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Periodic minimal surfaces embedded in $\mathbb{R}^3$ derived from the singly periodic Scherk minimal surface

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We construct three kinds of periodic minimal surfaces embedded in $\mathbb{R}^3$. We show the existence of a 1-parameter family of minimal surfaces invariant under the action of a translation by 2π, which seen from a distance look like m equidistant parallel planes intersecting orthogonally k equidistant parallel planes, $m, k \in \mathbb{N}$, $mk \geq 2$. We also consider the case where the surfaces are asymptotic to $m \in \mathbb{N}^+$ equidistant parallel planes intersecting orthogonally infinitely many equidistant parallel planes. In this case, the minimal surfaces are doubly periodic, precisely they are invariant under the action of two orthogonal translations. Last we construct triply periodic minimal surfaces which are invariant under the action of three orthogonal translations in the case of two stacks of infinitely many equidistant parallel planes which intersect orthogonally.

Keywords: Periodic minimal surfaces; gluing procedure; fixed point theorem.

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1. Introduction

Besides the plane and the helicoid, the first singly periodic minimal surface in $\mathbb{R}^3$ was discovered by Scherk [6] in 1835. This surface, known as singly periodic Scherk minimal surface, is a properly embedded minimal surface in $\mathbb{R}^3$ that is invariant under the action of one translation T and can be seen as the desingularization of two perpendicular planes. It has four ends which are asymptotic to four half-planes. In the quotient $\mathbb{R}^3/T$ by its shortest period, this surface has genus zero and four ends asymptotic to flat annuli. Such ends are called Scherk-type ends.

In [7], Traizet showed that for each finite set Π of vertical planes such that for each choice of three of them they do not have common intersection, there exists, for any $t > 0$ small enough, a complete minimal surface M_t of vertical period

$T = (0, 0, t)$ such that M_t converges to Π as t tends to zero, the quotient M_t/T has finite total curvature and Scherk-type ends.

In [3], Martin and Ramos Batista constructed a 1-parameter family of properly embedded singly periodic minimal surfaces which in the quotient have genus one and six Scherk-type ends.

In [1], the authors show the existence for each finite $k \geq 1$ of 1-parameter families of singly periodic minimal surfaces properly embedded in $\mathbb{R}^3$ which in the quotient have genus k and six and infinitely many Scherk-type ends.

In this paper, we show the existence of three kinds of periodic minimal surfaces which are properly embedded in $\mathbb{R}^3$. The surfaces of the first kind are singly periodic and seen from a distance they look like m equidistant parallel planes intersecting orthogonally k equidistant parallel planes. The numbers m and k satisfy the relation $2 \leq mk < +\infty$. The case $m = k = 1$ is not interesting because the minimal surface enjoying the properties above is the singly periodic Scherk surface. The surfaces are invariant under the action of a translation by 2π and they have $2k + 2m$ ends asymptotic to $2k + 2m$ half-planes. The quotient has $2k + 2m$ Scherk-type ends.

The surfaces of the second kind are doubly periodic and seen from a distance they look like $1 \leq m < +\infty$ equidistant parallel planes intersecting orthogonally infinitely many equidistant parallel planes. The surfaces are invariant under the action of two orthogonal translations and they have infinitely many ends. The quotient has two Scherk-type ends.

The surfaces of the third kind are triply periodic and seen from a distance they look like two stacks of infinitely many equidistant parallel planes which intersect orthogonally. The surfaces are invariant under the action of three orthogonal translations and the quotient is compact.

The technique we adopt to build these surfaces is the gluing procedure. Precisely we glue together pieces of the singly periodic Scherk minimal surface and the appropriate number of half-planes.

The statements of our results are the following.

Theorem 1.1. *For any value of* $m, k \in \mathbb{N}$, $2 \leq mk < +\infty$, *there exists a* 1-*parameter family of minimal surfaces properly embedded in* $\mathbb{R}^3$ *and invariant under the action of a translation by* 2π, *which seen from a distance look like* m *equidistant parallel planes intersecting orthogonally* k *equidistant parallel planes. The surfaces have* $2k + 2m$ *ends asymptotic to half-planes. In the quotient, the surfaces have* $2m + 2k$ *Scherk-type ends.*

Theorem 1.2. *For any finite value of* $m \in \mathbb{N}^+$, *there exists a* 1-*parameter family of minimal surfaces properly embedded in* $\mathbb{R}^3$ *and invariant under the action of two orthogonal translations, which seen from a distance look like* m *equidistant parallel planes intersecting orthogonally infinitely many equidistant parallel planes. The surfaces have infinitely many ends asymptotic to half-planes. In the quotient, the surfaces have two Scherk-type ends.*

Theorem 1.2 is an improvement with respect to the result of Traizet because we can handle the case of infinitely many planes and produce minimal surfaces with infinitely many ends.

Theorem 1.3. *There exists a* 1*-parameter family of minimal surfaces properly embedded in* $\mathbb{R}^3$ *and invariant under the action of three orthogonal translations, which seen from a distance look like two stacks of infinitely many equidistant parallel planes which intersect orthogonally. The quotient of the surfaces is compact.*

Here is how the paper is organized:

- In Sec. 2, we show there exists a family of singly periodic minimal graphs over a half-plane. Each surface in this family plays the role of end.
- In Sec. 4, we show there exists a family of singly periodic minimal perturbations of the singly periodic Scherk minimal surface.
- In Sec. 5, we show how to glue together the appropriate number of elements in the two families above in order to show Theorems 1.1–1.3.

2. The Ends

In this section, we will construct the ends of the surfaces. Precisely we will show the existence of a family of periodic minimal surfaces close to the half-plane

$$\Xi := \{(x, y, z) \in \mathbb{R}^3, z = 0, x \geq d_\varepsilon\}$$

where $d_\varepsilon = -2\ln\varepsilon$, $\varepsilon > 0$. To solve this problem, we will use the following approach. We will construct a family of singly periodic minimal graphs over Ξ with prescribed boundary data on $\partial\Xi$. Surfaces in this family will be glued to perturbations of the singly periodic Scherk surface.

Our first purpose is to find the expression of the euclidean mean curvature of the graph surface of a function over Ξ.

The half-plane Ξ can be identified with $J := [d_\varepsilon, +\infty) \times \mathbb{R} \subset \mathbb{R}^2$. Let u denote a function defined on J and let Σ_u denote its graph surface. It can be parametrized on J by

$$X(x, y) := (x, y, u(x, y)), \tag{1}$$

for $(x, y) \in J$ and $u \in \mathcal{C}^{2,\alpha}(J)$.

Σ_u is a minimal surface with respect to the euclidean metric if u is a solution to

$$2H_u = \operatorname{div}\left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}}\right) = 0, \tag{2}$$

where H_u denotes the mean curvature of Σ_u.

Let Δ_0 denote the Laplace operator. A computation allows us to prove the following result.

Lemma 2.1. *The following equality holds*:

$$2H_u = \frac{\Delta_0 u}{\sqrt{1+|\nabla u|^2}} + Q_0(\nabla u, \nabla^2 u), \tag{3}$$

where

$$Q_0(\nabla u, \nabla^2 u) = -\frac{\nabla u \cdot \nabla(|\nabla u|^2)}{2\sqrt{(1+|\nabla u|^2)^3}}.$$

In particular, Q_0 *has bounded coefficients and satisfies* $Q_0(0,0)=0, \nabla Q_0(0,0)=0$.

So Σ_u has vanishing mean curvature if u satisfies the equation

$$\Delta_0 u + \bar{Q}(\nabla u, \nabla^2 u) = 0 \tag{4}$$

where $\bar{Q}(\nabla u, \nabla^2 u) = \sqrt{1+|\nabla u|^2} Q_0(\nabla u, \nabla^2 u)$.

2.1. *A family of singly periodic minimal surfaces close to* Ξ

We set $S_a := \{(x,y) \in [a,+\infty) \times \mathbb{R}\}$. Consequently, $J = S_{d_\varepsilon}$.

Definition 2.2. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\rho \in (0,1)$, we define $C^{k,\alpha}_\rho(S_a)$ as the space of functions $u \in \mathcal{C}^{k,\alpha}_{\text{loc}}(S_a)$, which are even and 2π-periodic in the variable y and such that

$$\|u\|_{\mathcal{C}^{k,\alpha}_\rho(S_a)} := \sup_{x\in[a,+\infty)} \|e^{-\rho x} u\|_{C^{k,\alpha}([x,x+1)\times\mathbb{R})} < +\infty,$$

where $\|\cdot\|_{\mathcal{C}^{k,\alpha}}$ denotes the usual $\mathcal{C}^{k,\alpha}$ Hölder norm.

The following proposition deals with the mapping properties of the Laplace operator acting on $\mathcal{C}^{2,\alpha}_\rho(S_a)$.

Proposition 2.3. *Given* $\rho \in (0,1)$, *for each* $a \in \mathbb{R}$, *there exists an operator*

$$I : \mathcal{C}^{0,\alpha}_\rho(S_a) \to \mathcal{C}^{2,\alpha}_\rho(S_a)$$

such that for every $f \in \mathcal{C}^{0,\alpha}_\rho(S_a)$, *the function* $w := I(f)$ *solves*

$$\begin{cases} \Delta_0 w = f & \text{on } S_a, \\ w = 0 & \text{on } \partial S_a. \end{cases} \tag{5}$$

Moreover, $\|w\|_{\mathcal{C}^{2,\alpha}_\rho(S_a)} \leq c\|f\|_{\mathcal{C}^{0,\alpha}_\rho(S_a)}$, *for some constant* $c > 0$ *which does not depend on* a.

Proof. In order to show the existence of the solution we consider the Fourier series of f and w (we recall they are even functions):

$$f(x,y) = \sum_{m=0}^{+\infty} f_m(x)\cos(my), \quad w(x,y) = \sum_{m=0}^{+\infty} w_m(x)\cos(my).$$

If $\Delta_0 w = f$, then $w_m(x)$ is the solution to

$$L_m w_m := w_m''(x) - m^2 w_m(x) = f_m(x).$$

If $m = 0$, the general solution is

$$w_0(x) = c_1 + c_2 x + \int_a^x \int_a^s f_0(t)dtds.$$

If $m \geq 1$, we can determine the general solution by using the method of variation of parameters:

$$w_m(x) = C_1 e^{-mx} + C_2 e^{mx} - \frac{e^{-mx}}{2m}\int_a^x e^{ms} f_m(s)ds + \frac{e^{mx}}{2m}\int_a^x e^{-ms} f_m(s)ds.$$

Since we are looking for solution in $C_\rho^{2,\alpha}(S_a)$ with $\rho \in (0,1)$, then it is clear that $C_2 = 0$. If we impose $w_m(a) = 0$, $m \geq 1$, then $C_1 = 0$. The operator I is uniquely determined once we choose the values of c_1 and c_2. We set $c_1 = c_2 = 0$.

We now estimate the norm of the solution w in terms of the norm of f.

We can suppose $\|f\|_{\mathcal{C}_\rho^{0,\alpha}} < +\infty$. Since $M := \sup_{[a,+\infty)} e^{-\rho x}|f_0| < +\infty$ and

$$\begin{aligned}|w_0(x)| &\leq \int_a^x \int_a^s |f_0(t)|dtds = \int_a^x \int_a^s e^{\rho t} e^{-\rho t}|f_0(t)|dtds \\ &\leq M \int_a^x \int_a^s e^{\rho t} dtds = M\rho^{-2}e^{\rho x}(1 - e^{\rho(a-x)}(1-\rho(a-x))) \leq M\rho^{-2}e^{\rho x}\end{aligned}$$

holds true for any $x \geq a$, then

$$\sup_{[a,+\infty)} e^{-\rho x}|w_0(x)| \leq \rho^{-2} \sup_{[a,+\infty)} e^{-\rho x}|f_0(x)| \leq \rho^{-2}\|f_0\|_{\mathcal{C}_\rho^{0,\alpha}}.$$

We observe that, for $m \geq 1$, $L_m(e^{\rho x}) = (\rho^2 - m^2)e^{\rho x}$. If we define

$$v_m = \frac{\|f\|_{\mathcal{C}_\rho^{0,\alpha}(S_a)} e^{\rho x}}{m^2 - \rho^2} - w_m,$$

then $v_m(a) > 0$ and

$$L_m v_m = -\|f\|_{\mathcal{C}_\rho^{0,\alpha}(S_a)} e^{\rho x} - f_m \leq 0.$$

Consequently by the maximum principle $v_m(x) \geq 0$, that is

$$w_m \leq \frac{\|f\|_{\mathcal{C}_\rho^{0,\alpha}(S_a)} e^{\rho x}}{m^2 - \rho^2}.$$

Instead if we define

$$v_m = -\frac{\|f\|_{\mathcal{C}_\rho^{0,\alpha}(S_a)} e^{\rho x}}{m^2 - \rho^2} + w_m,$$

then using the same argument we can show

$$w_m \geq -\frac{\|f\|_{\mathcal{C}_\rho^{0,\alpha}(S_a)} e^{\rho x}}{m^2 - \rho^2}.$$

Combining the two estimates above, we get

$$|w_m(x)| \leq \frac{\|f\|_{\mathcal{C}^{0,\alpha}_\rho(S_a)} e^{\rho x}}{m^2 - \rho^2}.$$

From this estimate, summing up over m, we obtain

$$e^{-\rho x}|w(x)| \leq c\|f\|_{\mathcal{C}^{0,\alpha}_\rho(S_a)}.$$

Finally, using Schäuder estimates, we get the estimate for the derivatives of w. Combining these estimates with the previous one, we get

$$\|w\|_{\mathcal{C}^{2,\alpha}_\rho(S_a)} \leq c\|f\|_{\mathcal{C}^{0,\alpha}_\rho(S_a)}. \qquad \square$$

In the sequel, we will use the previous result under the assumption $a = d_\varepsilon = -2\ln\varepsilon$. In this case, S_a coincides with the set J.

Let $\varphi \in \mathcal{C}^{2,\alpha}(\mathbb{R})$ be even, 2π-periodic and L^2-orthogonal to the constant functions with $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leq \kappa\varepsilon^{1+\eta}$, $\eta \in (0,1)$ and $\kappa > 0$ is a constant to be determined later. Let $v_\varphi = H_{d_\varepsilon}(\varphi)$ be the harmonic extension of φ on J provided by Proposition A.1.

We look for a solution of Eq. (4) of the form $u = v + v_\varphi$.

The equation we are going to solve is

$$\Delta_0 v = -Q(v + v_\varphi),$$

where $Q(\cdot) = \bar{Q}(\nabla\cdot, \nabla^2\cdot)$ for the sake of simplicity. The resolution of the previous equation is obtained by solving the following fixed point problem

$$v = Z(\varphi, v),$$

where

$$Z(\varphi, v) = -I(Q(v + v_\varphi)).$$

I is the operator described by Proposition 2.3.

Proposition 2.4. *Given $\rho \in (0,1)$, $\eta \in (0,1)$, there exist $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that*

$$\|Z(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \leq c_\kappa \varepsilon^{2+2\eta+2\rho}$$

and for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|Z(\varphi, v_1) - Z(\varphi, v_2)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \leq c_\kappa \varepsilon^{1+\eta}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\rho(J)}$$

$$\|Z(\varphi_1, v) - Z(\varphi_2, v)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \leq c_\kappa \varepsilon^{2+2\eta}\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\rho(J)$ whose norm is bounded by $2c_\kappa\varepsilon^{2+2\eta+2\rho}$, for all boundary data $\varphi, \varphi_1, \varphi_2$ which are even, 2π-periodic, L^2-orthogonal to the constant functions and whose norm is bounded by a constant κ times $\varepsilon^{1+\eta}$.

Proof. Using Proposition 2.3, we get $\|I(f)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \leq c\|f\|_{\mathcal{C}^{0,\alpha}_\rho(J)}$, then

$$\|Z(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \leq c\|Q(v_\varphi)\|_{\mathcal{C}^{0,\alpha}_\rho(J)} \leq c\|v_\varphi^2\|_{\mathcal{C}^{2,\alpha}_\rho(J)}.$$

Here we use the fact that $Q(v_\varphi)$ consists of products of derivatives of v_φ. The $\mathcal{C}^{0,\alpha}_\rho$-norm of each factor is smaller than $c\|v_\varphi^2\|_{\mathcal{C}^{2,\alpha}_\rho(J)}$. In order to find an estimate of $\|Q(v_\varphi)\|_{\mathcal{C}^{0,\alpha}_\rho(J)}$ we observe that from Proposition A.1 we obtain

$$\|v_\varphi\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le ce^{-\rho d_\varepsilon}\|\varphi\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \le c_\kappa \varepsilon^{1+\eta+2\rho},$$

and similarly

$$\|v_\varphi^2\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le ce^{-\rho d_\varepsilon}\|\varphi\|^2_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \le c_\kappa \varepsilon^{2+2\eta+2\rho}.$$

As a consequence

$$\|Z(\varphi,0)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c_\kappa \varepsilon^{2+2\eta+2\rho}.$$

The bound for the norm of the functions v, v_1, v_2 follows from this estimate.

As for the second estimate, we observe that

$$\|Z(\varphi,v_2) - Z(\varphi,v_1)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c\|Q(v_\varphi+v_2) - Q(v_\varphi+v_1)\|_{\mathcal{C}^{0,\alpha}_\rho(J)}.$$

It is possible to show that

$$\begin{aligned}\|Q(v_\varphi+v_2) - Q(v_\varphi+v_1)\|_{\mathcal{C}^{0,\alpha}_\rho(J)} &\\ \le c\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\rho(J)}\|v_\varphi\|_{\mathcal{C}^{2,\alpha}(J)} &\\ \le c_\kappa\varepsilon^{1+\eta}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\rho(J)}.&\end{aligned}$$

Indeed, $Q(v_\varphi+v_i)$ consists in products of derivatives of $v_\varphi+v_i$ and the $\mathcal{C}^{0,\alpha}_\rho$-norm of the products in $Q(v_\varphi+v_2)-Q(v_\varphi+v_1)$ is smaller than

$$c\|(v_2-v_1)v_\varphi\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\rho(J)}\|v_\varphi\|_{\mathcal{C}^{2,\alpha}(J)}.$$

We used also the fact that $\|v_\varphi\|_{\mathcal{C}^{2,\alpha}(J)} \le \varepsilon^{-2\rho}\|v_\varphi\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c_\kappa\varepsilon^{1+\eta}$. Then

$$\|Z(\varphi,v_2) - Z(\varphi,v_1)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c_\kappa\varepsilon^{1+\eta}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\rho(J)}.$$

To show the third estimate, we proceed as above

$$\begin{aligned}\|Z(\varphi_2,v) - Z(\varphi_1,v)\|_{\mathcal{C}^{2,\alpha}_\rho(J)} &\\ \le c\|Q(v_{\varphi_2}+v) - Q(v_{\varphi_1}+v)\|_{\mathcal{C}^{0,\alpha}_\rho(J)} &\\ \le c\|v_{\varphi_2}-v_{\varphi_1}\|_{\mathcal{C}^{2,\alpha}_\rho(J)}\|v\|_{\mathcal{C}^{2,\alpha}(J)} &\\ \le c_\kappa\varepsilon^{2+2\eta}\|\varphi_2-\varphi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})}.&\end{aligned}$$

Indeed, $Q(v_{\varphi_i}+v)$ consists in products of derivatives of $v_{\varphi_i}+v$ and the $\mathcal{C}^{0,\alpha}_\rho$-norm of the products in $Q(v_{\varphi_2}+v)-Q(v_{\varphi_1}+v)$ is smaller than

$$c\|(v_{\varphi_2}-v_{\varphi_1})v\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c\|v_{\varphi_2}-v_{\varphi_1}\|_{\mathcal{C}^{2,\alpha}_\rho(J)}\|v\|_{\mathcal{C}^{2,\alpha}(J)}.$$

We used also the fact that $\|v\|_{\mathcal{C}^{2,\alpha}(J)} \le \varepsilon^{-2\rho}\|v\|_{\mathcal{C}^{2,\alpha}_\rho(J)} \le c_\kappa\varepsilon^{2+2\eta}$. □

Theorem 2.5. *Let $B_\kappa := \{v \in \mathcal{C}^{2,\alpha}_\rho(J) \mid \|v\|_{\mathcal{C}^{2,\alpha}_\rho} \le 2c_\kappa\varepsilon^{2+2\eta+2\rho}\}$. Then the nonlinear mapping $Z(\varphi,\cdot)$ defined above has a unique fixed point in B_κ.*

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping $Z(\varphi,\cdot)$ is a contraction mapping from the ball B_κ of radius $2c_\kappa\varepsilon^{2+2\eta+2\rho}$ in $\mathcal{C}^{2,\alpha}_\rho(J)$ into itself. This value follows from the estimate of the norm of $Z(\varphi,0)$. Consequently, thanks to Leray–Schäuder fixed point theorem, $Z(\varphi,\cdot)$ has a unique fixed point in this ball. □

We have proved that for each φ of norm bounded by $\kappa\varepsilon^{1+\eta}$, there exists a minimal surface which is close to J, and for (x,y) in a neighborhood of ∂J, it is the surface parametrized by (1) with $u=\bar{U}(x,y)$ given by

$$\bar{U}(x,y)=H_{d_\varepsilon}(\varphi)+\bar{V},\quad \text{with } \|\bar{V}\|_{\mathcal{C}^{2,\alpha}_\rho}\leq c\varepsilon^{2+2\eta+2\rho}.$$

We recall that $H_{d_\varepsilon}(\varphi)$ is a harmonic function and $d_\varepsilon=-2\ln\varepsilon$. The function $\bar{V}$ is the fixed point found above and it depends nonlinearly on ε,φ. We observe that

$$\|\bar{V}\|_{\mathcal{C}^{2,\alpha}}\leq e^{\rho d_\varepsilon}\|\bar{V}\|_{\mathcal{C}^{2,\alpha}_\rho}\leq c\varepsilon^{2+2\eta}.$$

Consequently, the dominant term in the formula for $\bar{U}$ is $H_{d_\varepsilon}(\varphi)$ because $\|H_{d_\varepsilon}(\varphi)\|_{\mathcal{C}^{2,\alpha}}\leq c_\kappa\varepsilon^{1+\eta}$. Furthermore using the third estimate of Lemma 2.4, we get

$$\begin{aligned}&\|\bar{V}(\varepsilon,\varphi)(\cdot,\cdot)-\bar{V}(\varepsilon,\varphi')(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(J)}\\&\quad\leq ce^{\rho d_\varepsilon}\|\bar{V}(\varepsilon,\varphi)(\cdot,\cdot)-\bar{V}(\varepsilon,\varphi')(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}_\rho(J)}\\&\quad\leq c\varepsilon^{2+2\eta-2\rho}\|\varphi-\varphi'\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})}.\end{aligned}\tag{6}$$

We recall that we identified J and the half-plane Ξ. So in conclusion, we constructed a periodic (invariant under the action of the horizontal translation by $2\pi e_2$, e_2 being one of the coordinate vectors) minimal graph $\Xi(\varphi)$ over Ξ.

3. The Singly Periodic Scherk Minimal Surfaces

The (symmetric) singly periodic Scherk minimal surface is an embedded singly periodic surface having four ends which are asymptotic to four half-planes. If $(O,\bar{x},\bar{y},\bar{z})$ denote the system of cartesian orthogonal coordinates of $\mathbb{R}^3$, then up to a translation, a rotation and a homothety of $\mathbb{R}^3$ the Scherk surface is invariant by the reflection in the three coordinate planes $\{\bar{x}=0\}$, $\{\bar{y}=0\}$, $\{\bar{z}=0\}$ and their period equals $2\pi e_2$. This surface can be viewed as the desingularization of two orthogonal planes whose intersection is the $\bar{y}$ axis. We consider the new system of coordinates (O,x,y,z) of $\mathbb{R}^3$ such that one of the four asymptotic half-planes is $\{(x,y,z)\in\mathbb{R}^3, z=0, x\geq 0\}$. Such half-plane can be identified with $W=[0,+\infty)\times\mathbb{R}$. Below we will give a description of the corresponding end of the Scherk surface. Furthermore, it is possible to represent each end of the surface as the graph surface of a function defined on W. Let S denote the Scherk surface. Let $E_{1,h}$ denote the horizontal end parametrized by

$$X_{1,h}(x,y):=(x,y,u_m(x,y))\in\mathbb{R}^3,\tag{7}$$

for $(x, y) \in [x_0, +\infty) \times \mathbb{R} \subset W$, where x_0 is big enough. The parametrizations $X_{1,v}, X_{2,h}, X_{2,v}$ of the remaining horizontal end $E_{2,h}$ and of the two vertical ends $E_{1,v}, E_{2,v}$ are obtained from $X_{1,h}$ by composition with appropriate isometries.

The function u_m satisfies the minimal surfaces equation

$$2H_u = \operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) = 0, \tag{8}$$

where H_u denotes the mean curvature of Σ_u, the graph surface of the function u, with respect to the euclidean metric.

The result we recall below is [7, Proposition 2.1].

Proposition 3.1. *For x big enough, the function u_m and its derivatives satisfy*

$$|D^k u_m| \leq c(k)e^{-x},$$

where $c(k)$ denotes a constant which depends on k. The Gauss curvature of the end $E_{1,h}$ at the point $X_{1,h}(x, y)$ satisfies: $c^{-1}e^{-2x} \leq |K(x, y)| \leq ce^{-x}$.

If we linearize at $u = 0$ the nonlinear equation (8) we obtain the expression of an operator which equals the Jacobi operator of the plane, that is $\mathcal{L}_{\mathbb{R}^2} = \Delta_0$. Indeed, Lemma 3.2 ensures that the linearization of (8) gives

$$L_u v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla u|^2}} - \nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1+|\nabla u|^2)^3}}\right). \tag{9}$$

We shall express H_{u+v}, the mean curvature of the graph surface of the function $u + v$, in terms of the mean curvature of Σ_u, that is H_u.

Lemma 3.2. *The following equality holds true*:

$$2H_{u+v} = 2H_u + L_u v + Q_u(\nabla v, \nabla^2 v), \tag{10}$$

where

$$Q_u(\nabla v, \nabla^2 v) := \operatorname{div}(\nabla(u+v)Q_u^*(v)) - \operatorname{div}\left(\nabla v \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}\right),$$

has bounded coefficients, it satisfies $Q_u(0, 0) = 0, \nabla Q_u(0, 0) = 0$, and $Q_u^*(v)$ is defined in (12).

Proof. We observe that

$$\frac{1}{\sqrt{1+|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\sqrt{(1+|\nabla u|^2)^3}} + Q_u^*(v). \tag{11}$$

$Q_u^*(v)$ has the following expression:

$$\frac{-|\nabla v|^2}{(1+|\nabla(u+\bar{t}v)|^2)^{3/2}} + \frac{3(\nabla u \cdot \nabla v + \bar{t}|\nabla v|^2)^2}{(1+|\nabla(u+\bar{t}v)|^2)^{5/2}}, \tag{12}$$

with $\bar{t} \in (0,1)$, and it satisfies $Q_u^*(0) = 0, \nabla Q_u^*(0) = 0$. To show (11), it is sufficient to set

$$f(t) := \frac{1}{\sqrt{1 + |\nabla(u + tv)|^2}},$$

to write down the Taylor series of order one of this function and to evaluate it at $t = 1$. That is $f(1) = f(0) + f'(0) + \frac{1}{2}f''(\bar{t})$, with $\bar{t} \in (0,1)$. We insert (11) in the expression of $2H_{u+v}$ getting

$$\begin{aligned} &\operatorname{div}\left(\frac{\nabla(u+v)}{\sqrt{1+|\nabla u|^2}} - \nabla(u+v)\frac{\nabla u \cdot \nabla v}{\sqrt{(1+|\nabla u|^2)^3}} + \nabla(u+v)Q_u^*(v)\right) \\ &\quad = 2H_u + \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla u|^2}} - \nabla u\frac{\nabla u \cdot \nabla v}{\sqrt{(1+|\nabla u|^2)^3}}\right) + Q_u(\nabla v, \nabla^2 v). \end{aligned}$$

From this, the wanted expression follows. □

Hereafter, we shall assume that $u = u_m$. Since $H_{u_m} = 0$ then the mean curvature of the graph surface of $u_m + v$ vanishes if

$$2H_{u_m+v} = \frac{1}{\sqrt{1+|\nabla u_m|^2}}(\Delta_0 v + \bar{L}_{u_m} v + \sqrt{1+|\nabla u_m|^2}Q_{u_m}(\nabla v, \nabla^2 v)) = 0, \quad (13)$$

where

$$\begin{aligned} \bar{L}_{u_m} v := \sqrt{1+|\nabla u_m|^2}\nabla v \cdot \nabla\left(\frac{1}{\sqrt{1+|\nabla u_m|^2}}\right) \\ - \sqrt{1+|\nabla u_m|^2}\operatorname{div}\left(\nabla u_m \frac{\nabla u_m \cdot \nabla v}{\sqrt{(1+|\nabla u_m|^2)^3}}\right) \end{aligned}$$

is a second-order linear operator with coefficients which are functions $\mathcal{O}_{\mathcal{C}^\infty}(e^{-2x})$, because $u_m = \mathcal{O}_{\mathcal{C}^\infty}(e^{-x})$.

If the function v satisfies $H_{u_m+v} = 0$, then the graph surface of $u_m + v$ is minimal.

We recall that S denotes the Scherk surface and that the parametrizations of the four ends of S are denoted by $X_{1,h}, X_{2,h}, X_{1,v}, X_{2,v}$.

We set

$$x_\varepsilon = -3 \ln \varepsilon.$$

We define S^T (respectively, S_0^T) as S from which we remove the images by $X_{1,h}, X_{2,h}, X_{1,v}, X_{2,v}$ of $[x_\varepsilon, +\infty) \times \mathbb{R}$ (respectively, $[x_0, +\infty) \times \mathbb{R}$, for the definition of x_0 see (7)).

In the sequel, we will show the existence of a family of minimal graphs over S^T. These minimal graphs are obtained from S^T by perturbation.

We define the normed space of functions on S we shall use later on.

Definition 3.3. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, we define $\mathcal{C}^{\ell,\alpha}_\delta(S)$ as the space of functions in $\mathcal{C}^{\ell,\alpha}_{\text{loc}}(S)$ for which the following norm is finite:

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(S)} := \|w\|_{\mathcal{C}^{\ell,\alpha}(\overline{S^T_0})} + \sum_{i=1,2;j=h,v} \|w \circ X_{i,j}\|_{\mathcal{C}^{\ell,\alpha}_\delta([x_0,+\infty)\times\mathbb{R})},$$

where

$$\|f\|_{\mathcal{C}^{\ell,\alpha}_\delta([x_0,+\infty)\times\mathbb{R})} := \sup_{x\geq x_0} \|e^{-\delta x} f\|_{\mathcal{C}^{\ell,\alpha}([x,x+1)\times\mathbb{R})},$$

and which are invariant by a horizontal translation of $2\pi e_2$, by the reflection in the plane $\{\bar{y} = 0\}$.

In order to construct a family of minimal perturbations of S^T, we shall consider graphs of functions of small norm over the original surface and we impose that their mean curvature vanishes. The equation to solve has the following structure:

$$\mathbb{L}_S v + Q_S(v) = 0,$$

where $\mathbb{L}_S$ denotes the Jacobi operator of S and Q_S is a nonlinear operator. We shall show the existence of a solution by solving a fixed point problem. This step requires the existence of an inverse of the Jacobi operator $\mathbb{L}_S$.

3.1. *The Jacobi operator of S*

We now study the mapping properties of the Jacobi operator. The Jacobi operator of S with respect to the euclidean metric is

$$\mathbb{L}_S := \Delta_S + |A_S|^2, \tag{14}$$

where $|A_S|$ is the norm of the second fundamental form of S.

We introduce the following operator acting on $\mathcal{C}^{2,\alpha}_\delta$ functions defined on S:

$$\mathcal{L}_S : \mathcal{C}^{2,\alpha}_\delta(S) \to \mathcal{C}^{0,\alpha}_\delta(S),$$
$$u \mapsto \mathbb{L}_S u.$$

It is linear and bounded.

To see that, let us set $u_m = 0$ in (13). We get the Jacobi operator of the plane

$$\mathcal{L}_{\mathbb{R}^2} := \Delta_0.$$

Such an operator maps the space $\mathcal{C}^{2,\alpha}([x_0,+\infty)\times\mathbb{R})$ into the space $\mathcal{C}^{0,\alpha}([x_0,+\infty)\times\mathbb{R})$. Since S is asymptotic to a half-plane at each end, then its Jacobi operator is close to $\mathcal{L}_{\mathbb{R}^2}$ at each end. We conclude that the operator $\mathcal{L}_S$ maps $\mathcal{C}^{2,\alpha}_\delta(S)$ into $\mathcal{C}^{0,\alpha}_\delta(S)$.

The Jacobi fields. It is known that any smooth 1-parameter group of isometries or dilations containing the identity generates a Jacobi field, that is a solution of the equation $\mathbb{L}_S u = 0$. Let p denote a point of S and $n(p)$ the unit normal vector to S

at p. The Jacobi fields of this type are listed below:

- the Killing vector field $\Xi(p) = p$ associated to the 1-parameter group of dilations, generates the Jacobi field $\Phi^d(p) := n(p) \cdot p$.
- the Jacobi field $\Phi^r(p) := n(p) \cdot (e_2 \times p)$ is generated by the Killing vector field $\Xi(p) = e_2 \times p$ which is associated to the group of rotations about the $\bar{y}$-axis.
- the Killing vector fields $\Xi(p) = e_j$, $j = 1, 2, 3$, associated to the 1-parameter group of translations in the direction e_j, generate the Jacobi fields $\Phi^j(p) := n(p) \cdot e_j$.

The first two of the Jacobi fields listed above are not bounded at the ends of S : indeed they grow linearly. The remaining are bounded. In particular, Φ^3 decays exponentially fast at the ends. Indeed, it easily follows from Proposition 3.1 that $\Phi^3 = \mathcal{O}_{L^\infty}(e^{-x})$ at the horizontal end $E_{1,h}$.

There is one more unbounded Jacobi field, F, which originates from the change of angle between adjacent asymptotic half-planes. Such a Jacobi field has a linear growth rate at the ends, consequently it belongs to $\mathcal{C}^{2,\alpha}_\delta(S)$ only if $\delta > 0$. It is possible to get a formula for F as follows. Let 2ϕ and $\pi - 2\phi$ denote the angles between adjacent asymptotic half-planes. For the symmetric Scherk minimal surface both of them are equal to $\pi/2$. If we change the value of ϕ into $\phi + \beta$, then we get a new surface $S(\beta)$. Let f denote the function on S whose normal graph coincides with $S(\beta)$. Then $F := \frac{df}{d\beta}_{|\beta=0}$.

There are no other bounded Jacobi fields. The dimension of the space of the bounded Jacobi fields about S equals 3, as can be showed using [4, Conclusion 1 and Corollary 15].

We observe that the quotient of S by its period has finite total curvature. The Gauss map of a surface with finite total curvature extends to a holomorphic map Ψ from a compact Riemann surface Σ to $\mathbb{S}^2$. In [4], it is proved that the nullity of Ψ, that is the dimension of $\{f \in C^\infty(\Sigma) : \Delta f + |\nabla\Psi|^2 f = 0\}$, is equal to the number of independent Jacobi fields of any complete minimal surface in $\mathbb{R}^3$ with finite total curvature which has Ψ as extended Gauss map. Second, the nullity of such a non-constant holomorphic map equals 3 if all of its branch values (that is the images of the branch points by the Gauss map) lie in an equator of $\mathbb{S}^2$.

The Weierstrass data of the symmetric Scherk surface, we consider are

$$g(w) = w, \quad dh = C\frac{wdw}{(w^2 - e^{i2\phi})(w^2 - e^{-i2\phi})}, \quad w \in \bar{\mathbb{C}}.$$

C is a constant which makes the period equal to $2\pi e_2$ and $2\phi = \pi/2$.

As a consequence, the Gauss map of the quotient of these surfaces has a unique branch point: $w = \infty$. It is easy to see that its image by the composition of the inverse Π^{-1} of the stereographic projection and g is located on an equator of $\mathbb{S}^2$. So the hypotheses of [4, Corollary 15] are satisfied.

None of the Jacobi fields belongs to $\mathcal{C}^{2,\alpha}_\delta(S)$ with $\delta < 0$, because they do not enjoy the invariance properties of functions in $\mathcal{C}^{2,\alpha}_\delta(S)$ or they do not have the appropriate decay rate at the ends. Indeed, if we choose $\delta < 0$, then the space

$\mathcal{C}^{2,\alpha}_\delta(S)$ contains only bounded functions which must decrease exponentially fast at the ends. That shows that the Jacobi operator $\mathbb{L}_S$ is injective.

Let S^q denote the quotient of S by its period. S^q is a manifold with 4 cylindrical ends. Its Jacobi operator $\mathcal{L}_{S^q}$ is $\mathcal{L}_S$ which we let act on functions defined on the quotient S^q. Now, we study the mapping properties of this operator acting on functions in $L^2_\delta(S^q)$ and $\mathcal{C}^{2,\alpha}_\delta(S^q)$.

We define $L^2_\delta(S^q)$ as the space of functions $u \in L^2_{\text{loc}}(S^q)$ for which the following norm is finite

$$\|u\|_{L^2_\delta(S^q)} = (\langle u, u\rangle_{L^2_\delta(S^q)})^{1/2} := \left(\int_{S^q} u^2\gamma^{-2} d\,\text{vol}_{S^q}\right)^{1/2},$$

where γ is a smooth function on S such that $\gamma = 1$ on S^T_0, $\gamma = e^{\delta x}$ on each end and invariant with respect to the isometries which leave S fixed.

We define the operator

$$A_\delta : L^2_\delta(S^q) \to L^2_\delta(S^q),$$
$$u \to \mathcal{L}_{S^q} u.$$

This operator is unbounded and has dense domain

$$\text{Dom}(A_\delta) := \{u \in L^2_\delta(S^q) : \mathcal{L}_{S^q} u \in L^2_\delta(S^q)\}.$$

The indicial roots of $\mathcal{L}_{S^q}$ at the ends are the real numbers m for which there exists a function $y \to v(y)$ such that

$$\mathcal{L}_{S^q}(e^{mx} v(y)) = \mathcal{O}_{L^\infty}(e^{m'x})$$

at the ends of S^q for $m' < m$ and $x \geq x_0$.

Using the fact that u_m decays like e^{-x} at the ends and (13), we get that $\mathcal{L}_{S^q} - \Delta_0$ is an operator whose coefficients can be estimated by $\mathcal{O}_{C^\infty}(e^{-2x})$ at the ends. From that it follows $\mathcal{L}_{S^q}(e^{jx}\cos(jy)) = \mathcal{O}_{L^\infty}(e^{(j-2)x})$, for $x \geq x_0$. Furthermore, a simple computation shows that if j is an indicial root, that is $\mathcal{L}_{S^q}(e^{jx}v(y)) = \mathcal{O}_{L^\infty}(e^{j'x})$, with $j' < j$, then $v(y)$ must be collinear to $\cos(jy)$. Last $j \in \mathbb{Z}$ otherwise $e^{jx}\cos(jy)$ would not be a function of period 2π with respect to the y-variable.

In conclusion, the set of indicial roots coincides with $\mathbb{Z}$.

It can be shown as in [2, Sec. 8.4] that A_δ is a Fredholm operator (that is it has closed range of finite codimension and kernel of finite dimension) if δ is not an indicial root. In order to use this property, it is convenient to identify the dual $(L^2_\delta(S^q))^*$ of $L^2_\delta(S^q)$ with $L^2_{-\delta}(S^q)$.

Lemma 3.4. *Given* $T \in (L^2_\delta(S^q))^*$, *there exists a unique* $w \in L^2_{-\delta}(S^q)$ *such that*

$$Tu = \int_{S^q} uw\,d\text{vol}_{S^q},$$

for every $u \in L^2_\delta(S^q)$. *Moreover,* $\|w\|_{L^2_{-\delta}(S^q)} = \|T\|_{(L^2_\delta(S^q))^*}$.

See [5, Sec. 4] for the proof of this and following results.

This allows us to identify the adjoint A^*_δ of A_δ with $A_{-\delta}$. Suppose $v \in \mathrm{Dom}(A_\delta)$ and $w \in L^2_{-\delta}(S^q)$. A^*_δ is the adjoint if

$$\int_{S^q} wA_\delta v \,\mathrm{dvol}_{S^q} = \int_{S^q} v(A^*_\delta w) \,\mathrm{dvol}_{S^q}.$$

If T_w is the operator such that $T_w(v) = \int_{S^q} vw \,\mathrm{dvol}_{S^q}$, then the right-hand side equals $T_{A^*_\delta w}(v)$. Since $v \in L^2_\delta(S^q)$, then by previous lemma, $T_{A^*_\delta w}$ can be thought as T_f with $f \in L^2_{-\delta}(S^q)$. That says

$$\int_{S^q} wA_\delta v \,\mathrm{dvol}_{S^q} = T_f v = \int_{S^q} fv \,\mathrm{dvol}_{S^q},$$

for any $v \in \mathrm{Dom}(A_\delta)$.

From the definition of A_δ, it follows that $\int_{S^q} w\mathcal{L}_{S^q} v \,\mathrm{dvol}_{S^q} = \int_{S^q} fv \,\mathrm{dvol}_{S^q}$. Since $\mathcal{L}_{S^q}$ is self-adjoint, that is $\int_{S^q} w\mathcal{L}_{S^q} v \,\mathrm{dvol}_{S^q} = \int_{S^q} v\mathcal{L}_{S^q} w \,\mathrm{dvol}_{S^q}$, this implies that $\mathcal{L}_{S^q} w = f$ in the sense of distributions. Since $w, f \in L^2_{-\delta}(S^q)$, we conclude that $w \in \mathrm{Dom}(A_{-\delta})$ and $f = A_{-\delta} w$. In conclusion, $T_{A^*_\delta w} = T_f = T_{A_{-\delta} w}$.

Conversely, if $w \in \mathrm{Dom}(A_{-\delta})$, then for all $v \in \mathrm{Dom}(A_\delta)$ it holds

$$\int_{S^q} wA_\delta v \,\mathrm{dvol}_{S^q} = \int_{S^q} w\mathcal{L}_{S^q_\varepsilon} v \,\mathrm{dvol}_{S^q} = \int_{S^q} v\mathcal{L}_{S^q_\varepsilon} w \,\mathrm{dvol}_{S^q} = \int_{S^q} vA_{-\delta} w \,\mathrm{dvol}_{S^q}.$$

The second identity uses integration by parts whose use is justified by the fact that $\nabla v \in L^2_\delta(S^q)$, $\nabla^2 v \in L^2_\delta(S^q)$, $\nabla w \in L^2_{-\delta}(S^q)$, $\nabla^2 w \in L^2_{-\delta}(S^q)$. In other terms v, w belong respectively to the weighted Sobolev spaces $W^{2,2}_\delta(S^q)$ and $W^{2,2}_{-\delta}(S^q)$. This is consequence of this result:

Proposition 3.5. *If $\delta \notin \mathbb{Z}$ then there exists a compact set $K \subset S^q$ and a constant $c > 0$ such that, if $u, f \in L^2_\delta(S^q)$ and*

$$\mathcal{L}_{S^q} u = f,$$

then

$$\|u\|_{L^2_\delta(S^q)} + \|\nabla u\|_{L^2_\delta(S^q)} + \|\nabla^2 u\|_{L^2_\delta(S^q)} \le c(\|f\|_{L^2_\delta(S^q)} + \|\nabla u\|_{L^2(K)}).$$

In other terms

$$\|u\|_{W^{2,2}_\delta(S^q)} \le c(\|f\|_{L^2_\delta(S^q)} + \|\nabla u\|_{L^2(K)}).$$

The proof is very similar to the one of [2, Proposition 8.15] in combination with the fact that at the ends $\mathcal{L}_{S^q}$ equals the sum of the Laplacian operator and an operator whose coefficients decay as fast as e^{-2x}.

In conclusion, $w \in \mathrm{Dom}(A^*_\delta)$ and $T_{A^*_\delta w} = T_{A_{-\delta} w}$.

This result also allows us to write $\mathrm{Dom}(A_\delta)$ as $\{u \in L^2_\delta(S^q) : \nabla u, \nabla^2 u \in L^2_\delta(S^q)\}$ or

$$\{u \in L^2_\delta(S^q) : \exists \{u_m\}_m \in \mathcal{C}^\infty_0, \{u_m\}_m \to u \in L^2_\delta(S^q), \{\mathcal{L}_{S^q} u_m\}_m \to h \in L^2_\delta(S^q)\}.$$

By using that it is easy to show that A_δ has dense domain and closed graph. If A_δ has dense domain, then $\mathrm{range}(A^*_\delta)$ is closed.

We recall that if $\delta \notin \mathbb{Z}$ then A_δ is a Fredholm operator. Then also A_δ^* has closed range. So from Fredholm theory for unbounded operators with dense domain and closed range, we get $\ker(A_\delta) = \text{range}(A_\delta^*)^\perp$, and if in addition $\text{range}(A_\delta)$ is closed, then $\text{range}(A_\delta) = \ker(A_\delta^*)^\perp$.

In view of the identification of A_δ^* with $A_{-\delta}$, we obtain

$$\ker(A_\delta) = \text{range}(A_{-\delta})^\perp, \quad \text{range}(A_\delta) = \ker(A_{-\delta})^\perp,$$

and

$$\dim\ker(A_\delta) = \text{codim range}(A_{-\delta}), \quad \dim \text{range}(A_\delta) = \text{codim}\ker(A_{-\delta}). \tag{15}$$

Consequently,

$$A_\delta \text{ is injective} \Leftrightarrow A_{-\delta} \text{ is surjective}.$$

In our case, A_δ is injective for $\delta < 0$ and a Fredholm operator only for $\delta \notin \mathbb{Z}$. As a consequence, A_δ is surjective for any $\delta > 0, \delta \notin \mathbb{N}$.

In the sequel, we explain how to extend this result to the operator $\mathcal{L}_{S^q}$ which acts on a weighted Hölder space.

We start by presenting an estimate for which the hypothesis $\delta \notin \mathbb{Z}$ is not necessary.

Proposition 3.6. *For any given $\delta \in \mathbb{R}$, there exists a constant $c > 0$ such that for all $u, f \in L^2_\delta(S^q)$ satisfying*

$$\mathcal{L}_{S^q} u = f,$$

if $f \in \mathcal{C}^{0,\alpha}_\delta(S^q)$ then $u \in \mathcal{C}^{2,\alpha}_\delta(S^q)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_\delta(S^q)} \leq c(\|f\|_{\mathcal{C}^{0,\alpha}_\delta(S^q)} + \|u\|_{L^2_\delta(S^q)}).$$

Using the previous proposition, it is possible to show the following result.

Proposition 3.7. *Let $\delta \in (0,1)$. The operator $\mathcal{L}_{S^q} : \mathcal{C}^{2,\alpha}_\delta(S^q) \to \mathcal{C}^{0,\alpha}_\delta(S^q)$ is surjective and has a kernel of dimension 4. Furthermore, there exists a right inverse whose norm is bounded.*

We observe that in [5] the kernel of the operator $\mathcal{L}_{S^q}$ has dimension 1. Indeed in [5], we consider a space of functions which are invariant under the action of several isometries of the Scherk surface. All the ends of the surface are in the same orbit under the action of these isometries. See [5, Propositions 4.7, 4.8 and Remark 4.9]. Instead in this work, we consider a space of functions which are invariant under the action of a smaller set of isometries. Consequently, we get one generator of the kernel for each end of the surface. That explains why the kernel is larger.

Proposition 3.8. *For any given $\delta' > \delta$, such that (δ, δ') does not contain any indicial root (i.e. it does not contain any integer number), there exists a constant $c > 0$ such that if $f \in \mathcal{C}^{0,\alpha}_\delta(S^q)$, $u, f \in L^2_{\delta'}(S^q)$, satisfying $\mathcal{L}_{S^q} u = f$, then $u \in \mathcal{C}^{2,\alpha}_\delta(S^q)$ and*

$$\|u\|_{\mathcal{C}^{2,\alpha}_\delta(S^q)} \leq c(\|f\|_{\mathcal{C}^{0,\alpha}_\delta(S^q)} + \|u\|_{L^2_{\delta'}(S^q)}). \tag{16}$$

Remark 3.9. Since the Jacobi operator of the surface is asymptotic to Δ_0 at the ends, then the Jacobi operator has a kernel which is related to the kernel of Δ_0. $\ker(\Delta_0)$ is generated by four functions having the following behavior at the ends. Each of them has a behavior close to the one of the functions $x - x_0$ at exactly one end and it is close to the function identically equal to zero at the remaining ends. We observe that the function which generates the kernel belongs to $\mathcal{C}^{2,\alpha}_\delta(S^q)$ only if $\delta > 0$.

The generators are clearly related to the Jacobi field F of the surface which is generated by the change of the parameter ϕ (half the angle between adjacent asymptotic half-planes). Such a Jacobi field is in $\mathcal{C}^{2,\alpha}_\delta(S^q)$ only if $\delta > 0$ because has a linear growth rate at the ends.

From Proposition 3.7, we get the corresponding result for $\mathcal{L}_S$.

Proposition 3.10. *Let* $\delta \in (0,1)$. *The operator* $\mathcal{L}_S : \mathcal{C}^{2,\alpha}_\delta(S) \to \mathcal{C}^{0,\alpha}_\delta(S)$ *is surjective, has a kernel of dimension* 4, *and there exists an inverse* $\mathcal{V}$ *of bounded norm.*

It is possible to show that we can recover the surjectivity and bijectivity of the Jacobi operator also for negative values of δ if we consider appropriate augmented functional spaces. Such a result will not be used in the sequel.

Let D denote the 8-dimensional space of functions on S defined as

$$D := \mathrm{span}\{\chi_{i,j}F, \chi_{i,j}, i = 1, 2, j = h, v\},$$

where $\chi_{i,j}$ denotes a cut-off function supported on the end $E_{i,j}$. We will use the same notation when considering the corresponding space of functions on the quotient S^q.

The following proposition is the result corresponding to Proposition 3.7. Note that since $\delta \in (-1,0)$, then the space D is not contained in $\mathcal{C}^{2,\alpha}_\delta(S^q)$.

Proposition 3.11. *Let* $\delta \in (-1,0)$.

(1) *Then the operator* $\mathcal{L}_{S^q} : \mathcal{C}^{2,\alpha}_\delta(S^q) \oplus D \to \mathcal{C}^{0,\alpha}_\delta(S^q)$ *is surjective, has a kernel of dimension* 4, *and there exists an inverse of bounded norm.*
(2) *Furthermore, the operator* $\mathcal{L}_{S^q} : \mathcal{C}^{2,\alpha}_\delta(S^q) \oplus \mathrm{span}\{\chi_{i,j}, i = 1, 2, j = h, v\} \to \mathcal{C}^{0,\alpha}_\delta(S^q)$ *is an isomorphism and the inverse has bounded norm.*

Definition 3.12. Let $v \in D$ be the function $v = a\chi_{i,j}F + b\chi_{i,j}$, $a, b \in \mathbb{R}$. We define the norm $\|\cdot\|_D$ as follows:

$$\|v\|_D := |a| + |b|.$$

We observe that since D is a finite dimensional space then all norms are equivalent.

The following proposition is the result corresponding to Proposition 3.8. The main differences are the fact that now the interval (δ, δ') contains 0, one of the indicial roots, and the use of the space D which contains functions which are in $\mathcal{C}^{2,\alpha}_{\delta'}(S^q)$, $\delta' \in (0,1)$, but do not belong to the smaller space $\mathcal{C}^{2,\alpha}_\delta(S^q)$, $\delta \in (-1,0)$.

Proposition 3.13. *For any given δ', δ, such that $-1 < \delta < 0 < \delta' < 1$, there exist a constant $c > 0$, a function $u \in \mathcal{C}^{2,\alpha}_\delta(S^q)$ and a function $v \in D$ such that if $f \in \mathcal{C}^{0,\alpha}_\delta(S^q)$, $w, f \in L^2_{\delta'}(S^q)$, satisfying $\mathcal{L}_{S^q} w = f$, then $w = u + v \in \mathcal{C}^{2,\alpha}_\delta(S^q) \oplus D$ and*

$$\|u\|_{\mathcal{C}^{2,\alpha}_\delta(S^q)} + \|v\|_D \leq c(\|f\|_{\mathcal{C}^{0,\alpha}_\delta(S^q)} + \|w\|_{L^2_{\delta'}(S^q)}). \tag{17}$$

Furthermore, the kernel of $\mathcal{L}_{S^q} : \mathcal{C}^{2,\alpha}_\delta(S^q) \oplus D \to \mathcal{C}^{0,\alpha}_\delta(S^q)$ has dimension 4.

4. A Family of Minimal Surfaces Close to S^T

In this section, we will show the existence of a family of minimal surfaces close to S^T. We recall we defined S^T as S from which we remove the images by $X_{1,h}, X_{2,h}, X_{1,v}, X_{2,v}$ of $[x_\varepsilon, +\infty) \times \mathbb{R}$ with $x_\varepsilon = -3 \ln \varepsilon$. Consequently, S^T has four boundary curves. We consider the part of the ends which is parameterized by (7) with $(x, y) \in [x_0, x_\varepsilon] \times \mathbb{R}$. Proposition 3.1 says that the function u_m is exponentially decreasing with respect to the x-variable as fast as e^{-x}. For values of x close to x_ε, it holds $u_m = \mathcal{O}(\varepsilon^3)$.

Remark 4.1. The choice of x_ε guarantees that the ends of the surface nearby the boundary curves are the graphs of functions of very small norm over the asymptotic half-planes.

We introduce a vector field on S^T. Let $\tilde{n}$ denote the vector field defined as the smooth interpolation of

- the constant unit normal vector to the asymptotic half-planes for $x \in [\frac{1}{2}x_\varepsilon, x_\varepsilon]$ (we assume $\frac{1}{2}x_\varepsilon$ to be big enough so that $\frac{1}{2}x_\varepsilon > x_0$);
- the unit normal vector n to S^T_0.

Such a vector field will replace the unit normal vector n to S^T in the construction of the family of minimal graphs over S^T. Indeed, given a function $w \in \mathcal{C}^{2,\alpha}_\delta(S^T)$, then Σ_w denotes the graph surface over S^T of w in the direction of the vector field $\tilde{n}$. It is a minimal surface with respect to the euclidean metric g_0 if and only if w is a solution of the following second-order nonlinear elliptic equation

$$2H(\Sigma_w) = \mathbb{L}_{S^T} w + \tilde{L} w + Q(w) = 0,$$

where $\mathbb{L}_{S^T}$ is defined in (14), Q is a nonlinear differential operator which is asymptotic at the Scherk type ends to the last term of (13); $\tilde{L}$ is a linear operator whose presence is consequence of the fact we consider a vector field $\tilde{n}$ which is not normal to S^T. The coefficients of this operator are nonzero where $\tilde{n} \neq n$. In view of (A.4), an estimate of its coefficients can be obtained from an estimate of $g_0(n, \tilde{n})$. Clearly, if $n = \tilde{n}$ then $g_0(n, \tilde{n}) = 1$. Instead at the ends, $n = \frac{(-\partial_x u_m, -\partial_y u_m, 1)}{\sqrt{1 + (\partial_x u_m)^2 + (\partial_y u_m)^2}}$, and, by Proposition 3.1, if $x \in [\frac{1}{2}x_\varepsilon, x_\varepsilon]$, then $|D^k u_m| \leq c\varepsilon^{\frac{3}{2}}$, and $(1 + (\partial_x u_m)^2 + (\partial_y u_m)^2)^{-\frac{1}{2}} = 1 + \mathcal{O}_{\mathcal{C}^\infty}(\varepsilon^3)$. Consequently,

$g_0(n, \tilde{n}) = 1 + \mathcal{O}_{\mathcal{C}^\infty}(\varepsilon^3)$ and it follows from (A.4) that the coefficients of $\tilde{L}$ are uniformly bounded by a constant times ε^3.

The equation to solve is

$$\mathbb{L}_{S^T} w + \tilde{L} w + Q(w) = 0. \tag{18}$$

We shall look for a solution of (18) of the form $w = w_\xi + v$, where w_ξ is a function defined below.

Definition 4.2. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, we define $\mathcal{C}^{\ell,\alpha}(\mathbb{R})$ as the space of functions w on $\mathbb{R}$ which are even, 2π-periodic, and such that the norm

$$\|w\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R})} := \sum_{k=0}^{\ell} \sup_{\mathbb{R}} \left| \frac{d^k w}{dy^k} \right| + \sup_{\mathbb{R}} \left| \frac{d^\ell w}{dy^\ell} \right|_\alpha$$

is finite.

Let ξ be the collection $\{\xi_{i,j}(y), i = 1, 2; j = h, v\}$ of four functions belonging to $\mathcal{C}^{2,\alpha}(\mathbb{R})$. They are even, 2π-periodic with respect to the y-variable, orthogonal to the constant functions and such that

$$\|\xi_{i,j}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \leq \kappa \varepsilon^{1+\eta},$$

κ being a positive constant and $\eta \in (0,1)$. We set

$$\|\xi\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} := \sum_{i,j} \|\xi_{i,j}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})}.$$

We define w_ξ to be the function in $C^{2,\alpha}_\delta(S^T)$ equal to

(1) $\chi \tilde{H}_{x_\varepsilon}(\xi_{i,j})$ on the images of $[x_0, x_\varepsilon] \times \mathbb{R}$ by each $X_{i,j}$, $i = 1, 2, j = h, v$,
(2) 0 on the remaining part of the surface S^T,

Here, χ is a cut-off function equal to 0 for $x \leq x_0 + 1$ and equal to 1 for $x \in [x_0 + 2, x_\varepsilon]$, and $\tilde{H}$ is the extension operator introduced in Proposition A.2.

Let us fix $\delta \in (0,1)$. To solve Eq. (18), we rewrite it as a fixed point problem:

$$v = T(\xi, v)$$

with

$$T(\xi, v) := -\mathcal{V} \circ \mathcal{E}_\varepsilon(\mathbb{L}_{S^T} w_\xi + \tilde{L}(v + w_\xi) + Q(v + w_\xi)), \tag{19}$$

where $\mathcal{V}$ denotes the operator introduced in Proposition 3.10. $\mathcal{E}_\varepsilon$ denotes the linear extension operator

$$\mathcal{E}_\varepsilon : \mathcal{C}^{0,\alpha}_\delta(S^T) \to \mathcal{C}^{0,\alpha}_\delta(S),$$

defined as follows. If $v_{E_{i,j}} := v \circ X_{i,j} \in \mathcal{C}^{0,\alpha}_\delta([x_0, x_\varepsilon] \times \mathbb{R})$ for $i = 1, 2$, $j = h, v$, then

$$(\mathcal{E}_\varepsilon v_{E_{i,j}})(x, z) := 0 \quad \text{for } (x, z) \in [x_\varepsilon + 1, +\infty) \times \mathbb{R},$$

$$(\mathcal{E}_\varepsilon v_{E_{i,j}})(x, z) := (1 + x_\varepsilon - x) v_{E_{i,j}}(x_\varepsilon, z) \quad \text{for } (x, z) \in [x_\varepsilon, x_\varepsilon + 1] \times \mathbb{R}.$$

$\mathcal{E}_\varepsilon$ is the identity operator elsewhere.

From the definition of $\mathcal{E}_\varepsilon$, it follows that

$$\|\mathcal{E}_\varepsilon(v \circ X_{i,j})\|_{\mathcal{C}^{0,\alpha}_\delta([x_0,+\infty)\times\mathbb{R})} \le c\|v \circ X_{i,j}\|_{\mathcal{C}^{0,\alpha}_\delta([x_0,x_\varepsilon]\times\mathbb{R})}.$$

The reason why we use the operator $\mathcal{E}_\varepsilon$ is that the inverse $\mathcal{V}$ acts on functions defined on the whole surface S.

Proposition 4.3. *Given* $\delta \in (0,1)$, $\eta \in (0,1)$, *there exist* $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that*

$$\|T(\xi,0)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa \varepsilon^{1+\mu},$$

with $\mu := 1 + 2\eta + 3\delta$, *and for all* $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|T(\xi,v_1) - T(\xi,v_2)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa \varepsilon^{1+\eta}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta(S)},$$

$$\|T(\xi_1,v) - T(\xi_2,v)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c\varepsilon^{1+\gamma}\|\xi_2 - \xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})},$$

with $\gamma := \min\{2 + 3\delta, \mu\}$, *for all* $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(S)$ *and satisfying* $\|v\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le 2c_\kappa\varepsilon^{1+\mu}$, *for all collections* ξ, ξ_1, ξ_2 *of functions which are even*, 2π-*periodic, orthogonal to the constant functions, and whose norm is bounded by* $\kappa\varepsilon^{1+\eta}$.

Proof. We recall that the Jacobi operator associated to S^T is asymptotic to the Laplacian at the ends. The function w_ξ is identically zero far away from the ends, that is where the explicit expression of $\mathbb{L}_S$ is not known: this is the reason of our particular choice in the definition of w_ξ. From Proposition 3.10, it follows

$$\|T(\xi,0)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c\|\mathcal{E}_\varepsilon(\mathbb{L}_{S^T} w_\xi)\|_{\mathcal{C}^{0,\alpha}_\delta(S)} + c\|\mathcal{E}_\varepsilon(\tilde{L}(w_\xi))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} + c\|\mathcal{E}_\varepsilon(Q(w_\xi))\|_{\mathcal{C}^{0,\alpha}_\delta(S)}.$$

We need to find an estimate of each summand. We recall that $\|\xi_{i,j}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \le \kappa\varepsilon^{1+\eta}$ and from Proposition A.2

$$\|w_\xi\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le ce^{-\delta x_\varepsilon}\sum_{i,j}\|\xi_{i,j}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \le c_\kappa\varepsilon^{1+\eta+3\delta}.$$

From the construction of w_ξ and from the fact that the coefficients of $\mathbb{L}_{S^T} - \Delta_0 = \bar{L}_{u_m}$ can be estimated by $\mathcal{O}_{C^\infty}(e^{-2x})$ at the ends (see (13)), we get

$$\begin{aligned}
&\|\mathcal{E}_\varepsilon(\mathbb{L}_{S^T} w_\xi)\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \\
&\quad\le c\sum_{i=1,2;j=h,v}\|\mathbb{L}_{S^T}(w_\xi)_{E_{i,j}}\|_{\mathcal{C}^{0,\alpha}_\delta([x_0+1,x_\varepsilon]\times\mathbb{R})} \\
&\quad\le \sum_{i=1,2;j=h,v} c\|e^{-2x}(w_\xi)_{E_{i,j}}\|_{\mathcal{C}^{0,\alpha}_\delta([x_0+1,x_\varepsilon]\times\mathbb{R})} \\
&\quad\le c_\kappa\varepsilon^3\varepsilon^{1+\eta+3\delta} = c_\kappa\varepsilon^{4+\eta+3\delta}.
\end{aligned}$$

In order to get this estimate (and in particular the factor ε^3) we used the fact that at each end $e^{-2x}w_\xi$ is the sum of terms equal to $e^{-2x}e^{k(x-x_\varepsilon)}g_k(y)$, $k \ge 1$, for

$x \in [x_0+1, x_\varepsilon]$ (see the proof of Proposition A.2). If x is close to x_ε then e^{-2x} is very small, so the estimate of $e^{-2x}w_\xi$ depends on the behavior of this function for x close to x_0. We write $|e^{(k-2)x-kx_\varepsilon}g_k(y)| = |e^{(k-2)x-(k-2)x_\varepsilon}e^{-2x_\varepsilon}g_k(y)| \le c\varepsilon^6|g_k(y)|$ if $k \ge 2$, because $e^{(k-2)x-(k-2)x_\varepsilon} \le 1$, for any $x \le x_\varepsilon$. Instead, if $k=1$ the function to consider is $e^{-x-x_\varepsilon}g_1(y)$. If x is close to x_0 then $|e^{-x_0-x_\varepsilon}g_1(y)| \le ce^{-x_\varepsilon}|g_1(y)| = c\varepsilon^3|g_1(y)|$.

Furthermore,

$$\|\mathcal{E}_\varepsilon(\tilde{L}w_\xi)\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \le c\varepsilon^3\|w_\xi\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa\varepsilon^{4+\eta+3\delta},$$

because the coefficients of $\tilde{L}$ are uniformly bounded by $c\varepsilon^3$,

$$\|\mathcal{E}_\varepsilon(Q(w_\xi))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \le c\|(w_\xi)^2\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa\varepsilon^{2+2\eta}\varepsilon^{3\delta} = c_\kappa\varepsilon^{2+2\eta+3\delta},$$

because $Q(w_\xi)$ consists in products of derivatives of w_ξ.

Then we conclude

$$\|T(\xi,0)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa\varepsilon^{1+\mu},$$

with $\mu := 1+2\eta+3\delta$. As for the second estimate, we observe

$$\begin{aligned}\|T(\xi,v_2)-T(\xi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} &\le c\|\mathcal{E}_\varepsilon(\tilde{L}(v_2-v_1))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \\ &\quad + c\|\mathcal{E}_\varepsilon(Q(w_\xi+v_2)-Q(w_\xi+v_1))\|_{\mathcal{C}^{0,\alpha}_\delta(S)}.\end{aligned}$$

Thanks to the considerations made above it holds

$$\begin{aligned}\|\mathcal{E}_\varepsilon(\tilde{L}(v_2-v_1))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} &\le c\varepsilon^2\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\delta(S)}, \\ \|\mathcal{E}_\varepsilon(Q(w_\xi+v_2)-Q(w_\xi+v_1))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} &\le c\|(v_2-v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(S)}\|w_\xi\|_{\mathcal{C}^{2,\alpha}_*(S)} \\ &\le c_\kappa\varepsilon^{1+\eta}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\delta(S)}.\end{aligned}$$

Here we used the estimate $\|w_\xi\|_{\mathcal{C}^{2,\alpha}_*(S)} \le c_\kappa\varepsilon^{1+\eta}$, where $\|\cdot\|_{\mathcal{C}^{2,\alpha}_*(S)}$ denotes the norm introduced in Definition 3.3 with $\delta=0$. Then

$$\|T(\xi,v_2)-T(\xi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c_\kappa\varepsilon^{1+\eta}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\delta(S)}.$$

To show the third estimate, we proceed in a similar way:

$$\begin{aligned}&\|T(\xi_2,v)-T(\xi_1,v)\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \\ &\quad\le \|\mathcal{E}_\varepsilon(\mathbb{L}_{S^T}(w_{\xi_2}-w_{\xi_1}))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} + \|\mathcal{E}_\varepsilon(\tilde{L}(w_{\xi_2}-w_{\xi_1}))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \\ &\qquad + \|\mathcal{E}_\varepsilon(Q(w_{\xi_2}+v)-Q(w_{\xi_1}+v))\|_{\mathcal{C}^{0,\alpha}_\delta(S)} \\ &\quad\le c\|e^{-2x}(w_{\xi_2}-w_{\xi_1})\|_{\mathcal{C}^{2,\alpha}_\delta(S)} + c\varepsilon^3\|(w_{\xi_2}-w_{\xi_1})\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \\ &\qquad + c\|(w_{\xi_2}-w_{\xi_1})\|_{\mathcal{C}^{2,\alpha}_\delta(S)}\|v\|_{\mathcal{C}^{2,\alpha}_*(S)} \\ &\quad\le c\varepsilon^3\varepsilon^{3\delta}\|\xi_2-\xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} + c\varepsilon^{1+\mu}\|\xi_2-\xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \\ &\quad\le c\varepsilon^{1+\gamma}\|\xi_2-\xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})},\end{aligned}$$

where $\gamma := \min\{2+3\delta, \mu\}$. We used the estimate $\|v\|_{\mathcal{C}^{2,\alpha}_*(S)} \le ce^{\delta x_\varepsilon}\|v\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \le c\varepsilon^{-3\delta}\varepsilon^{1+\mu} = c\varepsilon^{1+\mu-3\delta}$. Consequently,

$$\begin{aligned}\|(w_{\xi_2} - w_{\xi_1})\|_{\mathcal{C}^{2,\alpha}_\delta(S)}\|v\|_{\mathcal{C}^{2,\alpha}_*(S)} &\le c\varepsilon^{1+\mu-3\delta}\|w_{\xi_2} - w_{\xi_1}\|_{\mathcal{C}^{2,\alpha}_\delta(S)} \\ &\le c\varepsilon^{1+\mu-3\delta}\varepsilon^{3\delta}\|\xi_2 - \xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \\ &\le c\varepsilon^{1+\mu}\|\xi_2 - \xi_1\|_{\mathcal{C}^{2,\alpha}(\mathbb{R})}.\end{aligned}$$

□

Theorem 4.4. *Let* $\mathcal{B}_\kappa := \{v \in \mathcal{C}^{2,\alpha}_\delta(S) \,|\, \|v\|_{\mathcal{C}^{2,\alpha}_\delta} \le 2c_\kappa\varepsilon^{1+\mu}\}$, *with* κ *chosen large enough. Then the nonlinear mapping* $T(\xi,\cdot)$ *defined by* (19) *has a unique fixed point in* $\mathcal{B}_\kappa$.

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping $T(\xi,\cdot)$ is a contraction mapping from the ball $\mathcal{B}_\kappa$ of radius $2c_\kappa\varepsilon^{1+\mu}$ in $\mathcal{C}^{2,\alpha}_\delta(S)$ into itself. This value follows from the estimate of the norm of $T(\xi, 0)$. Consequently, by Leray–Schäuder fixed point theorem, $T(\xi,\cdot)$ has a unique fixed point in this ball. □

We have proved that, for any collection $\xi = \{\xi_{i,j}, i = 1, 2; j = h, v\} \in (\mathcal{C}^{2,\alpha}(\mathbb{R}))^4$ of norm sufficiently small, there exists a minimal surface, $S^T(\xi)$, which is a small deformation of S^T. More precisely, $S^T(\xi)$ is the graph surface of a function over S^T. Since each end is, up to an appropriate isometry, the graph surface of u_m, then each end $\bar{E}_{i,j}$ of $S^T(\xi)$ is the image of the immersion $(x, y) \to (x, y, U_{i,j}(x, y))$, where $U_{i,j}(x, y)$ is the restriction to the end of $u_m + w_\xi + v$, v being the fixed point whose existence is provided by Theorem 4.4. Let $\|\cdot\|_{\mathcal{C}^{2,\alpha}_*}$ denote the norm introduced in Definition 3.3 with $\delta = 0$. The norm of v can be estimated as follows $\|v\|_{\mathcal{C}^{2,\alpha}_*} \le c\varepsilon^{1+\mu-3\delta} = c\varepsilon^{2+2\eta}$. This estimate follows from

$$\|v\|_{\mathcal{C}^{2,\alpha}_*} \le ce^{\delta x_\varepsilon}\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \le c\varepsilon^{-3\delta}\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \le c\varepsilon^{1+\mu-3\delta},$$

the fact that $e^{\delta x_\varepsilon} = \varepsilon^{-3\delta}$, and the identity $\mu = 1 + 2\eta + 3\delta$. In the sequel, we will suppose $2\eta < 1$.

Since at the end $\bar{E}_{ij}$ it holds $w_\xi = \tilde{H}_{x_\varepsilon}(\xi_{i,j})$, then we can write

$$U_{i,j}(x, y) = \tilde{H}_{x_\varepsilon}(\xi_{i,j}) + V_{i,j},$$

with $(x, y) \in [x_0, x_\varepsilon] \times \mathbb{R}$. The term $V_{i,j}$ is the sum of the restriction of the fixed point to the end and u_m. In a sufficiently small neighborhood of the boundary curve of each end, that is for x say in $[x_\varepsilon - 1, x_\varepsilon + 1]$, the function u_m satisfies $u_m = \mathcal{O}(\varepsilon^3)$. That follows from Proposition 3.1. Consequently for those values of x, the dominant term in the expression for $V_{i,j}$ is v. Then the norm of $V_{i,j}$ can be estimated as follows: $\|V_{i,j}\|_{\mathcal{C}^{2,\alpha}_*([x_\varepsilon-1,x_\varepsilon]\times\mathbb{R})} \le c\varepsilon^{2+2\eta}$. Clearly $2 + 2\eta > 1 + \eta$, so the term of largest norm in the expression for $U_{i,j}$ is $\tilde{H}_{x_\varepsilon}(\xi_{i,j})$. Indeed, from the definition of the operator $\tilde{H}$, it follows that $\|\tilde{H}_{x_\varepsilon}(\xi_{i,j})\|_{\mathcal{C}^{2,\alpha}_*} \le c\|\xi_{i,j}\|_{\mathcal{C}^{2,\alpha}} \le c_\kappa\varepsilon^{1+\eta}$.

The function $V_{i,j}$ depends nonlinearly on ε, ξ. Furthermore from the third estimate of Proposition 4.3, we get

$$\|V_{i,j}(\varepsilon,\xi)(\cdot,\cdot) - V_{i,j}(\varepsilon,\xi')(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}_*} \le c\varepsilon^{1+\gamma-3\delta}\|\xi-\xi'\|_{\mathcal{C}^{2,\alpha}}. \tag{20}$$

We observe $1+\gamma-3\delta>0$ in (20) because $\gamma = \min\{2+3\delta, 1+2\eta+3\delta\}$.

5. The Gluing Procedure

In this section, we will prove Theorems 1.1–1.3.

5.1. *Proof of Theorem* 1.1

Let $m, k \in \mathbb{N}$ be two numbers such that $2 \le mk < +\infty$.

We will glue together mk minimal graphs over S^T and $2k+2m$ minimal graphs over Ξ along their boundary curves. They are arranged so that, in the case $m=3$, $k=5$, the cross section of the asymptotic planes is the one shown in Fig. 1. The surface has $2k+2m$ ends asymptotic to just as many half-planes.

We recall below the necessary results we proved in previous sections.

In Sec. 2.1, we showed the existence of minimal graphs $\Xi(\varphi)$ over Ξ, for each $\varphi \in C^{2,\alpha}(\mathbb{R})$ which is even, 2π-periodic, L^2-orthogonal to the constant functions and $\|\varphi\|_{C^{2,\alpha}} \le \kappa\varepsilon^{1+\eta}$. $\Xi(\varphi)$ is the graph surface of the function

$$\bar{U}(x,y) = H_{d_\varepsilon}(\varphi)(x,y) + \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{2+2\eta})$$

over Ξ. The function $\mathcal{O}_{C^{2,\alpha}}(\varepsilon^{2+2\eta})$ replaces the function $\bar{V}$ that appears at the end of Sec. 2.1. We recall that $d_\varepsilon = -2\ln\varepsilon$. It depends nonlinearly on the function φ and is bounded by a constant times $\varepsilon^{2+2\eta}$. The boundary curve corresponds to $x = d_\varepsilon$. These surfaces will be glued to a perturbation of S^T.

For each

$$\xi = (\xi_{1,h}, \xi_{2,h}, \xi_{1,v}, \xi_{2,v}) \in (C^{2,\alpha}(\mathbb{R}))^4,$$

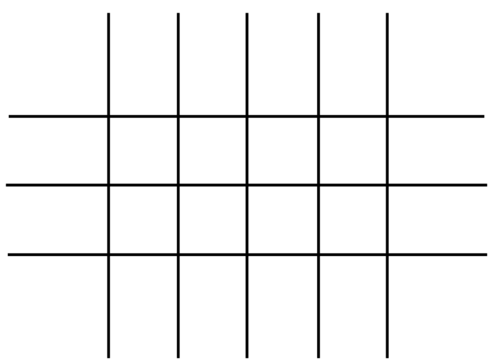

Fig. 1. Position of the asymptotic planes in the case $m=3$, $k=5$.

the result of Sec. 4 ensures the existence of a minimal surface $S^T_\varepsilon(\xi)$ which is close to S^T. The functions $\xi_{i,j}$, $i = 1, 2$, $j = h, v$, are even, 2π-periodic, L^2-orthogonal to the constant functions and $\|\xi_{i,j}\|_{C^{2,\alpha}} \leq \kappa\varepsilon^{1+\eta}$.

Each surface $S^T(\xi)$ nearby the boundary of its four ends is the graph surface of the function

$$U_{i,j}(x, y) = \tilde{H}_{x_\varepsilon}(\xi_{i,j})(x, y) + \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{2+2\eta}),$$

$i = 1, 2$, $j = h, v$. The functions $\mathcal{O}_{C^{2,\alpha}}(\varepsilon^{2+2\eta})$ replace the functions $V_{i,j}$ that appear at the end of Sec. 4. They depend nonlinearly on the functions $\xi_{i,j}$ and are bounded by a constant times $\varepsilon^{2+2\eta}$. The boundary curves correspond to $x = x_\varepsilon$.

In the sequel $S^T(\xi^{a,b})$, with $a \in \{1, \ldots, m\}$, $b \in \{1, \ldots, k\}$, denote the deformations of S^T involved in the gluing procedure. $\Xi(\varphi_n)$, with $n \in \{1, \ldots, 2k+2m\}$, are the minimal graphs over Ξ which play the role of ends. We assume that, for any $a \in \{1, \ldots, m\}$, $b \in \{1, \ldots, k\}$, and $n \in \{1, \ldots, 2k + 2m\}$ it holds

$$\|(\xi^{a,b})_{i,j}\|_{C^{2,\alpha}} \leq \kappa\varepsilon^{1+\eta}, \quad \|\varphi_n\|_{C^{2,\alpha}} \leq \kappa\varepsilon^{1+\eta}, \tag{21}$$

for a constant $\kappa > 0$ fixed large enough. It remains to show that, for all ε small enough, it is possible to choose these functions in such a way that the surface

$$\left(\bigcup_{a=1}^{m}\bigcup_{b=1}^{k} S^T(\xi^{a,b})\right) \bigcup \left(\bigcup_{n=1}^{2m+2k} \Xi(\varphi_n)\right) \tag{22}$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is actually smooth and by construction it has the desired properties. It is necessary to solve a system of equations. Such a system consists of several pairs of equations. Precisely one pair of equations for each pair of adjacent summands.

If we want to glue $S^T(\xi^{a,b})$ to $S^T(\xi^{c,d})$, $(c, d) = (a + 1, b)$ or $(c, d) = (a, b + 1)$, then we have to consider the parametrization of corresponding ends. We observe that there are two possible cases: to glue the horizontal end $E^{a,b}_{i,h}$ of $S^T(\xi^{a,b})$ to the horizontal end $E^{c,d}_{j,h}$ of $S^T(\xi^{c,d})$ with $i \neq j$ or to glue the vertical end $E^{a,b}_{i,v}$ of $S^T(\xi_{a,b})$ to the vertical end $E^{c,d}_{j,v}$ of $S^T(\xi_{c,d})$ with $i \neq j$. We remark that by construction if $i \neq j$ then the normal vectors to the ends $E^{a,b}_{i,h}$ and $E^{c,d}_{j,h}$ point in opposite directions. The same consideration holds true for the ends $E^{a,b}_{i,v}$ and $E^{c,d}_{j,v}$, if $i \neq j$. We suppose that $U^{a,b}_{l,m}$ denotes the function whose graph surface is the end $E^{a,b}_{l,m}$ of $S^T(\xi^{a,b})$. Then the equations to solve are

$$\begin{cases} U^{a,b}_{i,m}(x_\varepsilon, \cdot) = -U^{c,d}_{j,m}(x_\varepsilon, \cdot), \\ \partial_x U^{a,b}_{i,m}(x_\varepsilon, \cdot) = \partial_x U^{c,d}_{j,m}(x_\varepsilon, \cdot). \end{cases} \tag{23}$$

Here, $m = h$ or $m = v$ and $(a, b) \neq (c, d)$.

Instead if we want to glue an end of $S^T(\xi^{ab})$ to one of the minimal graphs $\Xi(\varphi_n)$ then the equations to solve are

$$\begin{cases} U_{i,j}^{a,b}(x_\varepsilon, \cdot) = \bar{U}^n(d_\varepsilon, \cdot), \\ \partial_x U_{i,j}^{a,b}(x_\varepsilon, \cdot) = \partial_x \bar{U}^n(d_\varepsilon, \cdot). \end{cases} \tag{24}$$

Here, $j = h$ or $j = v$ and $i = 1$ or $i = 2$. We assumed that $\bar{U}^n$ is the function whose graph surface is $\Xi(\varphi_n)$.

In view of the structure of $U_{l,m}^{a,b}$ and $\bar{U}^n$, we deduce that (23) is equivalent to

$$\begin{cases} \xi_{i,m}^{a,b} + \xi_{j,m}^{c,d} = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \\ \partial^* \xi_{i,m}^{a,b} - \partial^* \xi_{j,m}^{c,d} = \mathcal{O}_{C^{1,\alpha}}(\varepsilon^{1+\eta}) \end{cases} \tag{25}$$

and (24) is equivalent to

$$\begin{cases} \xi_{i,j}^{a,b} - \varphi_n = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \\ \partial^* \xi_{i,j}^{a,b} + \partial^* \varphi_n = \mathcal{O}_{C^{1,\alpha}}(\varepsilon^{1+\eta}). \end{cases} \tag{26}$$

Here, ∂^* denotes the operator which associates to $\phi = \sum_{i=1}^{\infty} \phi_i \cos(iy)$ the function $\partial^* \phi = \sum_{i=1}^{\infty} i\phi_i \cos(iy)$. In particular, we used the fact that $H_{d_\varepsilon}(\varphi_n)(d_\varepsilon, \cdot) = \varphi_n$, $\tilde{H}_{x_\varepsilon}(\xi_{i,j}^{a,b})(x_\varepsilon, \cdot) = \xi_{i,j}^{a,b}$ and we applied Lemmas A.3 and A.4.

In conclusion, we get a pair of equations of the form:

$$\begin{cases} \omega \pm \lambda = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \\ \partial^* \omega \mp \partial^* \lambda = \mathcal{O}_{C^{1,\alpha}}(\varepsilon^{1+\eta}). \end{cases} \tag{27}$$

Lemma 5.1. *The operator h defined by*

$$C^{2,\alpha}(\mathbb{R}) \to C^{1,\alpha}(\mathbb{R}),$$

$$\varphi \to \partial^* \varphi$$

acting on functions that are even, 2π-periodic, L^2-orthogonal to the constant functions, is invertible.

Proof. We observe that if we decompose $\varphi = \sum_{j=1}^{\infty} \varphi_j \cos(jy)$, then

$$h(\varphi) = \sum_{j=1}^{\infty} j\varphi_j \cos(jy),$$

is clearly invertible from $H^1_{2\pi\text{-periodic}}(\mathbb{R})$ into $L^2_{2\pi\text{-periodic}}(\mathbb{R})$. Now, elliptic regularity theory implies that this is also the case when this operator is defined between Hölder spaces. □

Using this result, the last equation of (27) is equivalent to $\omega \mp \lambda = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta})$. Consequently, the pairs of equations of the form (27) can be rewritten as

$$(\omega, \lambda) = (\mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta})). \tag{28}$$

We recall that the right-hand side depends nonlinearly on the functions ω, λ. If we consider all of such pairs of equations, we get the following system:

$$\begin{cases} (\xi^{a,b})_{i,j} = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \quad a = 1, \ldots, m;\ \ b = 1, \ldots, k;\ \ i = 1, 2;\ \ j = h, v, \\ \varphi_n = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \quad n = 1, \ldots, 2k + 2m. \end{cases} \tag{29}$$

We look at this system as a fixed point problem and fix the constant κ in (21) large enough. Thanks to estimates (6) and (20), we can use a fixed point theorem for contracting mappings in the ball of radius $\kappa\varepsilon^{1+\eta}$ in $(C^{2,\alpha}(\mathbb{R}))^{4km+2k+2m}$ to obtain, for all ε small enough, a solution of (29). This provides a set of functions $((\xi^{a,b})_{i,j}, \varphi_n), a = 1, \ldots, m; b = 1, \ldots, k; i = 1, 2; j = h, v; n = 1, \ldots, 2k + 2m$, such that the surface (22) is C^1. So the proof of Theorem 1.1 is complete.

5.2. *Proof of Theorem* 1.2

In this case, the surface we wish to construct is doubly periodic, that is invariant under the action of two orthogonal translations. We suppose that $S^T(\xi^{a,b})$, with $a \in \{1, \ldots, m\}$ and $b \in \mathbb{Z}$, denote the deformations of S^T and $\Xi(\varphi^{k,b})$, with $k = 1, 2$ and $b \in \mathbb{Z}$, denote the minimal graphs over Ξ which play the role of ends of the surface. The surface we construct is

$$\bigcup_{b\in\mathbb{Z}} \{\Xi(\varphi^{1,b}) \cup S^T(\xi^{1,b}) \cup \cdots \cup S^T(\xi^{m,b}) \cup \Xi(\varphi^{2,b})\}.$$

In order to get the additional periodicity we impose $S^T(\xi^{a,b}) = S^T(\xi^{a,b+1})$ and $\Xi(\varphi^{i,b}) = \Xi(\varphi^{i,b+1})$ for any $b \in \mathbb{Z}$. That is equivalent to impose $(\xi^{a,b})_{i,j} = (\xi^{a,b+1})_{i,j}$ and $\varphi^{i,b} = \varphi^{i,b+1}$ for any $b \in \mathbb{Z}$. Consequently, the number of independent equations and the number of independent unknowns are finite. By following the same technique used in the proof of Theorem 1.1, we get the following fixed point problem:

$$\begin{cases} (\xi^{a,0})_{i,j} = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \quad a = 1, \ldots, m;\ \ i = 1, 2;\ \ j = h, v, \\ \varphi^{k,0} = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \quad k = 1, 2. \end{cases} \tag{30}$$

Such a problem can be solved by using a fixed point theorem for contracting mappings in the ball of radius $\kappa\varepsilon^{1+\eta}$ in $(C^{2,\alpha}(\mathbb{R}))^{4m+2}$ as done for (29).

5.3. *Proof of Theorem* 1.3

In this case, the surface we wish to construct is triply periodic, that is invariant under the action of three orthogonal translations. Only one type of building blocks is considered. Indeed, the surface is given by $\bigcup_{a\in\mathbb{Z}} \bigcup_{b\in\mathbb{Z}} S^T(\xi^{a,b})$ and it has no ends. In order to get a triply periodic surface, we impose $S^T(\xi^{a,b}) = S^T(\xi^{a,b+1}) = S^T(\xi^{a+1,b})$ for any $a, b \in \mathbb{Z}$. That is equivalent to impose $(\xi^{a,b})_{i,j} = (\xi^{a,b+1})_{i,j} = (\xi^{a+1,b})_{i,j}$ for any $a, b \in \mathbb{Z}$. Consequently also in this case, the number of independent equations and the number of independent unknowns are finite. Indeed, we get that the fixed point problem to solve reduces to

$$\{(\xi^{0,0})_{i,j} = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^{1+\eta}), \quad i = 1, 2;\ \ j = h, v. \tag{31}$$

Such a problem can be solved by using a fixed point theorem for contracting mappings in the ball of radius $\kappa\varepsilon^{1+\eta}$ in $(C^{2,\alpha}(\mathbb{R}))^4$ as done for (29) and (30).

Appendix

A.1. *Harmonic extension operators*

Proposition A.1. *Let $\rho \in (0, 1)$. There exists an operator*

$$H_a : C^{2,\alpha}(\mathbb{R}) \to C^{2,\alpha}_\rho([a, +\infty) \times \mathbb{R}),$$

such that for each even 2π-periodic function $\varphi \in C^{2,\alpha}(\mathbb{R})$, which is L^2-orthogonal to the constant functions, then $w_\varphi = H_a(\varphi)$ solves

$$\begin{cases} \partial^2_{xx} w_\varphi + \partial^2_{yy} w_\varphi = 0 & \text{on } [a, +\infty) \times \mathbb{R} \\ w_\varphi = \varphi & \text{on } \{a\} \times \mathbb{R}. \end{cases}$$

Moreover then

$$\|H_a(\varphi)\|_{C^{2,\alpha}_\rho([a,+\infty)\times\mathbb{R})} \le c e^{-\rho a} \|\varphi\|_{C^{2,\alpha}(\mathbb{R})}, \tag{A.1}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(iy)\}_{i\in\mathbb{N}}$, that is

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(iy).$$

Then the solution w_φ is given by

$$w_\varphi(x, y) = \sum_{i=1}^{\infty} e^{-i(x-a)} \varphi_i \cos(iy).$$

Since

$$e^{-\rho x}|w_\varphi(x, y)| \le e^{-\rho x} e^{-(x-a)}|\varphi| \le e^{-\rho x} e^{\rho(x-a)}|\varphi|,$$

then we get $\|w_\varphi\|_{C^{2,\alpha}_\delta([a,+\infty)\times\mathbb{R})} \le c e^{-\rho a} \|\varphi\|_{C^{2,\alpha}(\mathbb{R})}$. □

Proposition A.2. *Let $\delta \in (0, 1)$. There exists an operator*

$$\tilde{H}_a : C^{2,\alpha}(\mathbb{R}) \to C^{2,\alpha}_\delta([x_0, a] \times \mathbb{R}),$$

such that for each even 2π-periodic function $\varphi \in C^{2,\alpha}(\mathbb{R})$, which is L^2-orthogonal to the constant functions then $w_\varphi = \tilde{H}_a(\varphi)$ solves

$$\begin{cases} \partial^2_{xx} w_\varphi + \partial^2_{yy} w_\varphi = 0 & \text{on } [x_0, a] \times \mathbb{R}, \\ w_\varphi = \varphi & \text{on } \{a\} \times \mathbb{R}. \end{cases}$$

Moreover,

$$\|\tilde{H}_a(\varphi)\|_{C^{2,\alpha}_\delta([x_0,a]\times\mathbb{R})} \le ce^{-\delta a}\|\varphi\|_{C^{2,\alpha}(\mathbb{R})}, \tag{A.2}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(iy)\}$, that is

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(iy).$$

Then the solution w_φ is given by

$$w_\varphi(x,y) = \sum_{i=1}^{\infty} e^{i(x-a)}\varphi_i \cos(iy).$$

It is easy to show that $\|w_\varphi\|_{C^{2,\alpha}_\delta([x_0,a]\times\mathbb{R})} \le ce^{-\delta a}\|\varphi\|_{C^{2,\alpha}(\mathbb{R})}$. □

Lemma A.3. *Let $w_\varphi(x,y)$ be the harmonic extension given by Proposition* A.1 *of the even 2π-periodic function $\varphi \in \mathcal{C}^{2,\alpha}(\mathbb{R})$. Then*

$$\partial^*\varphi(y) = -\partial_x w_\varphi(x,y)_{|x=a}.$$

Proof.

$$\partial_x w_\varphi(x,y) = -\sum_{i=1}^{\infty} ie^{-i(x-a)}\varphi_i \cos(iy).$$

Consequently,

$$\partial^*\varphi(y) = -\partial_x w_\varphi(x,y)_{|x=a}.$$ □

Lemma A.4. *Let $w_\varphi(x,y)$ be the harmonic extension given by Proposition* A.2 *of the even 2π-periodic function $\varphi \in \mathcal{C}^{2,\alpha}(\mathbb{R})$. Then*

$$\partial^*\varphi(y) = \partial_x w_\varphi(x,y)_{|x=a}.$$

Proof.

$$\partial_x w_\varphi(x,y) = \sum_{i=1}^{\infty} ie^{i(x-a)}\varphi_i \cos(iy).$$

Consequently,

$$\partial^*\varphi(y) = \partial_x w_\varphi(x,y)_{|x=a}.$$ □

A.2. *Jacobi operators of a surface with respect to different transverse vector fields*

Let Σ denote a smooth oriented surface embedded in a Riemannian manifold (M,g). Let $N, \tilde{N}$ be respectively the unit normal vector field compatible with the orientation of Σ and a vector field transverse to Σ.

Given a point $p_0 \in \Sigma$, by the implicit function theorem, there exist two neighbourhoods U, V of $(p_0, 0) \in \Sigma \times \mathbb{R}$ and a diffeomorphism

$$U \ni (p, s) \to (f(p, s), h(p, s)) \in V,$$

such that $f(p, 0) = p, h(p, 0) = 0$, and

$$\exp_p(s\tilde{N}(p)) = \exp_{f(p,s)}(h(p, s)N(f(p, s))). \tag{A.3}$$

By differentiation of this identity with respect to s at $s = 0$, we get

$$\tilde{N}(p) = \partial_s f(p, 0) + \partial_s h(p, 0)N(p).$$

From that we get

$$g(\tilde{N}(p), N(p)) = \partial_s h(p, 0).$$

Consequently, $h(p, s) = g(\tilde{N}(p), N(p))s + \mathcal{O}(s^2)$. Using the formula for $\tilde{N}(p)$, we get that its tangential component is equal to

$$[\tilde{N}(p)]^T = \partial_s f(p, 0).$$

A surface $\tilde{\Sigma}$ which is close enough to Σ can be parameterized at the same time as the graph of a function w over Σ in the direction of $\tilde{N}$ and the graph of a different function $\tilde{w}$ over Σ in the direction of N. From (A.3), we get

$$\tilde{w}(f(p, w(p))) = h(p, w(p)).$$

Furthermore, we observe that the mean curvatures of Σ at the point $\exp_p(w(p)\tilde{N}(p))$ and at the point $\exp_{\bar{p}}(\tilde{w}(\bar{p})N(\bar{p}))$ are equal provided $\bar{p} = f(p, w(p))$. In other terms,

$$H_{\tilde{N},w}(p) = H_{N,\tilde{w}}(\bar{p}).$$

Differentiating this identity with respect to w at $w = 0$, we get

$$D_w H_{\tilde{N},0}(v) = D_{\tilde{w}} H_{N,0}(\partial_s h v) + D_{\bar{p}} H_{N,0}(\partial_s f v).$$

If we insert the formulas for $\partial_s h$ and $\partial_s f$, we get the following result.

Proposition A.5.

$$D_w H_{\tilde{N},0}(v) = D_{\tilde{w}} H_{N,0}(g(N, \tilde{N})v) + (\nabla_{[\tilde{N}(p)]^T} H_{N,0})v.$$

If Σ has constant mean curvature, we get

$$D_w H_{\tilde{N},0}(v) = D_{\tilde{w}} H_{N,0}(g(N, \tilde{N})v). \tag{A.4}$$

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