A Double Well Potential Systems

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For a potential V on R^n which has only two global minimum points p, q, we consider the following elliptic system $\Delta u(x) - V_u(x,u(x))) = 0$, $x \in R \times \Omega \subset R^m$, $u(x) = p$ for $x_1 = -\infty$, $u(x) = q$ for $x_1 = \infty$, which is an ODE system, when m=1. An existence of a heteroclinic solution connecting p and q is one of the most fundamental question in the research on dynamical systems. First, the existence of the heterocliinc solution was proved under the optimal conditions for the potential V, which improves all previous results. Second, in the scalar case n=1, a gap condition is a necessary and sufficient conditions for existence of a real soution close to any shadowing chains. For our system above, we have found a corresponding "isolatedness condition" and under the condition we proved the chaotic structure of solutions, the existence of a real soution close to any shadowing chains.

1. Background

As for the movement of a pendulum, there three kinds of solutions : the first kind is a class of periodic solutions, the second of homoclinic or heteroclinic solutions, the third kind of rotating solutions. The potential governing the movement of a pendulum is given by the sine function. For the pendulum, we could classify completely the solutions by the phase plane analysis. On the other hand, for the movement of many objects interacting each other, it is described by a system of equations given through a potential. More generally, many similar models are described by a system of partial differential equations. For a systems of ordinary differential equations , even a proof for the existence of a specific kind of solutions is quite involved. It is much more involved for the case of a system of difficult partial differential equations, Thus, for the construction of each kind of solutions, many tools have been developed. In this research we studied heteroclinic or homoclinic solutions connecting equilibrium points.

2. Contents

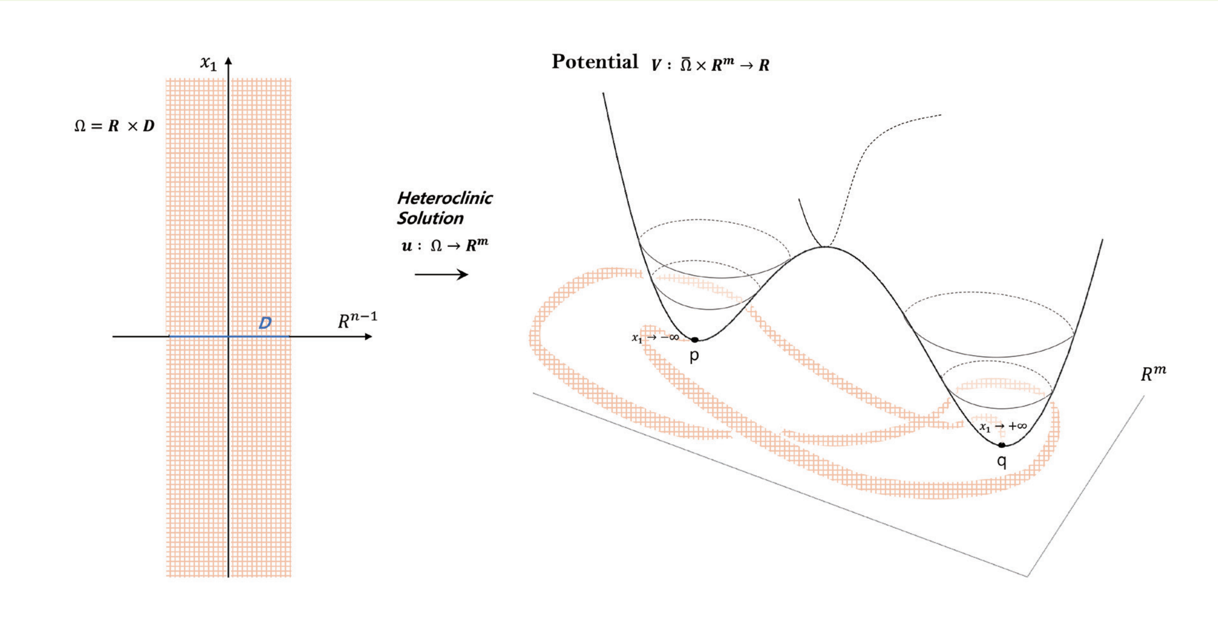
We consider the following system of partial differential equations for $u \in R^n$

$\Delta u(x) - V_u(x,u(x))) = 0$, $x \in R \times \Omega \subset R^m$, $u(x) = p$ for $x_1 = -\infty$, $u(x) = q$ for $x_1 = \infty$. For m=1, it is well known that there exists a heteroclinic solution connecting $p \neq q \in R^n$ when V(p)=V(q) and a local property of V near p and q holds ; typically,

$$\lim_{|x| \to \infty} V(x) > V(p),\ V(x) > V(p) = V(q) \text{ for } x \neq p,q$$

and $\nabla^2 V(p)$, $\nabla^2 V(q)$ are nondegenerate.

In this research we prove the existence of heteroclinic solution connecting p,q without any additional conditions, that is, when it holds that



$\lim_{|x|\to\infty} V(x) > V(p)$, $V(x) > V(p) = V(q)$ for $x \neq p,q$.

For this problem, Alikakos and Fusco [Journal of the European Mathematical Society, 2015] showed the existence of heteroclinic solution connecting p, q when V satisfies a local property near p,q and a monotone property outside of a ball containing p and q.

We could extend the result so that it holds when there is a monotonicity near the boundary of a convex set Ω including p, q and $V(x) > V(p) = V(q)$, $x \in \Omega \backslash \{p,q\}$.

Furthermore, in the scalar case, there is a well-known condition "gap condition" for a building block heteroclinic solution under which condition there exit real solutions close to any shadowing chains connecting two elements among {p,p,q,q}. For the system, we could find a corresponding condition "an isolatedness condition" under which condition the same existence result corresponding to any shadowing chains holds. This implies that when the isolatedness condition holds, the structure of heteoroclinic and homoclinic solutions is very complicated and there are solutions with chaotic behavior. This is the first result for the system studied in this work.

3. Expected effect

It is an utmost goal to prove the existence of solutions of differential equations for most general cases. This goal entails the development of new approaches or new tools for the resolution. We believe that the approach in our work could be applied to many other related problems, ODE systems in infinite dimensional spaces as a generalization of our PDE model and special type of solutions for liquid crystals, etc.

Research Funding

• This research was supported by through Mid-career Researcher Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (NRF-2013R1A2A2A05006371)

Research Outcomes

• Jaeyoung Byeon, Piero Monttechiari and Paul Rabnowitz, A double well potential system, Analysis and PDE, 9(2016), 1737-1772