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Two extensions of the Erdős–Szekeres problem

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Abstract. According to Suk's breakthrough result on the Erdős–Szekeres problem, any point set in general position in the plane, which has no n elements that form the vertex set of a convex n-gon, has at most $2^{n+O(n^{2/3}\log n)}$ points. We strengthen this theorem in two ways. First, we show that the result generalizes to convexity structures induced by pseudoline arrangements. Second, we improve the error term.

A family of n convex bodies in the plane is said to be in *convex position* if the convex hull of the union of no $n-1$ of its members contains the remaining one. If any *three* members are in convex position, we say that the family is in *general position*. Combining our results with a theorem of Dobbins, Holmsen, and Hubard, we significantly improve the best known upper bounds on the following two functions, introduced by Bisztriczky and Fejes Tóth and by Pach and Tóth, respectively. Let $c(n)$ (and $c'(n)$) denote the smallest positive integer N with the property that any family of N pairwise disjoint convex bodies in general position (resp., N convex bodies in general position, any pair of which share at most two boundary points) has an n-member subfamily in convex position. We show that $c(n) \le c'(n) \le 2^{n+O(\sqrt{n\log n})}$.

Keywords. Erdős–Szekeres conjecture, arrangements of pseudolines

1. Introduction

We say that a set of n points in the plane is in *convex position* if the convex hull of no $n-1$ of them contains the n-th point. If no three elements of the set are collinear (that is, any three points are in convex position), then the set is said to be in *general position*. According to a classical conjecture of Erdős and Szekeres [7], if P is a set of points in general position in the plane with $|P| \ge 2^{n-2}+1$, then it has n elements in convex

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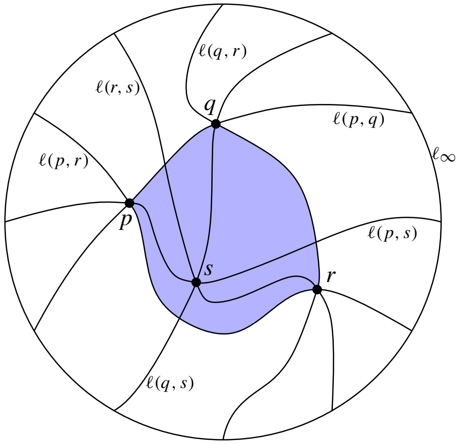

Fig. 1. A pseudoconfiguration of four points with the convex hull shaded.

position. This bound, if true, cannot be improved [8]. In a recent breakthrough, Suk [19] came close to proving the conjectured bound.

Theorem 1.1 (Suk, 2017). *Given any integer $n \geq 3$, let $e(n)$ denote the smallest number with the property that every family of at least $e(n)$ points in general position in the plane has n elements in convex position. Then*

$$e(n) \leq 2^{n+O(n^{2/3}\log n)}.$$

Pseudoconfigurations. A set of simple continuous curves in the Euclidean plane that start and end "at infinity" is called an *arrangement of pseudolines* if any two of them meet in precisely one point: at a proper crossing. A *pseudoconfiguration* is a finite set P of points in the Euclidean plane such that each pair of distinct points p and q in P span a unique *pseudoline*, denoted by $\ell(p,q)$, such that $L(P) = \{\ell(p,q) : p,q \in P,\ p \neq q\}$ form a pseudoline arrangement and for any $p,q \in P$, $p \neq q$ we have $\ell(p,q) = \ell(q,p)$ and $\ell(p,q) \cap P = \{p,q\}$; see [10].

This underlying pseudoline arrangement induces a convexity structure on the point configuration in a natural way. For any pair of points $p,q \in P$, the bounded portion of $\ell(p,q)$ between p and q is called the *pseudosegment* connecting p and q. If we delete from the plane all pseudosegments between the elements of P, the plane is divided into a number of connected components, precisely one of which is unbounded. The *convex hull* of the configuration is defined as the complement of the unbounded region, and is denoted by $\operatorname{conv} P$. We say that a subset $Q \subseteq P$ is in *convex position* if no point $p \in Q$ is in the convex hull of $Q \setminus \{p\}$.[1]

[1] Pseudoconfigurations also have a purely combinatorial characterization. They can be defined by several equivalent systems of axioms. Other names for pseudoconfigurations that can be found in the literature are *generalized configurations* [10], *uniform rank* 3 *acyclic oriented matroids* [4], and *CC-systems* [13].

It turns out that for four points there are only two combinatorially distinct pseudoconfigurations and both can be obtained from straight lines, but for five or more points there exist pseudoconfigurations that are not realizable by straight lines. Still, the number of possible pseudoconfigurations on five points is finite and we will leave the verification of some simple statements about at most five points in a pseudoconfiguration to the reader. (This applies in particular to Observations 2.2 and 2.3 below.)

Many basic theorems of convexity hold in this more general setting. For instance, a set of points is in convex position if and only if any four of its elements are in convex position [5]. This is Carathéodory's theorem in the plane.

Goodman and Pollack [11] proposed the generalization of the Erdős–Szekeres problem to pseudoconfigurations. The original "cup-cap" proof due to Erdős and Szekeres [7] readily generalizes to this setting:

Theorem 1.2. *Let P be a pseudoconfiguration. If $|P| \geq 4^n$, then P contains an n-element subset in convex position.*

The purpose of this note is to show that Suk's breakthrough result, Theorem 1.1, carries over to pseudoconfigurations. In the process we also improve on the error term.

Theorem 1.3. *Given any $n \geq 3$, let $b(n)$ denote the smallest number such that every pseudoconfiguration of size at least $b(n)$ has n members in convex position. Then*

$$b(n) \leq 2^{n+O(\sqrt{n \log n})}.$$

Clearly, $b(n) \geq e(n)$ for all n, thus our result also bounds the function $e(n)$ defined for the original Erdős–Szekeres problem (cf. Theorem 1.1).

Remark 1.4. In our proof of Theorem 1.3, for the sake of clarity, we do not focus on the constant in the error term in the bound on $b(n)$. However, with a less wasteful calculation (given at the end of the paper) we obtain the bound

$$b(n) \leq 2^{n+(8\sqrt{2}/3+o(1))\sqrt{n \log n}}.$$

Families of convex bodies. Bisztriczky and G. Fejes Tóth [2, 3] gave another (seemingly unrelated) generalization of the Erdős–Szekeres problem in 1989 by replacing point sets with families of pairwise disjoint convex bodies. They defined n convex bodies to be in *convex position* if the convex hull of no $n-1$ of them contains the remaining one. If any three members of a family of convex bodies are in convex position, then the family is in *general position*. In their pioneering paper, Bisztriczky and Fejes Tóth proved that for any $n \geq 3$, there exists a smallest integer $c(n)$ with the following property. If $\mathcal{F}$ is a family of pairwise disjoint convex bodies in general position in the plane with $|\mathcal{F}| \geq c(n)$, then it has n members in convex position. They conjectured that $c(n) = e(n)$. The first singly-exponential upper bound on $c(n)$ was established by Pach and Tóth [16]. They extended the statement to families of pairwise *noncrossing* convex bodies, that is, to convex bodies that may intersect, but any pair can share at most two boundary points [17]. This assumption is necessary.

Theorem 1.5 (Pach–Tóth, 2000). *For any integer $n \geq 3$, there exists a smallest number $c'(n)$ with the following property. Any family of at least $c'(n)$ pairwise noncrossing convex bodies in general position in the plane has n members in convex position.*

Clearly, $c'(n) \geq c(n) \geq e(n)$ for every n. The original upper bound on $c'(n)$ was subsequently improved by Hubard, Montejano, Mora, and Suk [12] and by Fox, Pach, Sudakov, and Suk [9] to $2^{O(n^2 \log n)}$, and later by Dobbins, Holmsen, and Hubard [6] to 4^n. More importantly from our point of view, they showed that there is an intimate relationship between the generalizations of the Erdős–Szekeres problem to noncrossing convex bodies and to pseudoconfigurations. The following is the union of Lemmas 2.4 and 2.7 in their paper.

Theorem 1.6 (Dobbins–Holmsen–Hubard, 2014). *Let $\mathcal{F}$ be a family of pairwise noncrossing convex bodies in general position in the plane. There exists a pseudoconfiguration P and a bijection $\varphi : P \to \mathcal{F}$ such that for any subset $S \subseteq P$ which is in convex position, the subfamily $\varphi(S)$ is also in convex position.*

It follows from this result that $c'(n) \leq b(n)$ for all n. (In fact, it was shown in [6] that $c'(n) = b(n)$ for all $n \geq 3$.) In view of this, Theorem 1.3 immediately implies the following.

Theorem 1.7. *Given any $n \geq 3$, let $c'(n)$ denote the smallest number such that every family of at least $c'(n)$ pairwise noncrossing convex bodies in general position in the plane has n members in convex position. Then*

$$c'(n) \leq 2^{n+O(\sqrt{n \log n})}.$$

Here is a summary of the known bounds on the various functions discussed above:

$$2^{n-2} + 1 \leq e(n) \leq c(n) \leq c'(n) = b(n) \leq 2^{n+O(\sqrt{n \log n})}.$$

(Note that none of the inequalities are known to be strict.)

The rest of this note is organized as follows. After highlighting two auxiliary results in Section 2, we present the proof of Theorem 1.3 in Section 3. In the end of Section 3 we show how to optimize the constant appearing in the error term.

2. Auxiliary results

To follow Suk's line of argument, we recall two results needed for the proof: a combinatorial version of the "cup-cap" theorem (Theorem 2.1) and a variant of a positive fraction Erdős–Szekeres theorem [1] (Theorem 2.4). For future reference, we also collect some simple observations on pseudoconfigurations in convex position (Observations 2.2 and 2.3).

Transitive colorings. Let S be a finite set with a given linear ordering $\prec$, and suppose the ordered triples $s_i \prec s_j \prec s_k$ are partitioned into two parts $T_1 \cup T_2$. This partition is called a *transitive coloring* if any $s_1 \prec s_2 \prec s_3 \prec s_4$ in S and $i \in \{1, 2\}$ satisfy

$$(s_1, s_2, s_3), (s_2, s_3, s_4) \in T_i \implies (s_1, s_2, s_4), (s_1, s_3, s_4) \in T_i.$$

Transitive colorings were introduced in [9] and [12]. The following statement can be proved in precisely the same way as the "cup-cap" theorem; see [14] for an alternative proof.

Theorem 2.1 ([9, 12]). *Let S be a finite set with a given linear ordering and let $T_1 \cup T_2$ be a transitive coloring of the triples of S. If*

$$|S| > \binom{k+l-4}{k-2}, \tag{2.1}$$

then there exists a k-element subset $S_1 \subseteq S$ such that every triple of S_1 is in T_1, or there exists an l-element subset $S_2 \subseteq S$ such that every triple of S_2 is in T_2.

Convex hulls of pseudoconfigurations. Below we collect a few simple observations on the convexity structure of pseudoconfigurations. These statements are trivial for the usual notion of convexity, and easy to prove in this more general context.

Observation 2.2. *Let P be a pseudoconfiguration.*

(i) *The convex hull is a monotone operation. That is, for any $X \subseteq Y \subseteq P$, we have* $\operatorname{conv} X \subseteq \operatorname{conv} Y$.

(ii) $\operatorname{conv} X$ *is a simply connected closed set, for any $X \subseteq P$.*

(iii) *If $X \subseteq P$ is in convex position, then all points of X appear on the boundary of* $\operatorname{conv} X$.

(iv) *Let $k \geq 3$, and assume that $X = \{x_1, \ldots, x_k\} \subseteq P$ is in convex position, where the points x_i appear on the boundary of* $\operatorname{conv} X$ *in this cyclic order. Then the boundary of* $\operatorname{conv} X$ *is the union of the pseudosegments* $\operatorname{conv}\{x_i, x_{i+1}\}$ *for $1 \leq i \leq k$. Furthermore, for each i, the pseudoline $\ell(x_i, x_{i+1})$ intersects* $\operatorname{conv} X$ *in the pseudosegment* $\operatorname{conv}\{x_i, x_{i+1}\}$, *and (the rest of)* $\operatorname{conv} X$ *lies entirely on one side of $\ell(x_i, x_{i+1})$. (Indices are understood modulo k.)*

Convex clusterings. Consider the pseudoconfiguration described in Observation 2.2(iv): Let $X = \{x_1, \ldots, x_k\}$ be a k-element subset of P in convex position, where $k \geq 3$, and suppose that the points x_i appear on the boundary of $\operatorname{conv} X$ in this cyclic order. We define the i-th *spike* of X, denoted by S_i, to be the open region consisting of the points of the plane separated from the interior of $\operatorname{conv} X$ by the pseudoline $\ell(x_i, x_{i+1})$, but not separated from $\operatorname{conv} X$ by $\ell(x_{i-1}, x_i)$ and by $\ell(x_{i+1}, x_{i+2})$. This is a connected region bounded by the pseudosegment $\operatorname{conv}\{x_i, x_{i+1}\}$ and by portions of the pseudolines $\ell(x_{i-1}, x_i)$ and $\ell(x_{i+1}, x_{i+2})$. It is either a triangle-like bounded region or an unbounded region of three sides; see Fig. 2.

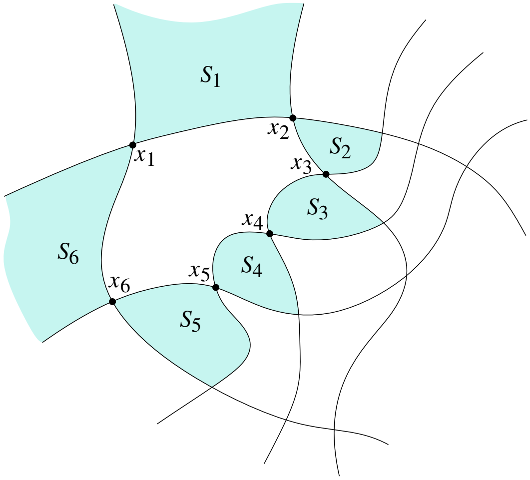

Fig. 2. A pseudoconfiguration in convex position where the spikes are shaded.

Observation 2.3. *Let* $1 \leq i \leq k$.

(i) *The line* $\ell(x_i, x_{i+1})$ *is disjoint from every spike and separates* S_i *from all other spikes* S_j $(j \neq i)$. *In particular, the spikes are pairwise disjoint.*

(ii) *A point* $p \in P \setminus X$ *belongs to the spike* S_i *if and only if* $X' = X \cup \{p\}$ *is in convex position and* p *appears on the boundary of* $\operatorname{conv} X'$ *between* x_i *and* x_{i+1}. *In particular, whether* $X \cup \{p\}$ *is in convex position is determined by which region* p *belongs to in the arrangement of pseudolines spanned by* X.

For the usual notion of convexity in the Euclidean plane, the following statement was proved by Pór and Valtr [18]. It is a slight strengthening of a result of Pach and Solymosi [15] that can be obtained by simple double counting. Since we will use this statement for pseudoconfigurations, to make our paper self-contained, we translate its proof into this setting.

Theorem 2.4. *Let* $k \geq 3$ *be an integer, and let* P *be a pseudoconfiguration with* $|P| = N \geq 2^{4k}$. *Then there exists a subset* $X = \{x_1, \dots, x_k\} \subset P$ *in convex position such that the sets* P_i *of all points of* P *lying in the* i*-th spike,* $i = 1, \dots, k$, *satisfy the inequality*

$$\prod_{i=1}^{k} |P_i| \geq \frac{N^k}{2^{8k^2}}. \tag{2.2}$$

Proof. Let P be a pseudoconfiguration with $|P| = N \geq 2^{4k}$. By Theorem 1.2, every 4^{2k}-element subset $Q \subseteq P$ contains a $2k$-element subset $R \subset Q$ in convex position. Therefore, by double counting, P has at least

$$\frac{\binom{N}{4^{2k}}}{\binom{N-2k}{4^{2k}-2k}} = \frac{\binom{N}{2k}}{\binom{4^{2k}}{2k}} > \left(\frac{N}{4^{2k}}\right)^{2k}$$

distinct $2k$-element subsets in convex position.

Given a $2k$-element subset Y in convex position, we say that a k-element subset $X \subset Y$ *supports* Y if the points of Y along the boundary of $\operatorname{conv} Y$ alternately belong to X and to $Y \setminus X$. Note that Y is supported by two subsets.

Since the number of k-element subsets of P in convex position is at most $\binom{N}{k}$, there exists a k-element subset X which supports at least

$$\frac{(\frac{N}{4^{2k}})^{2k}}{\binom{N}{k}} > \frac{N^k}{2^{8k^2}}$$

distinct $2k$-element subsets in convex position. By Observation 2.3(ii), if X supports Y, then the points of $Y \setminus X$ belong to distinct spikes of X, which implies inequality (2.2). □

3. Proof of Theorem 1.3

Consider a sufficiently large fixed pseudoconfiguration P, let $k \geq 4$ be an even integer, and let $X = \{x_1, \ldots, x_k\} \subset P$ be a k-element subset in convex position whose points appear on the boundary of $\operatorname{conv} X$ in this cyclic order. Suppose that X meets the requirements of Theorem 2.4. As before, let $S_1, \ldots, S_k$ denote the spikes of X and let $P_i = P \cap S_i$. The indices are taken modulo k.

Vertical and horizontal orderings on P_i. Let p and q be distinct points in P_i. We write

$$p \prec_i^v q \iff \operatorname{conv}\{x_{i-1}, p, x_{i+2}\} \subset \operatorname{conv}\{x_{i-1}, q, x_{i+2}\},$$
$$p \prec_i^h q \iff \operatorname{conv}\{x_{i-1}, q\} \cap \operatorname{conv}\{x_{i+2}, p\} \neq \emptyset,$$

where the superscripts v and h stand for "vertical" and "horizontal", respectively.

Observation 3.1. *Let $1 \leq i \leq k$.*

(i) *Both $\prec_i^v$ and $\prec_i^h$ are partial orders on P_i.*

(ii) *Any two distinct elements of P_i are comparable by either $\prec_i^v$ or $\prec_i^h$, but not by both.*

Proof. The definition of $\prec_i^v$ clearly implies that it is a partial order. To see that the same is true for $\prec_i^h$, one has to show that if $p \prec_i^h q \prec_i^h r$ for three points $p, q, r \in P_i$, then $p \prec_i^h r$. This can be done by checking the few possible pseudoconfigurations of the five points x_{i-1}, x_{i+2}, p, q and r.

To prove (ii), it is sufficient to consider the pseudoconfigurations consisting of only four points: x_{i-1}, x_{i+2}, and two points p and q from P_i. Using the fact that p and q lie on the same side of $\ell(x_{i-1}, x_{i+2})$, one can show that out of the four relations $p \prec_i^v q$, $q \prec_i^v p$, $p \prec_i^h q$, and $q \prec_i^h p$, precisely one will hold. Consider the four open regions into which the pseudolines $\ell(p, x_{i-1})$ and $\ell(p, x_{i+2})$ partition the plane. The region in which q lies uniquely determines which of the above four relations will hold. □

For $1 \leq i \leq k$, let v_i denote the length of the longest chain in P_i with respect to $\prec_i^v$, and let h_i denote the length of the longest chain in P_i with respect to $\prec_i^h$. By Observation 3.1 and by (the easy part of) Dilworth's theorem, we have

$$|P_i| \leq v_i h_i. \tag{3.1}$$

Further observations concerning points and spikes. As before, the following observations are trivial for the usual notion of convexity in the Euclidean plane. Here we show that they also hold for pseudoconfigurations.

Observation 3.2. *For any pair of distinct points $p, q \in P$, the pseudoline $\ell(p, q)$ intersects at most two spikes of X.*

Proof. Assume for contradiction that $\ell(p, q)$ intersects three separate spikes S_i, S_j, and S_l in this order. By Observation 2.3(i), this line should intersect $\ell(x_j, x_{j+1})$ twice, a contradiction. □

Observation 3.3. *Let p and q be distinct points of P_i. If $p \prec_i^v q$, then the pseudoline $\ell(p, q)$ separates spikes S_{i-1} and S_{i+1}.*

Proof. Since $p \in \operatorname{conv}\{x_{i-1}, q, x_{i+2}\}$, the pseudoline $\ell(p, q)$ intersects the pseudosegment $\operatorname{conv}\{x_{i-1}, x_{i+2}\}$. This implies that $\ell(p, q)$ has to intersect one of the spikes S_{i+2}, $S_{i+3}, \ldots, S_{i-2}$. By Observation 3.2, $\ell(p, q)$ intersects at most two spikes, one of which is S_i. Thus, it cannot intersect S_{i-1} and S_{i+1}, which implies that S_{i-1} and S_{i+1} must be separated by $\ell(p, q)$. □

Observation 3.4. *Let p and q be distinct points of P_i. If $p \prec_i^h q$, then all the spikes $S_{i+2}, S_{i+3}, \ldots, S_{i-2}$ must lie on the same side of the pseudoline $\ell(p, q)$.*

Proof. All spikes S_j with $j \notin \{i-1, i+1\}$ are on the same side of both pseudolines $\ell(x_{i-1}, x_i)$ and $\ell(x_{i+1}, x_{i+2})$. The angular region determined by these two pseudolines and containing the above spikes (and the interior of $\operatorname{conv} X$) is cut into two parts by the pseudosegment $\operatorname{conv}\{x_{i-1}, x_{i+2}\}$, so that S_i lies on one side and the spikes S_j with $j \notin \{i-1, i, i+1\}$ on the other. Our assumption $p \prec_i^h q$ implies that the pseudoline $\ell(p, q)$ does not intersect the pseudosegment $\operatorname{conv}\{x_{i-1}, x_{i+2}\}$, so the part of the angular region on the other side of this pseudosegment (including all relevant spikes) is on the same side of $\ell(p, q)$, as claimed. □

Vertical convex chains. Let $C \subseteq P_i$ be a chain with respect to $\prec_i^v$. If $\{x_{i-1}\} \cup C$ is in convex position, we call C a *left convex chain* in P_i. If $\{x_{i+2}\} \cup C$ is in convex position, we call C a *right convex chain* in P_i.

Note that if $|C| = 3$, then C is either a left convex chain or a right convex chain, but not both. This can be verified by checking the pseudoconfiguration $C \cup \{x_{i-1}, x_{i+2}\}$. Moreover, if $p_1 \prec_i^v p_2 \prec_i^v p_3 \prec_i^v p_4$ and both $\{p_1, p_2, p_3\}$ and $\{p_2, p_3, p_4\}$ are left [right] convex chains, then $\{p_1, p_2, p_3, p_4\}$ is also a left [right] convex chain. Therefore, the same holds for both $\{p_1, p_2, p_4\}$ and $\{p_1, p_3, p_4\}$. This can be verified by checking the pseudoconfiguration $\{p_1, p_2, p_3, p_4, x_{i-1}, x_{i+2}\}$. See Fig. 3.

Take a chain $C \subseteq P_i$ of maximal size $|C| = v_i$, totally ordered by $\prec_i^v$. Partition the triples of C into left and right convex chains. In this way, we obtain a transitive coloring. Letting a_i and b_i denote the length of the longest left convex chain and the length of the longest right convex chain in C, respectively, by Theorem 2.1 we have

$$v_i \leq \binom{a_i + b_i - 2}{a_i - 1}. \tag{3.2}$$

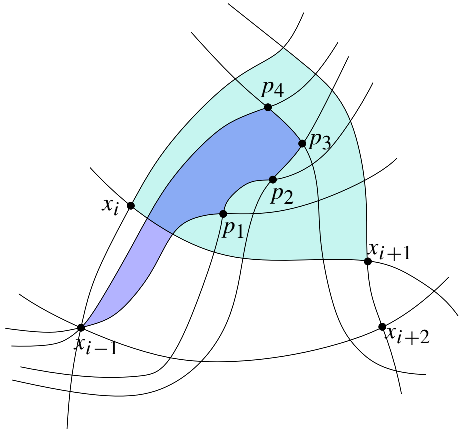

Fig. 3. A left convex chain $p_1 \prec_i^v p_2 \prec_i^v p_3 \prec_i^v p_4$ in P_i with conv $\{p_1, p_2, p_3, p_4, x_{i-1}\}$ in darker shade.

Actually, Theorem 2.1 only guarantees the existence of large subsets $C_1, C_2 \subseteq C$ such that all triples in C_1 are left convex chains and all triples in C_2 are right convex chains. However, using the above observations and the generalization of Carathéodory's theorem to pseudoconfigurations, it follows that C_1 and C_2 themselves must form a left convex chain and a right convex chain, respectively.

Observation 3.5. *If R is a right convex chain in P_i and L is a left convex chain in P_{i+1}, then $R \cup L$ is in convex position.*

Proof. First, note that for any pseudoconfiguration P consisting of four points, if a point $p \in P$ lies in the convex hull of $P \setminus \{p\}$, then any pseudoline passing through p and any other point of P crosses the pseudosegment determined by the remaining two points of P.

To prove the observation, it is enough to show that any four points $p, q, r, s \in R \cup L$ are in convex position. If all of them lie in one of R or L, then we are clearly done. Assume first that $r, s \in R$ and $p, q \in L$. By Observation 3.3, the pseudolines $\ell(p, q)$ and $\ell(r, s)$ do not intersect the pseudosegments conv $\{r, s\} \subset S_i$ and conv $\{p, q\} \subset S_{i+1}$, respectively. Therefore, by the discussion above, the points p, q, r, s are in convex position.

Now consider the case where $p, q, r \in L$ and $s \in R$. Again by Observation 3.3, none of the pseudolines $\ell(p, q)$, $\ell(p, r)$, and $\ell(q, r)$ intersects the spike S_i. Therefore, x_i and s lie in the same open region determined by the arrangement of these three pseudolines. By the assumption, the set $\{p, q, r, x_i\}$ is in convex position, so by the last statement of Observation 2.3(ii), $\{p, q, r, s\}$ is in convex position as well. The other case, $p \in L$ and $q, r, s \in R$, can be settled in a similar manner. See Fig. 4. □

Horizontal convex chains. Let $C \subseteq P_i$ be a chain with respect to $\prec_i^h$. If the set $\{p, q, r, x_{i-1}, x_{i+2}\}$ is in convex position for any three distinct elements p, q, r of C,

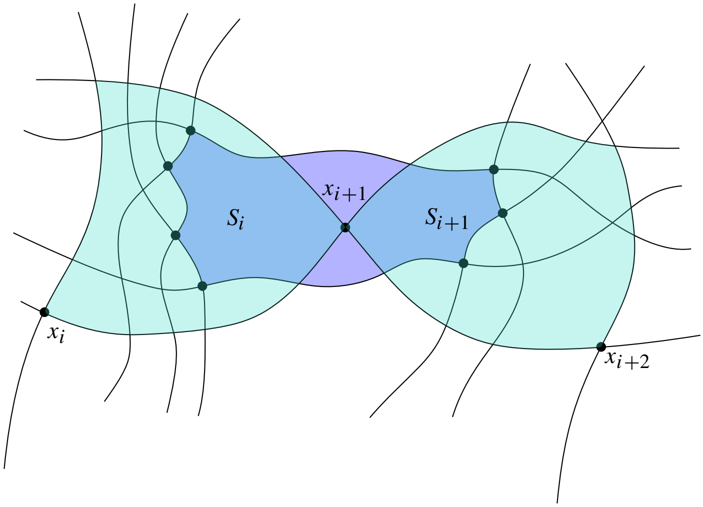

Fig. 4. Joining a right convex chain $R \subseteq P_i$ and a left convex chain $L \subseteq P_{i+1}$ to form a subset in convex position (convex hull in darker shade).

we call C an *inner convex chain*. If $\{p, q, r, x_{i-1}, x_{i+2}\}$ is *not* in convex position for any three distinct elements p, q, r of C, we call C an *outer convex chain*.

Note that chains of at most two elements are both inner and outer convex chains by this definition.

Observation 3.6. *Let* $1 \le i \le k$.

(i) *The partitioning of the triples in a horizontal chain* C *(ordered by* $\prec_i^h$*) into inner and outer convex chains is a transitive coloring.*
(ii) *The inner and outer convex chains in* P_i *are in convex position.*

Proof. Consider a horizontal chain $p \prec_i^h q \prec_i^h r$ in P_i. By checking the pseudoconfiguration $\{p, q, r, x_{i-1}, x_{i+2}\}$ we can verify that the following are equivalent:

- (p, q, r) is an outer [inner] convex chain.
- conv $\{x_{i-1}, x_{i+2}\}$ and r are separated by [lie on the same side of] $\ell(p, q)$.
- conv $\{x_{i-1}, x_{i+2}\}$ and p are separated by [lie on the same side of] $\ell(q, r)$.
- conv $\{x_{i-1}, x_{i+2}\}$ and q lie on the same side of [are separated by] $\ell(p, r)$.

Now consider a horizontal chain $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$. The pseudolines $\ell(x_{i-1}, p_4)$ and $\ell(x_{i+2}, p_1)$ divide the plane into four quadrants, each containing one of the pseudosegments conv $\{p_1, p_4\}$, conv $\{p_4, x_{i+2}\}$, conv $\{x_{i+2}, x_{i-1}\}$, conv $\{x_{i-1}, p_1\}$, in this cyclic order. By the ordering $\prec_i^h$, p_2 and p_3 are contained in the quadrant containing conv $\{p_1, p_4\}$. Furthermore, the pseudoline $\ell(p_2, p_3)$ must cross this quadrant, entering the boundary ray containing p_1, then meeting p_2 before p_3 and finally exiting the boundary ray containing p_4. If both (p_1, p_2, p_3) and (p_2, p_3, p_4) are outer (inner) convex chains, it follows by the observations above that conv $\{p_1, p_4\}$ and conv $\{x_{i-1}, x_{i+2}\}$

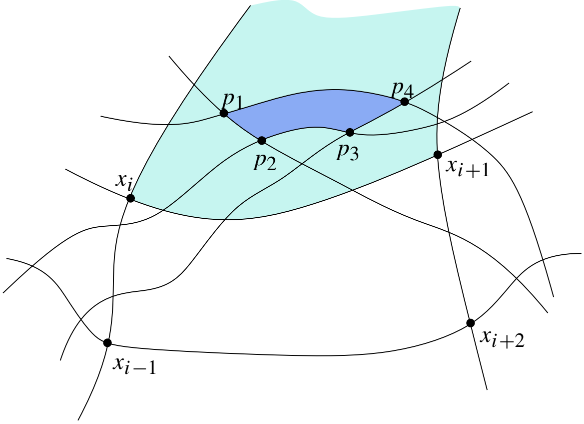

Fig. 5. An outer convex chain $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$ in P_i with conv $\{p_1, p_2, p_3, p_4\}$ in darker shade.

are separated by [lie on the same side of] $\ell(p_2, p_3)$. This implies that conv $\{x_{i-1}, x_{i+2}\}$ and p_4 are separated by [lie on the same side of] $\ell(p_1, p_2)$ and $\ell(p_1, p_3)$. Hence, (p_1, p_2, p_4) and (p_1, p_3, p_4) are both outer [inner] convex chains, which proves part (i). By Carathéodory's theorem, it suffices to check part (ii) for inner and outer convex chains $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$. However, it follows from the discussion above that $\ell(p_1, p_4)$ does not intersect conv $\{p_2, p_3\}$ and that $\ell(p_2, p_3)$ does not intersect conv $\{p_1, p_4\}$. As in the proof of Observation 3.5, we conclude that $\{p_1, p_2, p_3, p_4\}$ is in convex position. See Fig. 5. □

Letting c_i and d_i denote the length of the longest inner convex chain and the length of the longest outer convex chain in P_i, respectively, applying Theorem 2.1 to the longest horizontal chain in P_i and using Observation 3.6, we obtain

$$h_i \le \binom{c_i + d_i - 2}{c_i - 1}. \tag{3.3}$$

Observation 3.7. *Suppose that $k \ge 4$ is even, and let $A_1 \subseteq P_1, A_2 \subseteq P_2, \ldots, A_k \subseteq P_k$. If each A_i is an inner convex chain, then $A_1 \cup A_3 \cup \cdots \cup A_{k-1}$ is in convex position, and so is $A_2 \cup A_4 \cup \cdots \cup A_k$.*

Proof. The proof goes as that of Observation 3.5, and we repeatedly use the fact mentioned at the beginning of the latter. It suffices to prove that any four points $p_1, p_2, p_3, p_4 \in A_1 \cup A_3 \cup \cdots \cup A_{k-1}$ are in convex position. If all the points lie in one chain, we are done. Consider the case where three points belong to the same chain, say $p_1, p_2, p_3 \in A_{i_1}$, and $p_4 \in A_{i_2}$ with $i_1 \neq i_2$. By Observation 3.4, x_{i_1-1} and p_4 belong to the same open region determined by the pseudolines $\ell(p_1, p_2), \ell(p_1, p_3), \ell(p_2, p_3)$. Therefore, by the last statement of Observation 2.3(ii), the convexity of $\{p_1, p_2, p_3, x_{i_1-1}\}$ implies that $\{p_1, p_2, p_3, p_4\}$ is in convex position.

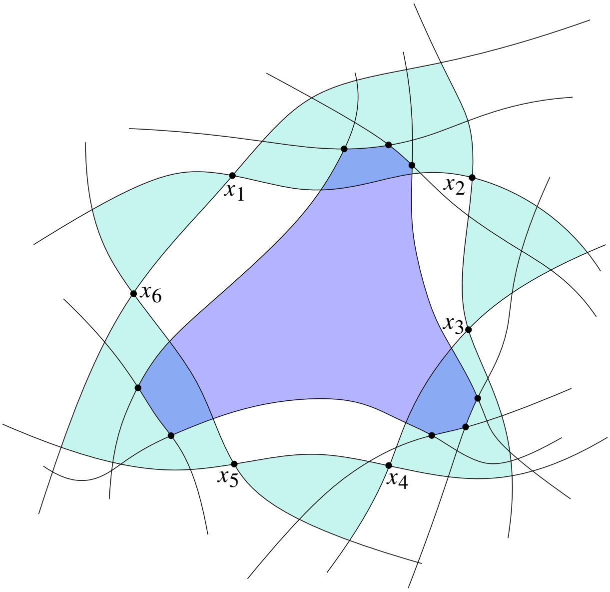

Fig. 6. Joining inner convex chains $A_1 \subseteq P_1$, $A_3 \subseteq P_3$, and $A_5 \subseteq P_5$ to form a subset in convex position (convex hull in darker shade).

If one of the chains contains exactly two of our points, say $p_1, p_2 \in A_i$, then neither p_1 nor p_2 can be in the convex hull of the other three points, as Observation 3.4 implies that the pseudoline $\ell(p_1, p_2)$ does not intersect the pseudosegment conv $\{p_3, p_4\}$.

To finish the proof, we need to verify that if one of the chains contains exactly one of our points, say $p_1 \in A_i$, then p_1 is not in the convex hull of the other three points. This follows from the fact that $\ell(x_i, x_{i+1})$ separates p_1 from p_2, p_3 and p_4. See Fig. 6. □

Proof of Theorem 1.3. Let P be a pseudoconfiguration on N points, and suppose that P does not contain n points in convex position. Let k be an even integer to be specified later, and let $X = \{x_1, \dots, x_k\} \subseteq P$ be a subset in convex position whose existence is guaranteed by Theorem 2.4. As above, for the sets P_i of all points of P contained in the i-th spike of X, $i = 1, \dots, k$, we define

$$\begin{aligned}
v_i &= \text{the length of the longest chain } C_i^v \text{ with respect to } \prec_i^v,\\
h_i &= \text{the length of the longest chain } C_i^h \text{ with respect to } \prec_i^h,\\
a_i &= \text{the length of the longest left convex chain in } C_i^v,\\
b_i &= \text{the length of the longest right convex chain in } C_i^v,\\
c_i &= \text{the length of the longest inner convex chain in } C_i^h,\\
d_i &= \text{the length of the longest outer convex chain in } C_i^h.
\end{aligned}$$

By Observation 3.6(ii), we have $d_i < n$. By Observation 3.5, we have

$$b_i + a_{i+1} < n \tag{3.4}$$

for all i, and, by Observation 3.7,

$$c_1 + \cdots + c_k < 2n. \tag{3.5}$$

Combining these with inequalities (2.2)–(3.3), we obtain

$$\begin{aligned}\frac{N^k}{2^{8k^2}} \le \prod_{i=1}^{k} |P_i| \le \prod_{i=1}^{k} v_i h_i \le \prod_{i=1}^{k} \binom{a_i + b_i - 2}{a_i - 1}\binom{c_i + d_i - 2}{c_i - 1} \\ < \prod_{i=1}^{k} 2^{a_i + b_i} d_i^{c_i} < 2^{kn + 2n\log n},\end{aligned}$$

which gives us

$$N < 2^{n + \frac{2n\log n}{k} + 8k}.$$

Setting k to be the smallest even integer greater than or equal to $\frac{1}{2}\sqrt{n\log n}$ gives the estimate

$$N = O\bigl(2^{n+8\sqrt{n\log n}}\bigr). \qquad \square$$

Optimizing the error term. Here we improve the error term in our previous estimate by showing the bound

$$b(n) \le 2^{n + (8\sqrt{2}/3 + o(1))\sqrt{n\log n}}.$$

The first improvement is a refinement of Theorem 2.4.

Proposition 3.8. *Let $k \ge 3$ be an integer, and let P be a pseudoconfiguration with $|P| = N \ge 2^{(1+o(1))4k}$. Then one of the following holds.*

(1) *There exists a subset $X = \{x_1, \ldots, x_k\} \subset P$ in convex position such that the sets P_i of all points of P in the i-th spike of X, $i = 1, \ldots, k$, satisfy*

$$\prod_{i=1}^{k} |P_i| \ge 2^{-\frac{8}{3}k^2} N^k.$$

(2) *There exists a subset $X' = \{x_1', \ldots, x_{2k}'\} \subset P$ in convex position such that the sets P_i' of all points of P in the i-th spike of X', $i = 1, \ldots, 2k$, satisfy*

$$\prod_{i=1}^{2k} |P_i'| \ge 2^{-\frac{40}{3}k^2 - o(k^2)} N^{2k}.$$

Proof. Let $f_j = f_j(P)$ denote the number of j-element subsets of P that are in convex position. Looking back at the proof of Theorem 2.4, we see that for the optimal set X, the quantity $\prod_{i=1}^{k} |P_i|$ is bounded below by f_{2k}/f_k. Similarly $\prod_{i=1}^{2k} |P_i'| \ge f_{4k}/f_{2k}$ for the optimal set X'. Trivially $f_k \le N^k$, and using our (preoptimized) bound on $b(n)$, with the same double counting as before we have $f_{4k} \ge 2^{-16k^2 - o(k^2)} N^{4k}$. The claim now follows from whether or not $f_{2k} \ge 2^{-\frac{8}{3}k^2} N^{2k}$. $\square$

Here is another improvement. In the proof of Theorem 1.3 we used the estimate $\binom{c_i+d_i-2}{d_i-1} < d_i^{c_i}$, where $d_i < n$ and $\sum_i c_i < 2n$, which gave us $\prod_i \binom{c_i+d_i-2}{d_i-1} < n^{2n}$. Instead, if we use the more precise estimate

$$\binom{c_i+d_i-2}{c_i-1} < \binom{2n}{c_i} < \left(\frac{2en}{c_i}\right)^{c_i} \le k^{c_i} e^{2n/k},$$

then we get

$$\prod_{i=1}^{k} \binom{c_i+d_i-2}{c_i-1} < (ek)^{2n}. \tag{3.6}$$

Now we combine these two improvements. Let P be a pseudoconfiguration on N points and suppose P does not contain n points in convex position. We apply the same argument as before to each of the cases in Proposition 3.8, using the better estimate from (3.6). In case (1) we obtain

$$2^{-\frac{8}{3}k^2} N^k \le \prod_{i=1}^{k} |P_i| < 2^{kn+2n\log(ek)},$$

and in case (2) we obtain

$$2^{-\frac{40}{3}k^2 - o(k^2)} N^{2k} \le \prod_{i=1}^{2k} |P_i'| < 2^{2kn+2n\log(2ek)}.$$

Setting k to be the smallest even integer greater than or equal to $\frac{\sqrt{n\log n}}{2\sqrt{2}}$, either case gives us the desired bound $N < 2^{n+(8\sqrt{2}/3+o(1))\sqrt{n\log n}}$.

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References

[1] Bárány, I., Valtr, P.: A positive fraction Erdős–Szekeres theorem. Discrete Comput. Geom. **19**, 335–342 (1998) Zbl 0914.52007 MR 1608874

[2] Bisztriczky, T., Fejes Tóth, G.: A generalization of the Erdős–Szekeres convex n-gon theorem. J. Reine Angew. Math. **395**, 167–170 (1989) Zbl 0654.52003 MR 0983064

[3] Bisztriczky, T., Fejes Tóth, G.: Nine convex sets determine a pentagon with convex sets as vertices. Geom. Dedicata **31**, 89–104 (1989) Zbl 0677.52003 MR 1009885

[4] Björner, A., Las Vergnas, M., Sturmfels, B., White, N., Ziegler, G. M.: Oriented Matroids. 2nd ed., Cambridge Univ. Press (1999) Zbl 0944.52006 MR 1744046

[5] Dhandapani, R., Goodman, J. E., Holmsen, A., Pollack, R., Smorodinsky, S.: Convexity in topological affine planes. Discrete Comput. Geom. **38**, 243–257 (2007) Zbl 1132.52002 MR 2343306

[6] Dobbins, M. G., Holmsen, A. F., Hubard, A.: The Erdős–Szekeres problem for non-crossing convex sets. Mathematika **60**, 463–484 (2014) Zbl 1298.52019 MR 3229499

[7] Erdős, P., Szekeres, G.: A combinatorial problem in geometry. Compos. Math. **2**, 463–470 (1935) Zbl 0012.27010 MR 1556929

[8] Erdős, P., Szekeres, G.: On some extremum problems in elementary geometry. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **3-4**, 53–62 (1960/1961) Zbl 0103.15502 MR 0133735

[9] Fox, J., Pach, J., Sudakov, B., Suk, A.: Erdős–Szekeres-type theorems for monotone paths and convex bodies. Proc. London Math. Soc. **105**, 953–982 (2012) Zbl 1254.05114 MR 2997043

[10] Goodman, J. E.: Pseudoline arrangements. In: Handbook of Discrete and Computational Geometry, 2nd ed., CRC, Boca Raton, LA, 97–128 (2004) Zbl 0914.51007

[11] Goodman, J. E., Pollack, R.: A combinatorial perspective on some problems in geometry. In: Combinatorics, Graph Theory and Computing, Vol. I (Baton Rouge, LA, 1981), Congress. Numer. 32, 383–394 (1981) Zbl 0495.05012 MR 0681897

[12] Hubard, A., Montejano, L., Mora, E., Suk, A.: Order types of convex bodies. Order **28**, 121–130 (2011) Zbl 1235.52004 MR 2774044

[13] Knuth, D. E.: Axioms and Hulls. Lecture Notes in Computer Sci. 606, Springer, Berlin (1992) Zbl 0777.68012 MR 1226891

[14] Moshkovitz, G., Shapira, A.: Ramsey theory, integer partitions and a new proof of the Erdős–Szekeres theorem. Adv. Math. **262**, 1107–1129 (2014) Zbl 1295.05255 MR 3228450

[15] Pach, J., Solymosi, J.: Canonical theorems for convex sets. Discrete Comput. Geom. **19**, 427–435 (1998) Zbl 0905.52001 MR 1608884

[16] Pach, J., Tóth, G.: A generalization of the Erdős–Szekeres theorem to disjoint convex sets. Discrete Comput. Geom. **19**, 437–445 (1998) Zbl 0903.52004 MR 1608885

[17] Pach, J., Tóth, G.: Erdős–Szekeres-type theorems for segments and noncrossing convex sets. Geom. Dedicata **81**, 1–12 (2000) Zbl 0959.52002 MR 1772191

[18] Pór, A., Valtr, P.: The partitioned version of the Erdős–Szekeres theorem. Discrete Comput. Geom. **28**, 625–637 (2002) Zbl 1019.52011 MR 1949905

[19] Suk, A.: On the Erdős–Szekeres convex polygon problem. J. Amer. Math. Soc. **30**, 1047–1053 (2017) Zbl 1370.52032 MR 3671936