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Spin analogues of superconductivity and the integer quantum Hall effect in an array of spin chains

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Motivated by the successful idea of using weakly-coupled quantum electronic wires to realize the quantum Hall effects and the quantum spin Hall effects, we theoretically study two systems composed of weakly-coupled quantum spin chains within the mean-field approximations, which can exhibit spin analogues of superconductivity and the integer quantum Hall effect. First, a certain bilayer of two arrays of interacting spin chains is mapped, via the Jordan-Wigner transformation, to an attractive Hubbard model that exhibits fermionic superconductivity, which corresponds to spin superconductivity in the original spin Hamiltonian. Secondly, an array of spin-orbit-coupled spin chains in the presence of a suitable external magnetic field is transformed to an array of quantum wires that exhibits the integer quantum Hall effect, which translates into its spin analogue in the spin Hamiltonian. The resultant spin superconductivity and spin integer quantum Hall effect can be characterized by their ability to transport spin without any resistance.

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Introduction.—In a metal under normal conditions, an electric current flows in the presence of a finite resistance engendered by, e.g., scattering with impurities. The lost electrical energy due to the resistance is dissipated into heat, which is referred to as Joule heating that opposes the efficient use of the energy. There are, however, two physical phenomena under special conditions that allow an electric current to flow without any resistance. The one is superconductivity occurring at low temperatures [1]. Its first microscopic theory was given in 1957 by Bardeen, Cooper, and Schrieffer [2], who showed that superconductivity can be understood as a property of macroscopic quantum wavefunction of condensed pairs of electrons subsequently termed Cooper pairs. The other is the set of quantum Hall effects exhibited in two-dimensional systems at low temperatures and strong magnetic fields [3]. The integer quantum Hall effect is the first of such that was discovered in 1980 by Klitzing *et al.* [4]. It occurs when the number of electrons per unit magnetic flux takes an integer value ν , leading to the situation in which the bulk is gapped, but the edge supports ν gapless modes with no resistance.

Spintronics aims at harnessing the spin degrees of freedom to advance from conventional charge-based electronics [5]. In particular, magnetic insulators that are free from Joule heating have been gaining attention in the field owing to their potential advantage of low-energy consumption. An efficient spin transport in such magnetic insulators is one of the important topics in spintronics, and researchers have been investigating possible ways to achieve it by borrowing some ideas from the aforementioned phenomena of dissipationless charge transport. For example, a bosonic spin-analogue of an electric supercurrent supported in easy-plane magnets has been theoretically investigated [6], which is shown to decay algebraically as a function of the distance from the spin-injection point contrary to an exponential decay

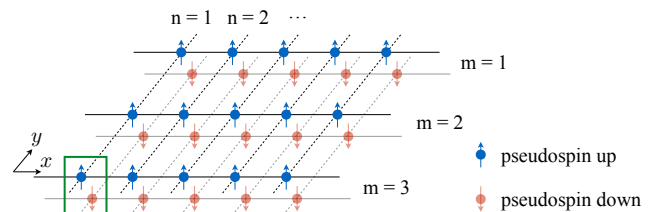


FIG. 1. Schematic of a bilayer of two arrays of weakly-coupled spin chains (shown as the solid lines indexed by m), each of which can be represented by a one-dimensional system of (spinless) Jordan-Wigner fermions. The top and bottom layer indices serve as the pseudospin up and down for the fermions, respectively. The green box represents a pseudospin-singlet Cooper pair of two fermions established by an Ising interlayer interaction.

of a diffusive spin current. Spin analogues of the integer and fractional quantum Hall effects have also been put forward in the studies of spin liquids [7, 8] and magnonic phases [9].

In this Rapid Communication, we theoretically construct two spin systems, which can exhibit spin analogues of superconductivity and the integer quantum Hall effect, by using weakly-coupled quantum spin chains. Our work is motivated by the successful theoretical realizations of the quantum Hall phases and the quantum spin Hall phases in an array of quantum electronic wires [10, 11]. Specifically, first, we show that an Ising-coupled bilayer of two arrays of weakly-coupled quantum XX spin chains can be mapped to a negative- U Hubbard model for electrons by the Jordan-Wigner (JW) transformation [12, 13] within a mean-field treatment of the interchain coupling. Since the particle current in the JW representation corresponds to the spin current polarized along the z axis, the established charge superconductivity of the negative- U Hubbard model [14] naturally translates into spin super-

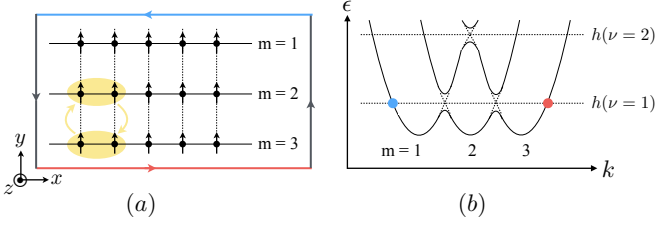


FIG. 2. (a) Schematic of an array of spin-orbit-coupled spin-1/2 spin chains, which can support chiral edge modes of the Jordan-Wigner fermions. The coupling of four spins (colored by yellow) illustrates the interchain interaction \mathcal{O} [Eq. (2)]. (b) A schematic plot showing how the interchain interaction gives rise to chiral edge modes with the gapped bulk. At the external magnetic field h corresponding to the filling factor $\nu = 1$, the JW fermion can flow in the left direction on the top chain (colored by blue) and in the right direction on the bottom chain (colored by red) in (a), which are represented by the left blue and right red dots in (b), respectively. The particle current in the JW representation corresponds to the spin current polarized along the z axis.

conductivity of our original spin system. See Fig. 1 for an illustration of the system. Secondly, we show that an array of weakly-coupled quantum XX spin chains with Dzyaloshinskii-Moriya (DM) intrachain interaction can be transformed to an array of quantum electronic wires subjected to an external magnetic field by the same approach taken for spin superconductivity. The integer quantum Hall effect of the latter electronic system [11] then translates into its spin analogue of the former spin system. See Fig. 2(a) for an illustration of the system. Our proposal to use coupled spin chains for quantum simulation of two-dimensional fermionic Hamiltonians can be an alternative to the other methods based on cold atoms in optical lattices or impurity centers in solids [15].

Spin superconductivity.—We consider the following spin Hamiltonian for two layers of weakly-coupled M spin-1/2 chains of length N :

$$\begin{aligned}
 H_{\text{sc}} = & J \sum_{n,m,\alpha} \boldsymbol{\sigma}_{n,m,\alpha}^{\parallel} \cdot \boldsymbol{\sigma}_{n+1,m,\alpha}^{\parallel} \\
 & - H \sum_{n,m,\alpha} \sigma_{n,m,\alpha}^z - U \sum_{n,m} \sigma_{n,m,\uparrow}^z \sigma_{n,m,\downarrow}^z \quad (1) \\
 & - K \sum_{n,m,\alpha} [\mathcal{O}_{n,m,\alpha} + \text{H.c.}] ,
 \end{aligned}$$

with

$$\mathcal{O}_{n,m,\alpha} = \sigma_{n,m,\alpha}^+ \sigma_{n+1,m,\alpha}^+ \sigma_{n,m+1,\alpha}^- \sigma_{n+1,m+1,\alpha}^- , \quad (2)$$

where the integers m and n are the indices for a spin chain within a layer and a spin within a chain, respectively, and $\alpha = \uparrow, \downarrow$ indexes the layer which will serve as the pseudospin of the JW fermions. A spin is represented by the three-dimensional Pauli matrices $\boldsymbol{\sigma}$; the symbol \parallel denotes the projection of the vector onto the xy

plane; $\sigma^{\pm} \equiv (\sigma^x \pm i\sigma^y)/2$. Here, the first term describes the quantum antiferromagnetic XX spin-1/2 chains with $J > 0$ [16]; the second term is the Zeeman energy; the third term is the ferromagnetic Ising interaction with $U > 0$ between the two layers; the last term represents a weak four-spin interaction with $0 < K \ll J$, which, in the JW representation, can engender the interchain tunneling and thereby make each layer an effective two-dimensional fermionic gas. Interchain interactions involving only two spins such as the Heisenberg XX exchange $\propto \boldsymbol{\sigma}_{n,m}^{\parallel} \cdot \boldsymbol{\sigma}_{n,m+1}^{\parallel}$ would also appear as tunneling between two chains. They, however, introduce nonlocal terms after the JW transformation, making it difficult to treat the interchain interaction [17]. Our goal, instead, is to construct simple spin systems that can be viewed as weakly-interacting simple fermionic wires. Therefore, by coupling neighboring spin chains by the four-spin interaction, we retain its locality after the JW transformation. The spin Hamiltonian H_{sc} respects the spin-rotational symmetry about the z axis, and thus the total spin projected onto the z axis is conserved.

The spin Hamiltonian H_{sc} can be transformed into the Hamiltonian for the spinless fermions by the multi-dimensional JW transformation [13]: $f_{n,m,\alpha} = \sigma_{n,m,\alpha}^- \left(\prod_{l < n} \sigma_{l,m,\alpha}^z \right) \left(\prod_{(k,\beta) < (n,\alpha)} \tau_{k,\beta}^y \right) \tau_{m,\alpha}^x$, and the analogous expression for $f_{n,m,\alpha}^{\dagger}$ with σ^- substituted by σ^+ , where the auxiliary Pauli-matrix vector, $\boldsymbol{\tau}_{m,\alpha}$, is introduced for each spin chain to make the fermion operators on different chains anticommute [18]. The interchain interaction $\mathcal{O}_{n,m,\alpha}$ [Eq. (2)] yields a quartic term in the fermion operators, $f_{n,m,\alpha}^{\dagger} f_{n+1,m,\alpha}^{\dagger} f_{n,m+1,\alpha} f_{n+1,m+1,\alpha}$, and thus we employ the Hartree-Fock decoupling [19] to study its effects. There are two potentially relevant mean-field order parameters [20]. One is an interchain-tunneling amplitude,

$$\chi = \frac{1}{2NM} \sum_{n,m,\alpha} \langle f_{n,m,\alpha}^{\dagger} f_{n,m+1,\alpha} \rangle . \quad (3)$$

The other is an intrachain pairing amplitude,

$$\Gamma = \frac{1}{2NM} \sum_{n,m,\alpha} \langle f_{n,m,\alpha} f_{n+1,m,\alpha} \rangle . \quad (4)$$

The mean-field Hamiltonian for a single layer of the pseudospin α is given by

$$\begin{aligned}
 \bar{H}_{\alpha} = & -t_x \sum_{n,m} [f_{n,m,\alpha}^{\dagger} f_{n+1,m,\alpha} + \text{H.c.}] \\
 & - 2K \sum_{n,m} [\chi f_{n,m,\alpha}^{\dagger} f_{n,m+1,\alpha} + \text{H.c.}] \\
 & - 2K \sum_{n,m} [\Gamma f_{n,m,\alpha}^{\dagger} f_{n+1,m,\alpha}^{\dagger} + \text{H.c.}] \\
 & - \mu \sum_{n,m} n_{n,m,\alpha} ,
 \end{aligned} \quad (5)$$

up to an additive constant, where $n_{n,m,\alpha} = f_{n,m,\alpha}^\dagger f_{n,m,\alpha}$ is the fermion-number operator, $t_x = 2J$, and $\mu = 2H - 2U$. (By \bar{H} , we will denote the Hamiltonians in the JW representation throughout.) Assuming the periodic boundary conditions, the self-consistency equations for the two mean-field order parameters χ and Γ in the momentum space are given by

$$\Gamma = \frac{1}{NM} \sum_{\mathbf{k}} \frac{2K\Gamma \sin^2 k_x}{\sqrt{\epsilon(\mathbf{k})^2 + |\Gamma(\mathbf{k})|^2}}, \quad (6)$$

$$\chi = \frac{1}{NM} \sum_{\mathbf{k}} \frac{\cos k_y}{2} \left(1 - \frac{\epsilon(\mathbf{k})}{\sqrt{\epsilon(\mathbf{k})^2 + |\Gamma(\mathbf{k})|^2}} \right), \quad (7)$$

where $\epsilon(\mathbf{k}) = -2t_x \cos k_x - 4K\chi \cos k_y - \mu$ and $\Gamma(\mathbf{k}) = 4iK\Gamma \sin k_x$. Here, the spatial coordinates x and y are related to n and m , respectively, as shown in Fig. 1. Since the coefficient for the interchain interaction is assumed to be positive, $K > 0$, the pairing amplitude vanishes, $\Gamma = 0$. We performed the numerical calculations to solve Eq. (7) for χ by varying the parameters K and μ , and they yielded the finite value of χ in a broad range of parameter values [21]. The self-consistent analytical solution for χ can be obtained when the effective chemical potential is close to the bottom of the band for a single chain, $\mu = -2t_x + \delta\mu$ with $|\delta\mu| \ll K$, allowing a parabolic band approximation for the dispersion $\epsilon(\mathbf{k})$ around the origin. The analytical solution is given by

$$\chi = \left(\int_0^{\pi/2} dk_y \cos^{3/2} k_y \right)^2 \frac{4K}{\pi^4 t_x} + \mathcal{O}\left(\frac{\delta\mu}{K}\right), \quad (8)$$

which can be approximated to $\chi \approx K/25t_x$. With the finite χ and vanishing $\Gamma_\alpha = 0$, the mean-field Hamiltonian \bar{H}_α (5) for a single layer describes a two-dimensional spinless fermion gas.

By combining the spin Hamiltonians \bar{H}_α for $\alpha = \uparrow$ and $\alpha = \downarrow$, we obtain the following mean-field Hamiltonian for the bilayer:

$$\begin{aligned} \bar{H}_{\text{sc}} = & -t_x \sum_{n,m,\alpha} [f_{n,m,\alpha}^\dagger f_{n+1,m,\alpha} + \text{H.c.}] \\ & -t_y \sum_{n,m,\alpha} [f_{n,m,\alpha}^\dagger f_{n,m+1,\alpha} + \text{H.c.}] \\ & -\mu \sum_{n,m,\alpha} n_{n,m,\alpha} - u \sum_{n,m} n_{n,m,\uparrow} n_{n,m,\downarrow}, \end{aligned} \quad (9)$$

where $t_y = 2K\chi$ and $u = 4U$, which describes the attractive Hubbard model [14]. It is known that the ground state of the Hamiltonian \bar{H}_{sc} away from the half-filling is in the superconducting phase composed of pseudospin-singlet Cooper pairs of the JW fermions [14] characterized by the finite superconducting gap, $\Delta = \sum_{n,m} \langle f_{n,m,\uparrow} f_{n,m,\downarrow} \rangle / NM$. Since the JW-fermion particle current corresponds to the spin current polarized along the z axis, the spin system described by the original

Hamiltonian H_{sc} (1) should exhibit a spin-analogue of charge superconductivity within the mean-field approximations [21].

Spin integer quantum Hall effect.—For a spin analogue of the integer quantum Hall effect, we take the following spin Hamiltonian:

$$\begin{aligned} H_{\text{qh}} = & J \sum_{n,m} \cos(m\phi) \boldsymbol{\sigma}_{n,m}^\parallel \cdot \boldsymbol{\sigma}_{n+1,m}^\parallel \\ & + J \sum_{n,m} \sin(m\phi) \hat{\mathbf{z}} \cdot \boldsymbol{\sigma}_{n,m} \times \boldsymbol{\sigma}_{n+1,m} \\ & - H \sum_{n,m} \sigma_{n,m}^z - K \sum_{n,m} [\mathcal{O}_{n,m} + \text{H.c.}], \end{aligned} \quad (10)$$

where the four-spin interaction $\mathcal{O}_{n,m}$ is given by Eq. (2) with α removed. Here, the first two terms describe the antiferromagnetic Heisenberg XY spin chains with the DM interaction; the third term is the Zeeman coupling; the last term is the weak interchain interaction, $0 < K \ll J$. See Fig. 2(a) for an illustration of the system. The DM interaction can exist if the reflection symmetry with respect to the xz plane is broken; the Hamiltonian respects the reflection symmetries through the xy and yz planes. The chain-dependent exchange coefficients can be realized by controlling the extent of the reflection-symmetry breaking associated with the DM interaction (see the discussion). We focus on the weak DM interactions, $0 < \phi J \ll K$, compared to the interchain coupling [22].

After employing the JW transformation [13], we take the mean-field approach for the interchain interaction. Since the gradient of spin-orbit coupling breaks the translational symmetry along the y axis of the system, it is difficult to obtain an analytical mean-field solution χ for arbitrary M . Instead, let us consider a special case of two weakly-coupled spin chains, which is described by H_{qh} [Eq. (10)] with $m = \pm 1$. Two possible order parameters pertain to the interchain tunneling, $\chi = \sum_n \langle f_{n,1}^\dagger f_{n,-1} \rangle / N$, and the intrachain pairing, $\Gamma = \sum_{n,m=\pm 1} \langle f_{n,m}^\dagger f_{n+1,m} \rangle / 2N$. The mean-field Hamiltonian for the JW fermions in the momentum space is given by

$$\begin{aligned} \bar{H} = & \sum_{k,m=\pm 1} \left[(-2t_x \cos(k+m\phi) - \mu) f_{k,m}^\dagger f_{k,m} \right] \\ & - 2K \sum_k \left[\chi f_{k,1}^\dagger f_{k,-1} + \text{H.c.} \right] \\ & - 2K \sum_{k,m=\pm 1} \left[\Gamma e^{ik} f_{k,m}^\dagger f_{-k,m}^\dagger + \text{H.c.} \right], \end{aligned} \quad (11)$$

where $t_x = 2J$ and $\mu = 2H$. We will assume that two phases with finite χ and Γ are mutually exclusive, and will treat them separately. For $K > 0$, which is assumed throughout, the self-consistency equation yields a vanishing pairing amplitude, $\Gamma = 0$, as in the case of spin superconductivity. With $\Gamma = 0$, the band structure of the

Hamiltonian is $\epsilon_{\pm}(k) = t_x k^2 - \delta\mu \pm 2\sqrt{(t_x \phi k)^2 + (K\chi)^2}$ for $|k|, |\phi| \ll 1$, where $\mu = -t_x(2 + \phi^2) + \delta\mu$. When the effective Fermi energy is at the band-crossing point, $\delta\mu = 0$, the analytical solution to the self-consistency equation for χ is given by $\chi \simeq K/2\pi^2 t_x$. The finite interchain tunneling $\chi > 0$ opens up the gap at the crossing point of the two bands of neighboring chains. See Fig. 2(b) for illustrations of the gap openings.

We adopt the above mean-field results for two spin chains to the cases of multiple spin chains, which results in the following mean-field Hamiltonian:

$$\begin{aligned} \bar{H}_{\text{qh}} = & -t_x \sum_{n,m} [e^{im\phi} f_{n,m}^{\dagger} f_{n+1,m} + \text{H.c.}] \\ & -t_y \sum_{n,m} [f_{n,m}^{\dagger} f_{n,m+1} + \text{H.c.}] \\ & -\mu \sum_{n,m} n_{n,m}, \end{aligned} \quad (12)$$

with $t_y = 2K\chi$, which describes an array of quantum electronic wires in the presence of an external magnetic field $\propto \phi$. The integer quantum Hall effect at the filling factor $\nu = 1$ arises when the Fermi energy is close to the crossing point of the two bands of adjacent chains, $\mu = -t_x(2 + \phi^2) + \delta\mu$ with $|\delta\mu| \ll t_x$. The integer quantum Hall effects at higher filling factors ν can be analogously obtained in the ν th order of the perturbative treatment of the interchain interaction [11].

The Hamiltonian \bar{H}_{qh} has been shown to exhibit the integer quantum Hall effect [10, 11]. Let us briefly explain how the integer quantum Hall effects arise in the model for an example of filling factor $\nu = 1$. See Fig. 2(b) for the JW fermion bands of spin chains and the gap openings by the interchain tunneling. When the Fermi energy μ lies in the bulk gap, there are one gapless mode in the top chain ($m = 1$ in the figure) and the other in the bottom chain ($m = 3$ in the figure). The two modes propagate in the opposite directions, and thus engender one chiral edge mode together. The integer quantum Hall effect at higher filling factors ν supports ν chiral edge modes by an analogous mechanism [11]. The state we obtained is different from the conventional quantum Hall phase in that the transported quantity is spin, not charge; it is also distinct from the traditional quantum spin Hall phase [23] in that the resultant spin transport does not accompany any charge transport.

Discussion.—Let us make some comments about possible experimental realizations. First, spin-1/2 chain systems Cs_2CoCl_4 [24] and PrCl_3 [25] are known to be approximately described by the isotropic antiferromagnetic Heisenberg XX model. Secondly, the DM interaction in a single chain can be induced by breaking the reflection symmetry through the xz plane. For a two-dimensional film in the xz plane, application of a normal electric field can induce the DM interaction, which, in turn, can be spatially modulated. Or, alternatively, the lithographic-

modulation of the proximate heavy-metal layer can also create the gradient of the DM interaction [26]. Lastly, the four-spin exchange interaction can arise as the fourth-order term in the strong-coupling expansion of the half-filled Hubbard model or due to the spin-lattice coupling, and its magnitude can be comparable to two-spin Heisenberg exchange in certain materials [27, 28].

The interchain interactions have been taken into account within the mean-field treatment, which can break down in certain cases, e.g., when the interchain coupling K is stronger than the energy scales of individual spin chains such as J . While we can investigate the quantum fluctuations around the mean-field solutions for the further analysis, it is beyond the scope of the present work.

We have theoretically constructed the two models of an array of weakly-coupled spin chains, which can exhibit spin analogues of charge superconductivity and the integer quantum Hall effect. To drive spin current through those systems, we can apply an external-magnetic-field gradient, which acts as an electric field on the JW fermions [8]. We can also attach the spin system to heavy metals such as platinum, which can directly inject a spin current to proximate magnets via spin Hall effects [29]. Reciprocally, a spin current out of the system can be measured via inverse spin Hall effects by putting it next to heavy metals. Spin superconductivity and spin integer quantum Hall effects can be characterized by the zero resistance in spin flow through the bulk and along the boundary, respectively, when neglecting spin dissipation due to, e.g., thermal fluctuations or spin-lattice coupling.

From the results obtained for quantum spin chains, we expect that an array of weakly-coupled classical Heisenberg spin chains (that are composed of large spins) with the DM interaction in the presence of a strong external magnetic field would support the magnonic chiral edge modes by forming a topological magnon insulator [9] under suitable conditions. More broadly, we envision that weakly-coupled one-dimensional magnetic materials would serve as a versatile platform to engineer various spin-related topological phases.

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- [17] The long-range interchain interactions may be treated within the framework of the coupled Luttinger liquids [30], but it is beyond the scope of our work.
- [18] The comparison of two tuples, (k, β) and (m, α) , representing two spin chains is determined by their orders in a serialization of spin chains. One can take an alternative JW transformation by restricting β to α in the product and introducing a new additional Pauli-matrix vector for maintaining anticommutation between the two layers.
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- [21] See Supplemental Material at [URL will be inserted by publisher] for the numerical solution to Eq. (7) for χ , the derivation of Eq. (8) from Eq. (7), and the discussion on the effects of the finite interlayer Cooper pairing, $\Delta \propto \sum_{n,m} f_{n,m,\uparrow} f_{n,m,\downarrow}$, on the interchain tunneling amplitude χ , which is shown to not change the results qualitatively for sufficiently small interlayer couplings, $0 < u \ll t_x$.
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