
18. Shimura, G.: Abelian Varieties with Complex Multiplication and Modular Functions. Princeton University Press, Princeton, NJ (1998)
19. Shimura, G.: Arithmeticity in the Theory of Automorphic Forms. Mathematical Surveys and Monographs, vol. 82. Amer. Math. Soc., Providence, RI (2000)
20. Siegel, C.L.: Advanced Analytic Number Theory. Tata Institute of Fundamental Research Studies in Mathematics, vol. 9, 2nd edn. Tata Institute of Fundamental Research, Bombay (1980)

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    $\boxed{\Delta}$ Dong Sung Yoon
    dsyoon@pusan.ac.kr
    Ja Kyung Koo
    jkgoo@kaist.ac.kr
    Gilles Robert
    gilles.rbrt@gmail.com
    Dong Hwa Shin
    dhshin@hufs.ac.kr
    1 Department of Mathematical Sciences, KAIST, Daejeon 34141, Republic of Korea
    2 Laboratoire de Mathématiques, Institut Fourier, B.P. 74, 38402 Saint-Martin-d'Hères, France
    3 Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Gyeonggi-do 17035, Republic of Korea
    4 Department of Mathematics Education, Pusan National University, Busan 46241, Republic of Korea

