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# Optimal Interpolation Formulas with Derivative in the Space $L_2^{(m)}(0,1)$

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**Abstract.** The paper studies the problem of construction of optimal interpolation formulas with derivative in the Sobolev space $L_2^{(m)}(0,1)$. Here the interpolation formula consists of the linear combination of values of the function at nodes and values of the first derivative of that function at the end points of the interval $[0,1]$. For any function of the space $L_2^{(m)}(0,1)$ the error of the interpolation formulas is estimated by the norm of the error functional in the conjugate space $L_2^{(m)*}(0,1)$. For this, the norm of the error functional is calculated. Further, in order to find the minimum of the norm of the error functional, the Lagrange method is applied and the system of linear equations for coefficients of optimal interpolation formulas is obtained. It is shown that the order of convergence of the obtained optimal interpolation formulas in the space $L_2^{(m)}(0,1)$ is $O(h^m)$. In order to solve the obtained system it is suggested to use the Sobolev method which is based on the discrete analog of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$. Using this method in the cases $m=2$ and $m=3$ the optimal interpolation formulas are constructed. It is proved that the order of convergence of the optimal interpolation formula in the case $m=2$ for functions of the space $C^4(0,1)$ is $O(h^4)$ while for functions of the space $L_2^{(2)}(0,1)$ is $O(h^2)$. Finally, some numerical results are presented.

## 1. Introduction and statement of the problem

It is well-known that most problems of science and technology are investigated based on mathematical models, which are differential, integro-differential, integral or functional equations. Solutions of such equations are sought in Banach or Hilbert spaces and they are mainly solved approximately using various type of approximation and interpolation methods of computational mathematics. In order to obtain effective approximate methods it is needed to study structures of the considered spaces. The internal properties of a space can be studied, for instance, by reproducing kernel functions in the theory of reproducing kernels in Hilbert spaces [7, 9] and by extremal functions in the theory of optimal formulas in Banach spaces [32, 36]. Presently, the reproducing kernel Hilbert space methods are widely used in numerical analysis, in particular, in numerical solution of fractional differential and integro-differential equations, see, for instance, [1–4] and references therein. The reproducing Hilbert spaces in probability and statistics are studied in [9]. In the recent work [22], authors studied the reproducing kernel functions in the standard Sobolev space $W_2^s$ and their application to tractability of integration. Using the extremal functions in various Hilbert spaces the

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optimal integration formulas were studied in the works [15, 17, 27, 29–31, 34, 35] and the interpolation formulas and splines were obtained in [13, 14, 16, 28, 33].

The present work is also devoted to construction of the optimal interpolation formula for functions from Sobolev space $L_2^{(m)}(0,1)$ using the extremal function, where $L_2^{(m)}(0,1)$ is the space of functions which are square integrable with $m$-th generalized derivative and equipped with the norm

$$\left\|\varphi|L_2^{(m)}(0,1)\right\| = \left\{\int_0^1 \left(\varphi^{(m)}(x)\right)^2 \mathrm{d}x\right\}^{1/2}$$

and $\int_0^1 \left(\varphi^{(m)}(x)\right)^2 \mathrm{d}x < \infty$. It is known that the interpolation problem is one of the typical problem of approximation theory. Classical method of its solution is construction of interpolation polynomials. However, interpolation polynomials have disadvantages as an approximation apparatus for functions with singularity and for functions with small smoothness. It is proved that the sequence of Lagrange interpolation polynomials constructed for specific continuous function does not converge to that function. Therefore in practice instead of high degree interpolation polynomials spline functions are used. The first spline functions were bonded from pieces of cubic polynomials. Further, this construction was modified, degree of polynomial was increased, boundary values are changed, but the idea of construction remains changeless. The next step in the spline theory is J.C.Holladay's [18] result connecting I.J.Schoenberg's natural cubic spline with the solution of the problem on minimum of the function norm from the space $L_2^{(2)}$. Further, C. de Boor [11] generalized J.C.Holladay's result. These results have aroused great interest and then appeared a large number of works where depending on the specific requirements the variational functional was modified. The theory of splines, based on variational methods, were studied and developed in works of J.Alberg, E.Nilson and J.Walsh [5], R.Arcangeli, M.C.Lopez de Silanes and J.J.Torres [6], M.Attea [8], A.Yu.Bezhaev and V.A.Vasilenko [10, 39], C. de Boor [11, 12], M.I.Ignatev and A.B.Pevniy [19], P.-J.Laurent [20], G.Mastroianni and G.V.Milovanović [21], L.L.Schumaker [25], S.L.Sobolev [33], S.B.Stechkin and Y.N.Subbotin [37] and others. A fairly complete bibliography in the theory of spline functions can be found, for example, in [25].

In the present work we study the problem of construction of an optimal interpolation formula based on variational method. Assume we are given the table of the values $\varphi(h\beta)$, $\beta = 0,1,...,N$, $h = 1/N$, $N$ is a natural number, of a function $\varphi$ and the values of the first derivative of that function at the end points of the interval $[0,1]$, i.e., $\varphi'(0)$ and $\varphi'(1)$. We consider the following approximation of a function $\varphi$ from the space $L_2^{(m)}(0,1)$ with $m \geq 2$ by another more simple function $P_\varphi$ as follows

$$\varphi(z) \cong P_\varphi(z) = \sum_{\beta=0}^{N} C_\beta(z)\varphi(h\beta) + A(z)\varphi'(0) + B(z)\varphi'(1), \tag{1}$$

where $C_\beta(z)$, $\beta = 0,1,...,N$, $A(z)$ and $B(z)$ are *the coefficients* of the approximation formula (1).

Further, in Section 3 we get that the optimal approximation formula of the form (1) is the optimal interpolation formula (see Remark 3.1).

The difference $\varphi - P_\varphi$ is called *the error* of the approximation formula (1). The value of this error at a certain point $z \in [0,1]$ is a linear functional on the space $L_2^{(m)}(0,1)$, i.e.,

$$\begin{aligned}(\ell,\varphi) &= \int_{-\infty}^{\infty} \ell(x,z)\varphi(x)\,\mathrm{d}x = \varphi(z) - P_\varphi(z) = \\ &= \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z)\varphi(h\beta) - A(z)\varphi'(0) - B(z)\varphi'(1), \end{aligned} \tag{2}$$

where $\delta$ is Dirac's delta-function and

$$\ell(x,z) = \delta\,(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x-h\beta) + A(z)\delta'(x) + B(z)\delta'(x-1) \tag{3}$$

is *the error functional* of the approximation formula (1) and belongs to the space $L_2^{(m)*}(0,1)$. The space $L_2^{(m)*}(0,1)$ is the conjugate space to the space $L_2^{(m)}(0,1)$. Further, for simplicity the error functional $\ell(x,z)$ we denote as $\ell(x)$.

By the Cauchy-Schwarz inequality the absolute value of the error (2) is estimated as follows

$$|(\ell,\varphi)| \leq \|\varphi|L_2^{(m)}\| \cdot \|\ell|L_2^{(m)*}\|,$$

where

$$\left\|\ell|L_2^{(m)*}\right\| = \sup_{\varphi,\|\varphi\|\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi|L_2^{(m)}\|}.$$

Therefore, in order to estimate the error of the approximation formula (1) on functions of the space $L_2^{(m)}(0,1)$ it is required to find the norm $\|\ell\|$ of the error functional $\ell$ in the conjugate space $L_2^{(m)*}(0,1)$.

From here we get the following.

**Problem 1.1.** *Find the norm $\|\ell\|$ of the error functional $\ell$ of the approximation formula (1) in the space $L_2^{(m)*}(0,1)$.*

It is clear that the norm of the error functional $\ell$ depends on coefficients $C_\beta(z), \beta = 0,1,...,N$, $A(z)$ and $B(z)$. We consider the minimization problem of the quantity $\|\ell\|$ by coefficients $C_\beta(z), \beta = 0,1,...,N$, $A(z)$ and $B(z)$.

The coefficients $\mathring{C}_\beta(z), \beta = 0,1,...,N$, $\mathring{A}(z)$ and $\mathring{B}(z)$ (if there exist) satisfying the equality

$$\left\|\mathring{\ell}|L_2^{(m)*}\right\| = \inf_{C_\beta(z),A(z),B(z)} \left\|\ell|L_2^{(m)*}\right\| \tag{4}$$

are called *the optimal coefficients*, the corresponding approximation formula

$$\mathring{P}_\varphi(z) = \sum_{\beta=0}^{N} \mathring{C}_\beta(z)\varphi(h\beta) + \mathring{A}(z)\varphi'(0) + \mathring{B}(z)\varphi'(1)$$

is called *the optimal approximation formula* and the difference $\varphi - \mathring{P}_\varphi$ is said to be *the error of the optimal approximation formula* (1) in the space $L_2^{(m)}(0,1)$.

Thus, in order to construct optimal approximation formula of the form (1) in the space $L_2^{(m)}(0,1)$ we need to solve the following problem.

**Problem 1.2.** *Find the coefficients $\mathring{C}_\beta(z), \beta = 0,1,...,N$, $\mathring{A}(z)$ and $\mathring{B}(z)$ which satisfy equality (4).*

It should be noted that first such type of problem was stated and studied by S.L. Sobolev in [33], where the extremal function of the interpolation formula was found in the $L_2^{(m)}$ space. The connection between interpolation formulas with derivatives and classical Euler-Maclaurin quadrature formulas were studied in [23, 24, 38].

The rest of the paper is organized as follows. In Section 2 the extremal function which corresponds to the error functional $\ell$ is found and with its help representation of the norm of the error functional (3) is calculated, i.e., Problem 1.1 is solved; in Section 3 in order to find the minimum of the quantity $\|\ell\|^2$ by coefficients $C_\beta(z), \beta = 0,1,...,N$, $A(z)$ and $B(z)$ the system of linear equations is obtained for the coefficients of optimal approximation formula (1) in the space $L_2^{(m)}(0,1)$, moreover existence and uniqueness of the solution of this system are discussed; in Section 4 some preliminaries are given; Section 5 is devoted to calculation of coefficients of the optimal interpolation formula (1) for the cases $m = 2$ and $m = 3$; finally, in Section 6 some numerical results which confirm the theoretical results of the paper are presented.

## 2. The extremal function and the norm of the error functional

In this section we find explicit form of the norm of the error functional $\ell$, i.e., we solve Problem 1.1.

For finding explicit form of the norm of the error functional $\ell$ in the space $L_2^{(m)}$ we use its extremal function [32, 33]. The function $\psi_\ell$ from $L_2^{(m)}(0,1)$ space is called *the extremal function* for the error functional $\ell$ if the following equality is fulfilled

$$(\ell, \psi_\ell) = \left\| \ell \left| L_2^{(m)*} \right\| \cdot \left\| \psi_\ell \right| L_2^{(m)} \right\|.$$

The space $L_2^{(m)}(0,1)$ is a Hilbert space and the inner product in this space is defined by the following formula

$$\langle \varphi, \psi \rangle = \int_0^1 \varphi^{(m)}(x)\psi^{(m)}(x)\, \mathrm{d}x. \tag{5}$$

According to the Riesz theorem any linear continuous functional $\ell$ in a Hilbert space is represented in the form of a inner product. Therefore, in our case, for any function $\varphi$ from $L_2^{(m)}(0,1)$ space, taking (5) into account, we have

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle. \tag{6}$$

Here $\psi_\ell$ is the function from $L_2^{(m)}(0,1)$, is defined uniquely by the functional $\ell$ and is the extremal function.

It is easy to see from (6) that the error functional $\ell$, defined on the space $L_2^{(m)}(0,1)$, satisfies the following equalities

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, ..., m-1. \tag{7}$$

Hence, it is clear that for existence of the approximation formula (1) the condition $N + 3 \geq m$ has to be met.

The equalities (7) mean that the approximation formula (1) is exact for any polynomial of degree $\leq m-1$. Then we conclude that for functions $\varphi$ of the space $L_2^{(m)}(0,1)$ the order of convergence of the approximation formula (1) is $O(h^m)$ .

The equation (6) was solved in [33] and for the extremal function $\psi_\ell$ was obtained the following expression

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-1}(x), \tag{8}$$

where

$$G_m(x) = \frac{x^{2m-1}\mathrm{sgn} x}{2(2m-1)!}, \tag{9}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$ and $*$ is the operation of convolution which for the functions $f$ and $g$ is defined as follows

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, \mathrm{d}y = \int_{-\infty}^{\infty} f(y)g(x-y)\, \mathrm{d}y.$$

Now we obtain the norm of the error functional $\ell$ which is uniquely defined by the extremal function $\psi_\ell$. Indeed, since $L_2^{(m)}(0,1)$ is the Hilbert space then by the Riesz theorem we have

$$(\ell, \psi_\ell) = ||\ell|| \cdot ||\psi_\ell|| = ||\ell||^2.$$

Hence, using (3) and (8), taking (7) into account, we get

$$\begin{aligned} \|\ell\|^2 &= (\ell,\psi_\ell) = (\ell(x), (-1)^m \ell(x) * G_m(x)) = \\ &= \int_{-\infty}^{\infty} \ell(x) \left( (-1)^m \int_{-\infty}^{\infty} \ell(y) G_m(x-y)\, \mathrm{d}y \right) \mathrm{d}x. \end{aligned}$$

Whence, keeping in mind that $G_m(x)$, defined by (9), is the even function, we have

$$\begin{aligned} \|\ell\|^2 &= (\ell,\psi_\ell) = (-1)^m \Bigg[ \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta(z) C_\gamma(z) \frac{\left|h\beta - h\gamma\right|^{2m-1}}{2(2m-1)!} - 2\sum_{\beta=0}^{N} C_\beta(z) \frac{(z-h\beta)^{2m-1}\mathrm{sgn}(z-h\beta)}{2(2m-1)!} \\ &\quad -2\sum_{\beta=0}^{N} C_\beta(z) \left( A(z) \frac{(h\beta)^{2m-2}}{2(2m-2)!} - B(z)\ \frac{(1-h\beta)^{2m-2}}{2(2m-2)!} \right) \\ &\quad + A(z) \frac{z^{2m-2}}{(2m-2)!} - B(z) \frac{(1-z)^{2m-2}}{(2m-2)!} - \frac{A(z)\cdot B(z)}{(2m-3)!} \Bigg]. \end{aligned} \tag{10}$$

Thus Problem 1.1 is solved.

Further, in next sections, we solve Problem 1.2.

## 3. Existence and uniqueness of the optimal interpolation formula

The error functional (3) satisfies conditions (7). The norm of the error functional $\ell$ is a multi variable quadratic function with respect to the coefficients $C_\beta(z), \beta = 0, 1, ..., N$, $A(z)$ and $B(z)$. For finding the point of the conditional minimum of the expression (10) under the conditions (7) we apply the Lagrange method.

Consider the function

$$\begin{aligned} &\Psi(C_0(z), C_1(z), ..., C_N(z), A(z), B(z), \lambda_0(z), ..., \lambda_{m-1}(z)) \\ &\quad = \|\ell\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-1} \lambda_\alpha(z)\,(\ell, x^\alpha)\,. \end{aligned}$$

Equating to 0 the partial derivatives of the function $\Psi$ by $C_\beta(z), \beta = 0, 1, ..., N, A(z), B(z)$ and $\lambda_0(z), \lambda_1(z), ..., \lambda_{m-1}(z)$, we get the following system of linear equations

$$\begin{aligned} &\sum_{\gamma=0}^{N} C_\gamma(z) \frac{\left|h\beta - h\gamma\right|^{2m-1}}{2(2m-1)!} - A(z) \frac{(h\beta)^{2m-2}}{2(2m-2)!} \\ &+ B(z) \frac{(h\beta-1)^{2m-2}}{2(2m-2)!} + \sum_{\alpha=0}^{m-1} \lambda_\alpha(z) (h\beta)^\alpha = \frac{\left|z - h\beta\right|^{2m-1}}{2(2m-1)!},\ \beta = 0, 1, ..., N, \end{aligned} \tag{11}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) \frac{(h\gamma)^{2m-2}}{2(2m-2)!} + \frac{B(z)}{2(2m-3)!} - \lambda_1(z) = \frac{z^{2m-2}}{2(2m-2)!}, \tag{12}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) \frac{(h\gamma-1)^{2m-2}}{2(2m-2)!} - \frac{A(z)}{2(2m-3)!} + \sum_{\alpha=1}^{m-1} \alpha\lambda_\alpha(z) = \frac{(1-z)^{2m-2}}{2(2m-2)!}, \tag{13}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) = 1, \tag{14}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma) + A(z) + B(z) = z, \tag{15}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma)^\alpha + \alpha B(z) = z^\alpha, \quad \alpha = 2, 3, ..., m-1. \tag{16}$$

The system (11)-(16) is called *the discrete system of Wiener-Hopf type* for optimal coefficients [32, 36]. In the system (11)-(16) the coefficients $C_\beta(z), \beta = 0, 1, ..., N, A(z), B(z)$ and $\lambda_\alpha(z), \alpha = 0, 1, ..., m-1$ are unknowns. The system (11)-(16) has a unique solution and this solution gives the minimum to $\|\ell\|^2$ under the conditions (14)-(16) when $N + 3 \geq m$. Here we omitted the proof of the existence and uniqueness of the solution of the system (11)-(16). The existence and uniqueness of the solution of this system can be proved similarly the existence and uniqueness of the solution of the discrete Wiener-Hopf type system for coefficients of optimal quadrature formulas in the space $L_2^{(m)}(0, 1)$ (see [32, 36]).

Therefore, the square of the norm of the error functional $\ell$, being quadratic function of the coefficients $C_\beta(z)$, $A(z)$ and $B(z)$ has a unique minimum in some concrete value $C_\beta(z) = \mathring{C}_\beta(z)$, $A(z) = \mathring{A}(z)$ and $B(z) = \mathring{B}(z)$.

As it was said above the approximation formula (1) with the coefficients $\mathring{C}_\beta(z)$, $\mathring{A}(z)$ and $\mathring{B}(z)$ corresponding to this minimum is called *the optimal approximation formula* and $\mathring{C}_\beta(z)$, $\mathring{A}(z)$ and $\mathring{B}(z)$ are called *the optimal coefficients*.

**Remark 3.1.** *It is easy to check that for the optimal coefficients $\mathring{C}_\beta(z)$, $\mathring{A}(z)$, $\mathring{B}(z)$ and $\mathring{\lambda}_\alpha(z)$ the following are true*

$$\mathring{C}_\beta(h\gamma) = \begin{cases} 1, & \gamma = \beta, \\ 0, & \gamma \neq \beta, \end{cases} \quad \gamma = 0, 1, ..., N, \ \beta = 0, 1, 2, ..., N,$$

$$\mathring{A}(h\beta) = 0, \ \mathring{B}(h\beta) = 0, \ \mathring{\lambda}_\alpha(h\beta) = 0, \ \beta = 0, 1, ..., N.$$

*Then we get that the optimal approximation formula (1) satisfies the following interpolation conditions*

$$\varphi(h\beta) = \mathring{P}_\varphi(h\beta), \ \ \beta = 0, 1, ..., N,$$

*which mean that the optimal approximation formula (1) is the interpolation formula. Therefore, further in this paper, the optimal approximation formula (1) we call the optimal interpolation formula (1).*

**Remark 3.2.** *It should be noted that by integrating both sides of the system (11)-(16) by z from* 0 *to* 1 *we get the system (4.1)-(4.6) of the work [30]. This means that by integrating both sides of the approximate equality (1) we get optimal quadrature formulas of the form (1.4) of the work [30]. In particular, when $m = 2$ and $m = 3$, we obtain the classical Euler-Maclaurin quadrature formulas. This confirms the result on connection between interpolation formulas with derivatives and classical Euler-Maclaurin quadrature formulas of works [23, 24, 38] in the case $m = 2$.*

One can solve the system (11)-(16) for coefficients of optimal interpolation formula (1) by direct or iterative methods. But here we use the method suggested by Sobolev for construction of optimal quadrature formulas in the space $L_2^{(m)}$ which is based on the discrete analogue of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ (see, for instance, [35]). This method allows to get explicit formulas for optimal coefficients and reduces the size of the system (11)-(16). Further we demonstrate this method in the case $m = 2$. Before that we give some preliminaries.

## 4. Preliminaries

Below, mainly the concept of discrete argument functions is used. The theory of discrete argument functions is given in [32, 36]. For completeness we give definitions about functions of discrete argument.

Assume that the nodes $x_\beta$ are equally spaced, i.e., $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ....$

**Definition 4.1.** *The function $\varphi(h\beta)$ is a function of discrete argument, if it is given on some set of integer values of $\beta$.*

**Definition 4.2.** *The inner product of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by*

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

*if the series on the right hand side converges absolutely.*

**Definition 4.3.** *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Furthermore, in our computations we need the discrete analogue $D_m(h\beta)$ of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ which satisfies the equality

$$hD_m(h\beta) * G_m(h\beta) = \delta_d(h\beta),$$

where $G_m(h\beta)$ is the discrete argument function corresponding to the function $G_m(x)$ defined by (9), $\delta_d(h\beta)$ is the discrete delta-function, i.e., $\delta_d(h\beta) = 0$ for $\beta \neq 0$ and $\delta_d(0) = 1$.

In [26] the discrete analogue $D_m(h\beta)$ was constructed and the following was proved.

**Theorem 4.4.** *The discrete analogue of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ has the form*

$$D_m(h\beta) = \frac{(2m-1)!}{h^{2m}} \begin{cases} \sum\limits_{k=1}^{m-1} \dfrac{(1-q_k)^{2m+1} q_k^{|\beta|}}{q_k E_{2m-1}(q_k)} & \text{for} \quad |\beta| \geq 2, \\ 1 + \sum\limits_{k=1}^{m-1} \dfrac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)} & \text{for} \quad |\beta| = 1, \\ -2^{2m-1} + \sum\limits_{k=1}^{m-1} \dfrac{(1-q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} & \text{for} \quad \beta = 0, \end{cases} \tag{17}$$

*where $E_{2m-1}(x)$ is the Euler-Frobenius polynomial of degree $2m-1$, $q_k$ are roots of the Euler-Frobenius polynomial $E_{2m-2}(x)$, $|q_k| < 1$, $h$ is small positive parameter.*

Several properties of the discrete argument function $D_m(h\beta)$ were proved in [26, 32, 36]. Here we give the following.

**Theorem 4.5.** *The discrete argument function $D_m(h\beta)$ and monomials $(h\beta)^k$ are related to each other as follows*

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & \text{for} \quad 0 \leq k \leq 2m-1, \\ (2m)! & \text{for} \quad k = 2m, \\ 0 & \text{for} \quad 2m+1 \leq k \leq 4m-1, \\ \dfrac{h^{2m}(4m)! B_{2m}}{(2m)!} & \text{for} \quad k = 4m, \end{cases} \tag{18}$$

*where $B_{2m}$ is the Bernoulli number.*

## 5. Coefficients for the optimal interpolation formula (1) in the cases $m = 2$ and $m = 3$

In this section we give solution of Problem 1.2 for the cases $m = 2$ and $m = 3$ and we find explicit formulas for optimal coefficients $\mathring{C}_\beta(z)$, $\beta = 0, 1, ..., N$, $\mathring{A}(z)$ and $\mathring{B}(z)$ of the optimal interpolation formula (1) using the discrete analogue of the operator $\mathrm{d}^{2m}/\,\mathrm{d}x^{2m}$ in the cases $m = 2$ and $m = 3$.

The following hold

**Theorem 5.1.** *The coefficients $\mathring{C}_\beta(z)$, $\beta = 0, 1, ..., N$, $\mathring{A}(z)$ and $\mathring{B}(z)$ of the optimal interpolation formula (1) in the space $L_2^{(2)}(0,1)$ have the following forms*

$$\begin{aligned}
\mathring{C}_0(z) &= \frac{1}{2h^3}\Big[6\sqrt{3}\sum_{\gamma=0}^{N} q^{\gamma}|z-h\gamma|^3 + |z-h|^3 + h^3 - z^3(4+3\sqrt{3}) \\
&\qquad +3z^2h(1-\sqrt{3}) + q^N N_1(z)\Big], \qquad (19)\\
\mathring{C}_\beta(z) &= \frac{1}{2h^3}\Big[6\sqrt{3}\sum_{\gamma=0}^{N} q^{|\beta-\gamma|}|z-h\gamma|^3 + |z-h(\beta-1)|^3 - 8|z-h\beta|^3 \\
&\qquad +|z-h(\beta+1)|^3 + q^{\beta}M_1(z) + q^{N-\beta}N_1(z)\Big],\ \ \beta = 1,2,...,N-1, \qquad (20)\\
\mathring{C}_N(z) &= \frac{1}{2h^3}\Big[6\sqrt{3}\sum_{\gamma=0}^{N} q^{N-\gamma}|z-h\gamma|^3 + |z-h(N-1)|^3 + h^3 \\
&\qquad -(1-z)^3(4+3\sqrt{3}) + 3h(1-z)^2(1-\sqrt{3}) + q^N M_1(z)\Big], \qquad (21)\\
\mathring{A}(z) &= \frac{f_1(z)}{q(1-q^{2N})}, \qquad (22)\\
\mathring{B}(z) &= \frac{f_2(z)}{q(1-q^{2N})}, \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}
M_1(z) &= 3z(z+h)(z-h-\sqrt{3}z) + 6h^2\frac{f_1(z)}{q(1-q^{2N})},\\
N_1(z) &= 3(1-z)(1-z+h)(1-z-h-\sqrt{3}+\sqrt{3}z) - 6h^2\frac{f_2(z)}{q(1-q^{2N})},\\
f_1(z) &= \frac{1}{2h^2}\Big[2\sqrt{3}\sum_{\gamma=0}^{N}(q^{\gamma+1}+q^{2N+1-\gamma})|z-h\gamma|^3 + h^2zq(1-q^{2N})\\
&\qquad +hz^2(2q+1)(1+q^{2N}) + z^3(q+1) + z^3q^{2N}(3q+1)\\
&\qquad +2h(1-z)^2(2q+1)q^N + (1-z)^3q^N(4q+2)\Big],\\
f_2(z) &= -\frac{1}{2h^2}\Big[2\sqrt{3}\sum_{\gamma=0}^{N}(q^{N-\gamma+1}+q^{N+1+\gamma})|z-h\gamma|^3\\
&\qquad +h^2(1-z)q(1-q^{2N}) + h(1-z)^2(2q+1)(1+q^{2N}) + (1-z)^3(q+1)\\
&\qquad +(1-z)^3q^{2N}(3q+1) + 2hz^2(2q+1)q^N + z^3q^N(4q+2)\Big],\\
q &= \sqrt{3}-2.
\end{aligned}$$

**Theorem 5.2.** *The coefficients $\mathring{C}_\beta(z)$, $\beta = 0,1,...,N$ of the optimal interpolation formula (1) in the space $L_2^{(3)}(0,1)$ have the following forms*

$$
\begin{aligned}
\mathring{C}_0(z) &= \frac{1}{2h^5}\Big[-32z^5 + |z-h|^5 + (z+h)^5 - 10A(z)h^4 + 240(\lambda_1(z)+2\lambda_2(z))h^2 \\
&\quad + \sum_{k=1}^{2}\frac{A_k}{q_k}\left(\sum_{\gamma=0}^{N} q_k^{\gamma}|z-h\gamma|^5 + M_k(z) + q_k^N N_k(z)\right)\Big],\\
\mathring{C}_\beta(z) &= \frac{1}{2h^5}\Big[|z-h(\beta-1)|^5 - 32|z-h\beta|^5 + |z-h(\beta+1)|^5 \\
&\quad + \sum_{k=1}^{2}\frac{A_k}{q_k}\left(\sum_{\gamma=0}^{N} q_k^{|\beta-\gamma|}|z-h\gamma|^5 + q_k^{\beta}M_k(z) + q_k^{N-\beta}N_k(z)\right)\Big],\ \ \beta = 1,2,...,N-1,\\
C_N(z) &= \frac{1}{2h^5}\Big[-32(1-z)^5 + |z-h(N-1)|^5 + (h+1-z)^5 + 10B(z)h^4 \\
&\quad -240h^2\lambda_1(z) + \sum_{k=1}^{2}\frac{A_k}{q_k}\left(\sum_{\gamma=0}^{N} q_k^{N-\gamma}|z-h\gamma|^5 + q_k^N M_k(z) + N_k(z)\right)\Big],
\end{aligned}
$$

*where*

$$
\begin{aligned}
A_k &= \frac{(1-q_k)^7}{q_k^5 + 57q_k^4 + 302q_k^3 + 302q_k^2 + 57q_k + 1},\\
M_k(z) &= \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left((z+h\gamma)^5 - 10\mathring{A}(z)(h\gamma)^4 + 240\left(\mathring{\lambda}_1(z) + 2\mathring{\lambda}_2(z)\right)(h\gamma)^2\right),\\
N_k(z) &= \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left((h\gamma+1-z)^5 + 10\mathring{B}(z)(h\gamma)^4 - 240\mathring{\lambda}_1(z)(h\gamma)^2\right),\\
&\quad k = 1,2,
\end{aligned}
$$

*the coefficients $\mathring{A}(z)$, $\mathring{B}(z)$ and $\mathring{\lambda}_1(z)$, $\mathring{\lambda}_2(z)$ satisfy the system of linear equations*

$$
\begin{aligned}
&A(z)\left[-\frac{h^4}{24}\sum_{\gamma=1}^{\infty} D_3(h\gamma+h\beta)\gamma^4\right] + B(z)\left[\frac{h^4}{24}\sum_{\gamma=1}^{\infty} D_3(h(N+\gamma)-h\beta)\gamma^4\right]\\
&\quad +\lambda_1(z)\left[h^2\sum_{\gamma=1}^{\infty} D_3(h\gamma+h\beta)\gamma^2 - h^2\sum_{\gamma=1}^{\infty} D_3(h(N+\gamma)-h\beta)\gamma^2\right]\\
&\quad +2\lambda_2(z)\left[h^2\sum_{\gamma=1}^{\infty} D_3(h\gamma+h\beta)\gamma^2\right] = -\frac{1}{240}\sum_{\gamma=-\infty}^{\infty} D_3(h\beta-h\gamma)|z-h\gamma|^5,\\
&\quad \beta=-2,\ \beta=-1,\ \beta=N+1,\ \beta=N+2,
\end{aligned}
$$

*here $D_3(h\beta)$ is defined by (17) and $q_k$, $(k = 1,2)$ are roots of the Euler-Frobenius polynomial $E_4(x) = x^4 + 26x^3 + 66x^2 + 26x + 1$ for which $|q_k| < 1$.*

Here we give the proof of Theorem 5.1. Theorem 5.2 can be proved similarly as Theorem 5.1.

**Proof of Theorem 5.1.** In the case $m = 2$ from the system (11)-(16) we have the following

$$\sum_{\gamma=0}^{N} C_\gamma(z)\frac{\left|h\beta - h\gamma\right|^3}{12} - A(z)\frac{(h\beta)^2}{4} + B(z)\frac{(h\beta-1)^2}{4} + \lambda_1(z)(h\beta) + \lambda_0(z) = \frac{\left|z-h\beta\right|^3}{12}, \quad \beta = 0,1,...,N, \tag{24}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma)^2 + 2B(z) - 4\lambda_1(z) = z^2, \tag{25}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(1-h\gamma)^2 - 2A(z) + 4\lambda_1(z) = (1-z)^2, \tag{26}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) = 1, \tag{27}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma) + A(z) + B(z) = z. \tag{28}$$

Using (25), (27) and (28) from (25) we get

$$\lambda_1(z) = 0. \tag{29}$$

Then, taking (29) into account, the system (24)-(28) can be written as follows

$$\sum_{\gamma=0}^{N} C_\gamma(z)\frac{\left|h\beta - h\gamma\right|^3}{12} - A(z)\frac{(h\beta)^2}{4} + B(z)\frac{(h\beta-1)^2}{4} + \lambda_0(z) = \frac{\left|z-h\beta\right|^3}{12}, \quad \beta = 0,1,...,N, \tag{30}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma)^2 + 2B(z) = z^2, \tag{31}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) = 1, \tag{32}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma) + A(z) + B(z) = z. \tag{33}$$

Further, we solve the system (30)-(33).
We introduce the following denotations

$$v_2(h\beta) = \sum_{\gamma=0}^{N} C_\gamma(z)\frac{\left|h\beta - h\gamma\right|^3}{12}, \tag{34}$$

$$u_2(h\beta) = v_2(h\beta) - A(z)\frac{(h\beta)^2}{4} + B(z)\frac{(h\beta-1)^2}{4} + \lambda_0(z). \tag{35}$$

Now we can express the coefficients $C_\beta(z)$, $\beta = 0,1,...,N$ by the function $u_2(h\beta)$. For this we use the discrete analogue $D_2(h\beta)$ of the differential operator $\frac{d^4}{dx^4}$ which satisfies the equation

$$hD_2(h\beta) * \frac{|h\beta|^3}{12} = \delta_d(h\beta),$$

where $\delta_d(h\beta)$ is the discrete delta function. From Theorem 4.4 in the case $m = 2$ we get the following

$$D_2(h\beta) = \frac{6}{h^4}\begin{cases} 6\sqrt{3}q^{|\beta|} & \text{for} \quad |\beta| \geq 2, \\ 19 - 12\sqrt{3} & \text{for} \quad |\beta| = 1, \\ 6\sqrt{3} - 8 & \text{for} \quad \beta = 0, \end{cases} \tag{36}$$

where $q = \sqrt{3} - 2$.

Using equality (18) and the discrete analogue (36) for coefficients $C_\beta(z)$, $\beta = 0, 1, ..., N$, of the optimal interpolation formula (1) we get the following equality

$$C_\beta(z) = hD_2(h\beta) * u_2(h\beta). \tag{37}$$

Hence we conclude that if we find the function $u_2(h\beta)$, then the coefficients $C_\beta(z)$, $\beta = 0, 1, ..., N$, of the formula (1) will be found from (37).

Now we find explicit representation of the function $u_2(h\beta)$. Since $C_\beta(z) = 0$ for $h\beta \notin [0, 1]$, then from (37) we get

$$C_\beta(z) = hD_2(h\beta) * u_2(h\beta) = 0 \text{ for } h\beta \notin [0, 1]. \tag{38}$$

Consider equality (34) for $h\beta \notin [0, 1]$.

Suppose $\beta < 0$. Then, taking (31)-(33) into account, we have

$$v_2(h\beta) \quad = \quad -\frac{1}{12}\Big((h\beta)^3 - 3(h\beta)^2(z - A(z) - B(z)) + 3(h\beta)(z^2 - 2B(z)) - \sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma)^3\Big). \tag{39}$$

Now, we assume $\beta > N$, then using (31)-(33) from (34) we get

$$v_2(h\beta) \quad = \quad \frac{1}{12}\Big((h\beta)^3 - 3(h\beta)^2(z - A(z) - B(z)) + 3(h\beta)(z^2 - 2B(z)) - \sum_{\gamma=0}^{N} C_\gamma(z)(h\gamma)^3\Big). \tag{40}$$

Further, using (39) and (40), from (35) we obtain

$$u_2(h\beta) = \begin{cases} -\frac{(h\beta-z)^3}{12} - \frac{A(z)}{2}(h\beta)^2, & \beta < 0, \\ \frac{|z-h\beta|^3}{12}, & 0 \le \beta \le N, \\ \frac{(h\beta-z)^3}{12} + \frac{B(z)}{2}(h\beta - 1)^2, & \beta > N. \end{cases} \tag{41}$$

Here $A(z)$ and $B(z)$ are unknowns. We find them from equation (38).

Now, taking (41) into account, from (38) we get

$$D_2(h\beta) * u_2(h\beta) = 0, \qquad \beta < 0, \;\; \beta > N,$$

that is, for $\beta < 0$ and $\beta > N$ we obtain the following

$$\sum_{\gamma=1}^{\infty} D_2(h\gamma + h\beta)\left(\frac{(h\gamma + z)^3}{12} - \frac{A(z)}{2}(h\gamma)^2\right) + \sum_{\gamma=0}^{N} D_2(h\gamma - h\beta)\frac{|z - h\gamma|^3}{12}$$
$$+ \sum_{\gamma=1}^{\infty} D_2(h(N + \gamma) - h\beta)\left(\frac{(h\gamma + 1 - z)^3}{12} + \frac{B(z)}{2}(h\gamma)^2\right) = 0.$$

Hence, for $\beta < 0$ and $\beta > N$ we have

$$A(z)\left(-\frac{h^2}{2}\sum_{\gamma=1}^{\infty} D_2(h\gamma + h\beta)\gamma^2\right) + B(z)\left(\frac{h^2}{2}\sum_{\gamma=1}^{\infty} D_2(h(N + \gamma) - h\beta)\gamma^2\right)$$
$$= -\sum_{\gamma=0}^{N} D_2(h\gamma - h\beta)\frac{|z - h\gamma|^3}{12} - \sum_{\gamma=1}^{\infty} D_2(h\gamma + h\beta)\left(\frac{(h\gamma + z)^3}{12}\right)$$
$$-\sum_{\gamma=1}^{\infty} D_2(h(N + \gamma) - h\beta)\left(\frac{(h\gamma + (1 - z))^3}{12}\right). \tag{42}$$

Thus, in order to find two unknowns $A(z)$ and $B(z)$ it is sufficient to get two linear equations from (42) when $\beta = -1$ and $\beta = N+1$. Then from (42) for $\beta = -1$ and $\beta = N+1$, after some simplifications, we get the following system of two linear equations

$$A(z)q + B(z)q^{N+1} = \frac{1}{2h^2}\Bigg[2\sqrt{3}\sum_{\gamma=0}^{N} q^{\gamma+1}|z-h\gamma|^3 + h^2zq + 3hz^2(2q+1) + z^3(q+1)$$
$$-h^2(1-z)q^{N+1} + h(1-z)^2(2q+1)q^N + (1-z)^3(3q+1)q^N\Bigg], \tag{43}$$
$$A(z)q^{N+1} + B(z)q = -\frac{1}{2h^2}\Bigg[2\sqrt{3}\sum_{\gamma=0}^{N} q^{N+1-\gamma}|z-h\gamma|^3 - 3h^2zq^{N+1} + hz^2(2q+1)q^N$$
$$+z^3(3q+1)q^N + +h^2(1-z)q + 3h(1-z)^2(2q+1) + (1-z)^3(q+1)\Bigg]. \tag{44}$$

Solving system (43)-(44) of equations we get explicit forms (22) and (23) of $A(z)$ and $B(z)$, respectively.

Further, from (37), using (36) and (41), for $\beta = 0,1,2,...,N$ we get analytic formulas (19)-(21) for optimal coefficients $C_\beta(z), \beta = 0,1,...,N$. Theorem 5.1 is proved. □

**Remark 5.3.** *We note that equations (32), (33), (31) and equation (30) with $\beta = 0$ mean that in the case $m = 2$ the optimal interpolation formula of the form (1) is exact for the monomials $1$, $z$, $z^2$ and $z^3$, respectively. Hence we conclude that for functions $\varphi$ from the space $C^4(0,1)$ the order of convergence of the optimal interpolation formula (1) obtained in Theorem 5.1 is $O(h^4)$ instead of $O(h^2)$ as stated in Section 2.*

## 6. Numerical results

In this section we give some numerical results using Theorem 5.1.

We consider the case $m = 2$ and $N = 5$. In this case the optimal interpolation formula (1) has the form

$$\varphi(z) \cong \mathring{P}_\varphi(z) = \sum_{\beta=0}^{5} \mathring{C}_\beta(z)\varphi(0.2\beta) + \mathring{A}(z)\varphi'(0) + \mathring{B}(z)\varphi'(1). \tag{45}$$

The optimal coefficients $\mathring{C}_\beta(z), \beta = \overline{0,5}$ and $\mathring{A}(z), \mathring{B}(z)$ are defined by (19)-(23). The graphs of these coefficients are presented in Figure 1. From Figure 1 one can see that for $\gamma = 0,1,...,5$

$$\begin{aligned} \mathring{C}_\beta(0.2\gamma) &= \begin{cases} 1 & \text{if } \gamma = \beta, \\ 0 & \text{if } \gamma \neq \beta, \end{cases} \quad \beta = 0,1,...,5, \\ \mathring{A}(0.2\gamma) &= 0, \\ \mathring{B}(0.2\gamma) &= 0. \end{aligned}$$

Integrating both sides of (45) by $z$ from 0 to 1 we get the corresponding Euler-Maclaurin quadrature formula.

In numerical examples we consider the functions $\varphi_1(z) = \sin z$ and $\varphi_2(z) = z^4$ since the optimal interpolation formula (1) in the case $m = 2$ is exact for monomials $1$, $z$, $z^2$ and $z^3$. We denote corresponding optimal interpolation formulas (1) by $\mathring{P}_{\varphi_1}(z)$ and $\mathring{P}_{\varphi_2}(z)$, respectively. Graphs of the corresponding absolute errors for the cases $N = 5$ and $N = 10$ are displayed in Figures 2-3. From these results we can see that the errors of the optimal interpolation formula (1) decreases as $N$ increases.

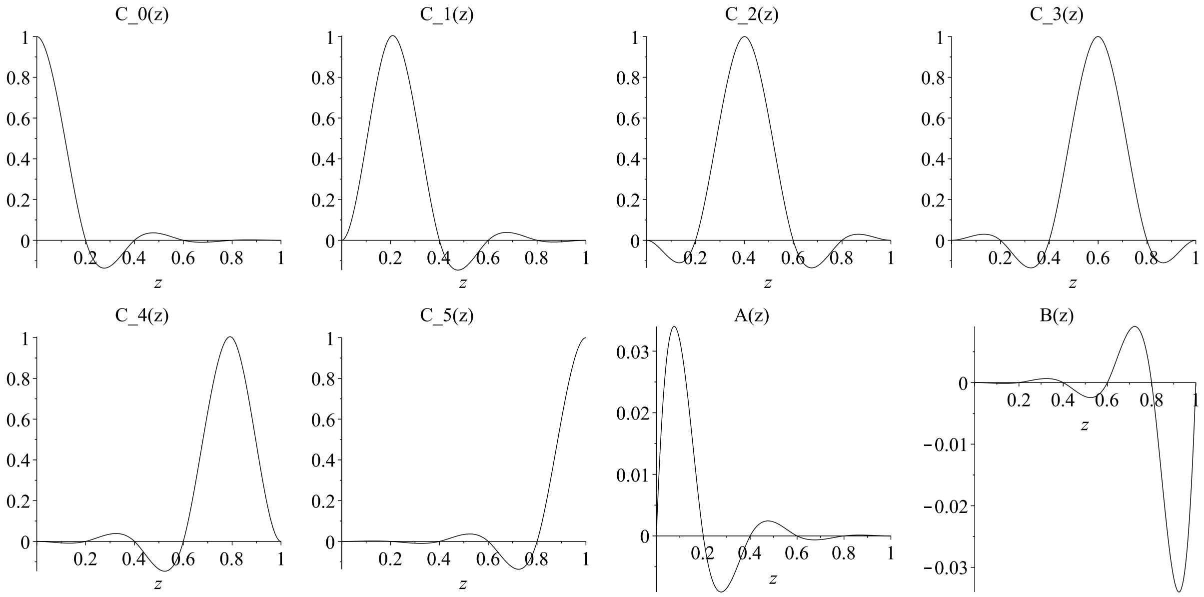


Figure 1: The graphs of the optimal coefficients $\mathring{C}_\beta(z)$, $\beta = \overline{0,5}$ and $\mathring{A}(z)$, $\mathring{B}(z)$ of the optimal interpolation formula (45).

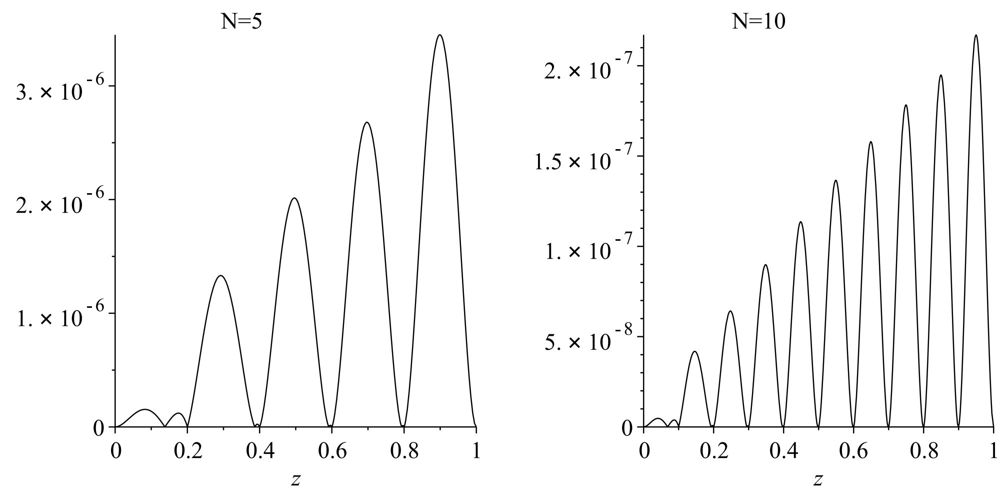


Figure 2: Graphs of the absolute errors $|\varphi_1(z) - \mathring{P}_{\varphi_1}(z)|$ for the cases $N = 5$ and $N = 10$.

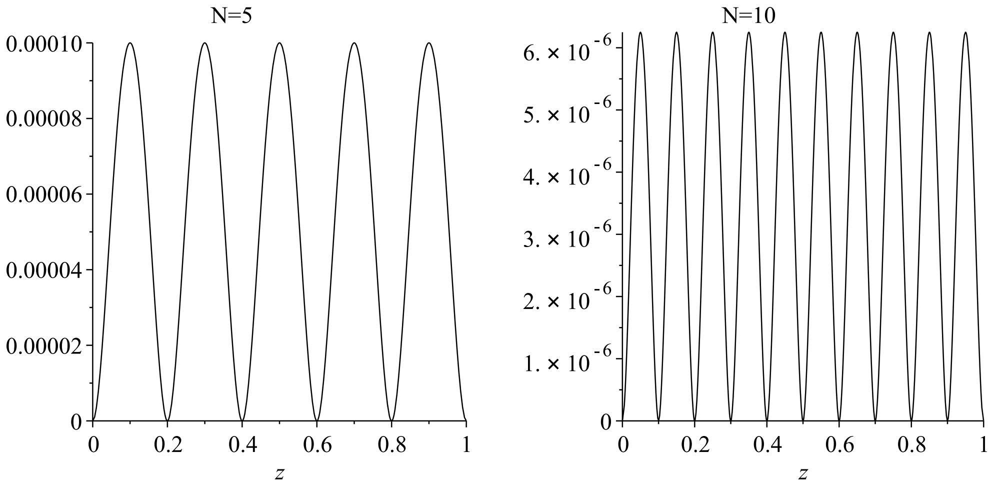


Figure 3: Graphs of the absolute errors $|\varphi_2(z) - \mathring{P}_{\varphi_2}(z)|$ for the cases $N = 5$ and $N = 10$.


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## References


[1] O. Abu Arqub, Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions, Computers and Mathematics with Applications 73 (2017) 1243–1261.

[2] O. Abu Arqub, B. Maayah, Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana-Baleanu fractional operator Chaos, Solitons and Fractals 117 (2018) 117-124.

[3] O. Abu Arqub, M. Al-Smadi, Atangana-Baleanu fractional approach to the solutions of Bagley-Torvik and Painlevé equations in Hilbert space, Chaos, Solitons and Fractals 117 (2018) 161-167.

[4] O. Abu Arqub, M. Al-Smadi, Numerical algorithm for solving time-fractional partial integro differential equations subject to initial and Dirichlet boundary conditions, Numer Methods for Partial Differential Equations 34 (2018) 1577–1597.

[5] J.H. Ahlberg, E.N. Nilson, J.L. Walsh, The theory of splines and their applications, Mathematics in Science and Engineering, New York: Academic Press, 1967.

[6] R. Arcangeli, M.C. Lopez de Silanes, J.J. Torrens, Multidimensional minimizing splines, Kluwer Academic publishers, Boston, 2004.

[7] N. Aronszajn, Theory of reproducing kernels, Trans Am Math Soc 68 (1950) 337–404 .

[8] M. Attea, Hilbertian kernels and spline functions, Studies in Computational Matematics 4, C. Brezinski and L.Wuytack eds, North-Holland, 1992.

[9] A. Berlinet, C. Thomas-Agnan. Reproducing Kernel Hilbert Space in Probability and Statistics, Springer, Dordrecht, 2004.

[10] A.Yu. Bezhaev, V.A. Vasilenko, Variational theory of splines, Springer, New York, 2001.

[11] C. de Boor, Best approximation properties of spline functions of odd degree, J. Math. Mech. 12 (1963) 747–749.

[12] C. de Boor, A practical guide to splines, Springer-Verlag, New York, 1978.

[13] A. Cabada, A.R. Hayotov, Kh.M. Shadimetov, Construction of $D^m$-splines in $L_2^{(m)}(0,1)$ space by Sobolev method, Applied Mathematics and Computation 244 (2014) 542–551

[14] A.R. Hayotov, Construction of interpolation splines minimizing the semi-norm in the space $K_2(P_m)$, Journal of Siberian Federal University. Mathematics and Physics 11 (2018) 383–396.

[15] A.R. Hayotov, G.V. Milovanović, Kh.M. Shadimetov, On an optimal quadrature formula in the sense of Sard, Numerical Algorithms 57 (2011) 487–510.
[16] A.R. Hayotov, G.V. Milovanović, Kh.M. Shadimetov, Interpolation splines minimizing a semi-norm, Calcolo 51 (2014) 245–260.
[17] A.R. Hayotov, G.V. Milovanović, Kh.M. Shadimetov, Optimal quadratures in the sense of Sard in a Hilbert space, Applied Mathematics and Computation 259 (2015) 637–653.
[18] J.C. Holladay, Smoothest curve approximation, Math. Tables Aids Comput. 11 (1957) 223–243.
[19] M.I. Ignatev, A.B. Pevniy, Natural splines of many variables, Nauka, Leningrad, 1991 (in Russian).
[20] P.-J. Laurent, Approximation and Optimization, Mir, Moscow, 1975 (in Russian).
[21] G. Mastroianni, G.V. Milovanović, Interpolation Processes – Basic Theory and Applications, Springer Monographs in Mathematics, Springer – Verlag, Berlin – Heidelberg, 2008.
[22] E. Novak, M. Ullrich, H. Woźniakowski, Sh. Zhang. Reproducing kernels of Sobolev spaces on $\mathbb{R}^d$ and applications to embedding constants and tractability, Analysis and Applications 16 (2018) 693–715.
[23] I.J. Schoenberg, On equadistant cubic spline interpolation, Bulletin of the American Mathematical Sociaty 77 (1971) 1039–1044.
[24] I.J. Schoenberg, Cardinal spline interpolation, SIAM, Philadelphia, 1973.
[25] L.L. Schumaker, Spline functions: basic theory, (3d edition), Cambridge University Press, 2007.
[26] Kh.M. Shadimetov. The discrete analogue of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ and its construction, Questions of Computations and Applied Mathematics, Tashkent, (1985) 22–35. ArXiv:1001.0556.v1 [math.NA] Jan. 2010.
[27] Kh.M. Shadimetov, A.R. Hayotov, Optimal quadrature formulas with positive coefficients in $L_2^{(m)}(0,1)$ space, Journal of Computational and Applied Mathematics 235 (2011) 1114–1128.
[28] Kh.M. Shadimetov, A.R. Hayotov, Construction of interpolation splines minimizing semi-norm in $W_2^{(m,m-1)}(0,1)$ space, BIT Numer Math 53 (2013) 545–563.
[29] Kh.M. Shadimetov, A.R. Hayotov, Optimal quadrature formulas in the sense of Sard in $W_2^{(m,m-1)}$ space, Calcolo 51 (2014) 211–243.
[30] Kh.M. Shadimetov, A.R. Hayotov, F.A. Nuraliev, On an optimal quadrature formula in Sobolev space $L_2^{(m)}(0,1)$, Journal of Computational and Applied Mathematics 243 (2013) 91–112.
[31] Kh.M. Shadimetov, A.R. Hayotov, F.A. Nuraliev, Optimal quadrature formulas of Euler-Maclaurin type, Applied Mathematics and Computation 276 (2016) 340–355.
[32] S.L. Sobolev, Introduction to the theory of cubature formulas, Nauka, Moscow, 1974 (in Russian).
[33] S.L. Sobolev, On interpolation of functions of $n$ variables, in: Selected works of S.L. Sobolev, Springer US, (2006) 451–456.
[34] S.L. Sobolev, Formulas of mechanical cubature in $n$- dimensional space, in: Selected Works of S.L.Sobolev, Springer US, (2006) 445–450.
[35] S.L. Sobolev, The coefficients of optimal quadrature formulas, in: Selected works of S.L.Sobolev, Springer US (2006) 561–566.
[36] S.L. Sobolev, V.L. Vaskevich, The theory of cubature formulas, Kluwer Academic Publishers Group, Dordrecht, 1997.
[37] S.B. Stechkin, Yu.N. Subbotin, Splines in computational mathematics, Nauka, Moscow, 1976 (in Russian).
[38] Z. Udovičić, D. Barrera, A second look at the interpolatory background of the Euler-Macalurin quadrature formula, Applied Mathematics and Computation 220 (2013) 608–615.
[39] V.A. Vasilenko, Spline-functions: Theory, Algorithms, Programs, Nauka, Novosibirsk, 1983 (in Russian).