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SPIKE LAYER SOLUTIONS FOR A SINGULARLY PERTURBED NEUMANN PROBLEM: VARIATIONAL CONSTRUCTION WITHOUT A NONDEGENERACY

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Abstract. We consider the following singularly perturbed problem

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Existence of a solution with a spike layer near a min-max critical point of the mean curvature on the boundary $\partial\Omega$ is well known when a nondegeneracy for a limiting problem holds. In this paper, we use a variational method for the construction of such a solution which does not depend on the nondengeneracy for the limiting problem. By a purely variational approach, we construct the solution for an optimal class of nonlinearities f satisfying the Berestycki-Lions conditions.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega \in C^4$. In this paper, we consider the following singularly perturbed nonlinear Neumann problem

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1}$$

which corresponds to steady states of a chemotaxis model of Keller and Segel [23] and the shadow system of Gierer and Meinhardt [18] for a pattern formation. For the background of the models originated from the ground breaking idea of Turing [36], refer to the survey article [33] by Ni.

Problem (1) has a mountain pass solution (refer to [2]) when $f \in C^1(\mathbb{R})$ satisfies the following conditions:

(f1): $f(t) = 0$ for $t \leq 0$ and $\lim_{t\to 0} f(t)/t = 0$;

(f2): there exist some $a, b > 0$ and $p \in (1, \frac{n+2}{n-2})$ such that $|f(t)| \leq at + bt^p$ for $t \geq 0$;

(f3-1): there exist $\mu > 2$ and $t_0 > 0$ such that $\mu\int_0^t f(s)ds < f(t)t$ for $t > t_0$.

In a series of papers [28, 34, 35], Ni and his collaborators studied an asymptotic profile of the mountain pass solution as $\varepsilon \to 0$. In fact, they proved very elegant results that for sufficiently small $\varepsilon > 0$, there exists a unique maximum point

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$x_\varepsilon \in \partial\Omega$ of the mountain pass solution u_ε and constants $C, c > 0$, independent of $\varepsilon > 0$, satisfying

(i): $0 < \liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) \le \limsup_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) < \infty, \quad u_\varepsilon(x) \le C\exp(-c\frac{|x-x_\varepsilon|}{\varepsilon})$,

(ii): for a diffeomorphism Ψ from $\overline{\mathbb{R}^n_+}$ to a neighborhood B of x_ε in $\overline{\Omega}$ satisfying $\Psi(\partial\mathbb{R}^n_+) = \overline{B} \cap \partial\Omega$ and $\nabla\Psi(0) \in SO(n)$, a transformed solution $v_\varepsilon(x) \equiv u_\varepsilon \circ \Psi(\varepsilon x)$ converges uniformly to a radially symmetric least energy solution U of the following limit problem

$$\begin{aligned} &\Delta u - u + f(u) = 0, \;\; u > 0 \text{ in } \mathbb{R}^n_+, \\ &\frac{\partial u}{\partial x_n} = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+ \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0, \end{aligned} \tag{2}$$

(iii): for the mean curvature H of $\partial\Omega$ with respect to the outward unit normal vector field,

$$\lim_{\varepsilon\to 0} H(x_\varepsilon) = \min_{x\in\partial\Omega} H(x)$$

when f satisfies the following additional conditions

(f3-2): there exists $\mu > 2$ such that $\mu\int_0^t f(s)ds < f(t)t$ for $t > 0$,

(f4): $f(t)/t$ is non-decreasing on $(0, \infty)$,

(f5): there exists a unique radially symmetric solution $U \in H^{1,2}(\mathbb{R}^n)$ for $\Delta u - u + f(u) = 0, u > 0$ in $\mathbb{R}^n$ such that if $\Delta V - V + f'(U)V = 0$ and $V \in H^{1,2}(\mathbb{R}^n)$, then $V = \sum_{i=1}^n a_i \frac{\partial U}{\partial x_i}$ for some $a_1, \cdots, a_n \in \mathbb{R}$.

This shows that for small $\varepsilon > 0$, the mountain pass solution u_ε develops a peak (or spike layer) at x_ε which approaches to a minimum point of H as $\varepsilon \to 0$.

Here, we note that in all previous papers related to the studies on problem (1), they use a mean curvature of $\partial\Omega$ with respect to the inward unit normal vector field. Since we use a min-max argument in this paper, for convenience's sake, we use instead the mean curvature H of $\partial\Omega$ with respect to the outward unit normal vector field; thus the mean curvature of the unit sphere is -1 in this paper.

As for the mountain pass solutions, there have been endeavors to weaken the additional conditions (f3-2),(f4), (f5) to get the asymptotic behaviors (i), (ii), (iii). In the insightful paper [16], del Pino and Felmer proved (i)-(iii) for the least energy solution of (1) without condition (f5) in a simpler way. (See also [21] for a variational construction of multi-peak solutions corresponding local minimum points of H). In [9], the first author and Park proved the asymptotic behaviors (i), (ii), (iii) for the mountain pass solution of the same type of problem on a compact $n-$dimensional manifold, $n \ge 3$, further without the assumptions (f4) and (f5). On the other hand, in a classical paper [4], Berestycki and Lions proved the existence of radially symmetric least energy solution U of the limiting problem (2) when (f1),(f2) and the following (f3) are satisfied:

(f3): there exists $T > 0$ satisfying $F(T) \equiv \int_0^T f(t)dt > T^2/2$,

which is a necessary condition for existence of a solution of (2). In [22], Jeanjean and Tanaka observed that the least energy solutions of (2) obtained in [4] are mountain pass solutions. It is our intuitive understanding that the mountain pass solutions are (structurally) stable under a perturbation, this is, the solutions persists to exist for a perturbation. Thus, since condition (f1)-(f3) are almost necessary and sufficient for the existence of a least energy solution of the limiting problem (2), it would be natural to expect that the perturbed problem (1) would have a solution under the conditions (f1)-(f3). In fact, the first author proved in [6] that when $f \in C^1$ satisfies (f1)-(f3) and $n \geq 3$, for any connected component K of local minimum points of the mean curvature H on $\partial\Omega$, there exists a solution u_ε of (1) whose peak point approaches to K as $\varepsilon \to 0$. This result was extended for $f \in C^0$ satisfying (f1)-(f3) and $n \geq 2$ in [8].

As for the solutions different from (local) mountain pass solutions, there have been numerous further studies. Existence of solutions with one spike layer near nondegenerate critical points of H was proved in [37] and C^1 stable critical points of H in [27] and variationally characterized critical points of H in [17], respectively. For the solutions with multiple spikes layer, refer to [14, 29] and references therein; for the solutions with higher dimensional layer, to [31, 30, 32] and references therein. All the previous works mentioned above related to the construction of solutions different from (local) mountain pass solutions strongly depends on the nondegeneracy condition, (f5), since they use the Liapunov-Schmidt reduction method. In many perturbation problems in geometry and analysis, such a nondegeneracy condition for limiting problems is widely assumed. On the other hand, for some perturbation problems, such a nondegeneracy condition for limiting problems is very difficult to prove in general; even worse, the nondegeneracy does not hold in some cases(refer to [11]). For our problem in this paper, the nondegeneracy condition, (f5), was verified for the most standard nonlinearity $f(u) = u^p, p \in (1, (n+2)/(n-2))$ by Kwong [24]. More generally, it is known in [12] that under a monotone increasing property of $(f'(u)u - u)/(f(u) - u)$ on $[u_0, \infty)$ for $f(u_0) = u_0$, (f5) holds. On the other hand, the monotone property is not preserved for a small perturbation of a nonlinearity f satisfying the monotonicity. In biological or physical modelings, it is desirable that such a small perturbation for nonlinearities representing reactions is allowed in the sense of stability of the modelings. Even for some typical cases $f(u) = u^p + \lambda u^q$ with $1 < q < 3 < p < 5, n = 3$ and large $\lambda > 0$, it was proved recently by Davila, del Pino and Guerra in [15] that there are at least three radially symmetric solutions of (2); the result of [15] suggests that it is almost impossible to prove (f5) for such a simple nonlinearity $f(u) = u^p + \lambda u^q$ for some $\lambda \in (0, \infty)$.

Now, since (f1)-(f3) are stable under small perturbation of f and almost optimal conditions for existence of a solution for (2), it is very desirable to prove that for any saddle or maximum point x_0 of H, there exists a solution u_ε of (1) with a peak point x_ε approaching to x_0 as $\varepsilon \to 0$. The purpose of this paper is to complete the task. In [17, 27, 37], through the Liapunov-Schmidt reduction method, they could distinguish ε order energy difference of their energies for approximating solutions depending on energy concentration points near the critical point of H on $\partial\Omega$; this good approximation leads to an existence of a solution for (1) with a spike layer near the critical point. When (f5) does not hold, we can not use the Liapunov-Schmidt reduction method. Recently, the first author and Tanaka developed a variational method in [10] for construction of semiclassical standing waves of nonlinear Schrödinger equations corresponding to critical points of a potential V which can be applied under only conditions (f1)-(f3) without the nondegeneracy condition (f5). For construction of the standing waves for nonlinear Schrödinger equations, we needed to distinguish only *zero* order difference of their energies for functions close to approximate solutions for small $\varepsilon > 0$. On the other hand, in our problem (1), when we adopt a variational method, we need to distinguish ε order energy difference of their energies for functions close to approximate solutions for small $\varepsilon > 0$. To obtain this subtle estimate is the main difficulty when we apply the scheme developed in [10] to problem (1). One of key differences with the previous variational method in [10] is that here we introduce a new concept of a transplantation flow in Ω along the boundary $\partial\Omega$ instead of the translation flow used in [10]. Using the transplantation flow, we prove in Proposition 19 that for any solution with one spike layer $x_\varepsilon \in \partial\Omega$, $\lim_{\varepsilon\to 0}\nabla_{\partial\Omega}H(x_\varepsilon) = 0$, which is known only when the additional condition (f5) is satisfied (refer to [37]).

For a statement of our main result, we use the following notations. For a set $A \subset \mathbb{R}^n$ and $d > 0$, we define $A^d \equiv \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \le d\}$. For $c_1 < c_2$, $H_{c_1}^{c_2} \equiv \{x \in \partial\Omega \mid c_1 \le H(x) \le c_2\}$. For each $a > 0$ and a set $M \subset \partial\Omega$, we define $M^a \equiv \{x \in \partial\Omega \mid \operatorname{dist}_{\partial\Omega}(x, M) < a\}$, where $\operatorname{dist}_{\partial\Omega}(x, M)$ is the minimal distance from x to M on $\partial\Omega$. Now, for a set $\mathcal{M}$ of critical points of H, we assume the following conditions:

(H1): There exists a connected open set $O \subset \partial\Omega$, a k-dimensional C^1-manifold L in O with a boundary $\partial L \equiv L_0 \subset O$ and $k \in \{1, \cdots, n-1\}$, a point $z_0 \in L$ and a continuous projection map $\Pi : \overline{O} \to L$ such that for a class

$$\Lambda_L \equiv \{\varphi \in C(L, \overline{O}) \mid \varphi(z) = z \text{ for } z \in L_0\},$$

$$m \equiv \max_{z\in L} H(z) = \inf_{\varphi\in\Lambda_L} \max_{z\in L} H(\varphi(z)) > \max_{z\in L_0} H(z) \equiv m_0,$$

and that

$$H(x) \ge m \text{ for } x \in \Pi^{-1}(z_0);$$

(H2): The set $\mathcal{M} \equiv \{X \in O \mid H(X) = m, \nabla H(x) = 0\}$ is a compact subset of O and for each small $q > 0$, there exists a neighborhood $\mathcal{N} \subset \partial\Omega$ of $\mathcal{M}$ with a smooth boundary in O such that $\mathcal{N} \subset \mathcal{M}^q$ and $|\nabla H(x)| \ne 0$ for $x \in \partial\mathcal{N}$.

(H3): For some $d_0 > 0$, there are linearly independent C^3 tangent vector fields $\{e_1, \cdots, e_{n-1}\}$ of $\partial\Omega$ on $\mathcal{M}^{d_0} \setminus \mathcal{M}$.

If (H1) holds, we see that $\mathcal{M}$ is not empty since $z_0 \in \mathcal{M}$. It is easy to see that if z_0 is an isolated critical point of H on $\partial\Omega$ which is a type of saddle or maximum, (H1)-(H3) hold.

We define a diffeomorphism Ψ_p for each $p \in \partial\Omega$ as follows. Let ν_p be the inward unit normal vector. Then there exists $E_p \in O(n)$ such that $E_p\nu_p = (0, ..., 0, 1)$. For small $r > 0$, there exists a neighborhood N of 0 such that $N \cap E_p(\partial\Omega)$ can be expressed as a graph of a function $\psi_p : \partial\mathbb{R}^n_+ \cap B(0, r) \to \mathbb{R}^n$. We define

$$\Psi_p(x_1, \cdots, x_n) = (E_p)^{-1}(x_1, ..., x_{n-1}, x_n + \psi_p(x_1, ..., x_{n-1})) + p.$$

Then, for sufficiently small $r > 0$, Ψ_p is a diffeomorphism between $B(0, r) \cap \mathbb{R}^n_+$ and a neighborhood of p in Ω. Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^4-smooth boundary $\partial\Omega$. Suppose that $f \in C^1$ satisfies conditions (f1)-(f3), and that (H1) - (H3) hold. Then for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε of (1) with a maximum point $z_\varepsilon \in \partial\Omega$ of u_ε such that there exist constants $C, c > 0$, independent of small $\varepsilon > 0$, satisfying $u_\varepsilon(x) \le C\exp(-\frac{c}{\varepsilon}|x - z_\varepsilon|)$ and*

$$\lim_{\varepsilon\to 0} dist(z_\varepsilon, \mathcal{M}) = 0.$$

Moreover, a transformed solution $w_\varepsilon(x) \equiv u_\varepsilon(\Psi_{z_\varepsilon}(\varepsilon x))$, $x \in B(0, r/\varepsilon) \cap \mathbb{R}^n_+$, converges, up to a subsequence, uniformly to a radially symmetric least energy solution w of (2).

This paper is organized as follows. In Section 2, we prepare some preliminaries containing coordinate transforms used for transplantation flows. In Section 3, we define a set of approximate solutions and the center of mass. In Section 4, we define a new neighborhood of the approximate solution set. In Section 5, we define a tail-minimizing operator which does not increase the energy and makes functions decay exponentially away from the center of mass. In Section 6, we construct an initial surface and we get an upper energy estimate. In Section 7, we get a lower estimate of the gradient norm of the energy functional in a tubular set. In Section 8, we introduce a transplantation flow which does not increase the energy. Finally, in Section 9, we prove the main theorem. As in [10], we use an iteration argument involving the gradient flow, the tail-minimizing operator and the transplantation flow.

2. **Preliminaries.** From now on, we suppose that conditions (f1)-(f3) and (H1)-(H3) hold. For any given set A in $\mathbb{R}^n$ and positive numbers $\varepsilon > 0, c > 0$, we define $A_\varepsilon \equiv \{x \in \mathbb{R}^n \mid \varepsilon x \in A\}$ and $A^c \equiv \{x \in \mathbb{R}^n \mid \inf_{y\in A} |x-y| < c\}$. By the change of variable, the problem (1) is equivalent to the equation:

$$\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon. \tag{3}$$

For $u \in H^1(\Omega_\varepsilon)$, we define its norm $\|u\|_\varepsilon \equiv (\int_{\Omega_\varepsilon} |\nabla u|^2 + u^2 dx)^{1/2}$ and an energy functional

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + u^2 dx - \int_{\Omega_\varepsilon} F(u)dx.$$

Then, if (f1)-(f3) hold, $\Gamma_\varepsilon \in C^1(H^1(\Omega_\varepsilon))$ and a critical point of Γ_ε is a solution of (3). For any set $A \subset H^1$ and $\delta > 0$, we denote the δ-neighborhood of A in H^1 by $N_\delta(A)$. For any $c_1, c_2 \in \mathbb{R}$, we define

$$(\Gamma_\varepsilon)^{c_1} \equiv \{u \in H^1(\Omega_\varepsilon) \mid \Gamma_\varepsilon(u) \le c_1\}, (\Gamma_\varepsilon)_{c_2} \equiv \{u \in H^1(\Omega_\varepsilon) \mid \Gamma_\varepsilon(u) \ge c_2\},$$

and

$$(\Gamma_\varepsilon)^{c_1}_{c_2} \equiv (\Gamma_\varepsilon)^{c_1} \cap (\Gamma_\varepsilon)_{c_2}.$$

2.1. **Gradient flow of the mean curvature on $\partial\Omega$.** We take any small $q \in (0, d_0)$ such that $\mathcal{M}^{20q} \subset \mathcal{M}^{d_0} \subset O, L_0 \cap \mathcal{M}^{20q} = \emptyset$. By condition (H2), for each small $q > 0$, there exists a neighborhood $\mathcal{N} \subset \partial\Omega$ of $\mathcal{M}$ with a smooth boundary such that $\mathcal{N} \subset \mathcal{M}^q$ and $|\nabla H(x)| > 0$ for $x \in \partial\mathcal{N}$. We take $d \in (0, q)$ such that $|\nabla H(x)| > 0$ for $x \in \mathcal{N}^{12d} \setminus \mathcal{N}$.

We find a C^1 function φ on $\partial\Omega$ such that $\varphi(X) = 1$ in $\mathcal{N}^{9d}$ and $\varphi(X) = 0$ in $O \setminus \mathcal{N}^{10d}$ and consider the following ODE

$$\frac{d\Phi}{dt}(t, X) = -\varphi(\Phi(t, X))\nabla H(\Phi(t, X)), \quad \Phi(0, X) = X. \tag{4}$$

Proposition 1. *There exists a global solution $\Phi : [0, \infty) \times O \to O$ of (4) such that for some $\alpha, \mu, c_1 > 0$,*

(i): *for each $X \in O$, $H(\Phi(\cdot, X))$ is nonincreasing on $[0, \infty)$;*
(ii): *$\Phi(t, X) = X$ if $t = 0$ or $X \in O \setminus \mathcal{N}^{10d} \supset L_0$;*
(iii): *$t \in [0, \mu]$ and $X \in (\mathcal{N}^{8d} \setminus \mathcal{N}^d)$,*

$$H(\Phi(t, X)) \le H(X) - \alpha t.$$

Proof. Since $\partial\Omega \in C^4$, there exists a global solution of (4). Since Φ satisfies the equation (4), (i) and (ii) are obvious. There exists $\mu > 0$ such that for $X \in (\mathcal{N}^{8d} \setminus \mathcal{N}^d)$ and $t \in [0, \mu]$, $\Phi(t, X) \in (\mathcal{N}^{9d} \setminus \mathcal{N})$. Moreover, there exists a constant $\alpha > 0$, such that $|\nabla H(x)| \geq \sqrt{\alpha}$ if $X \in (\mathcal{N}^{9d} \setminus \mathcal{N})$. This implies that for $t \in [0, \mu]$ and $X \in (\mathcal{N}^{8d} \setminus \mathcal{N}^d)$,

$$\frac{dH(\Phi(t,X))}{dt} = -\varphi(\Phi(t,X))|\nabla H(\Phi(t,X))|^2 \leq -\alpha.$$

This implies (iii). ☐

2.2. **A set of approximate solutions and a truncation of the nonlinearity.** We define an energy functional related to the limit problem (2) by

$$\Gamma(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 dx - \int_{\mathbb{R}^n} F(u)dx, \;\; u \in H^1(\mathbb{R}^n).$$

For $u \in H^1(\mathbb{R}^n)$, we define the norm $\|u\| \equiv (\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 dx)^{1/2}$. A nonzero critical point of Γ is a solution of

$$\Delta U - U + f(U) = 0, \; U > 0 \;\text{ in }\; \mathbb{R}^n, \; \lim_{|x|\to\infty} U(x) = 0. \tag{5}$$

We define $H^1_r(\mathbb{R}^n) = \{u \in H^1(\mathbb{R}^n) \mid u(gx) = u(x), g \in O(n)\}$. Berestycki and Lions proved in [4] that there exists a least energy solution of the equation (2) in $\mathbb{R}^n$ if (f1)-(f3) are satisfied. They also proved that each solution U satisfies Pohozaev's identity

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla U|^2 dx + n\int_{\mathbb{R}^n} \frac{U^2}{2} - F(U)dx = 0. \tag{6}$$

By the symmetry result in [19], any least energy solution in $H^1(\mathbb{R}^n)$ of (2) is, up to a translation, radially symmetric and monotone decreasing with respect to $|x|$. Let $\mathcal{S}$ be the set of least energy solutions $U \in H^1_r(\mathbb{R}^n)$ of (5). The following result is well known [7, Proposition 1].

Proposition 2. *The set $\mathcal{S}$ is compact and there exist $C, c > 0$ such that for any $U \in \mathcal{S}$,*

$$U(x) + |\nabla U(x)| \leq C\exp(-c|x|), \quad x \in \mathbb{R}^n.$$

Now we define a set of approximate solutions. We find a smooth radially symmetric function $\phi_\varepsilon \in C^\infty_0(\mathbb{R}^n)$ such that

$$\phi_\varepsilon(x) = 1 \text{ for } |x| \leq 1/2\varepsilon^{1/3}, \; \phi_\varepsilon(x) = 0 \text{ for } |x| \geq 1/\varepsilon^{1/3} \;\text{ and }\; |\nabla\phi_\varepsilon| \leq 3\varepsilon^{-1/3}. \tag{7}$$

We define

$$\mathcal{Z}^{10d}_\varepsilon \equiv \{\phi_\varepsilon(\cdot - \frac{z}{\varepsilon})U(\cdot - \frac{z}{\varepsilon}) \in H^1(\Omega_\varepsilon) \mid z \in \mathcal{N}^{10d}, U \in \mathcal{S}\}. \tag{8}$$

For $\delta > 0$, we consider a δ neighborhood $N_\delta(\mathcal{Z}^{10d}_\varepsilon)$ of $\mathcal{Z}^{10d}_\varepsilon$ in $H^1(\Omega_\varepsilon)$.

We see from L^∞-norm estimates ([5, Proposition 3.5]) that there exists a large constant $K > 0$ such that for any solution $u_\varepsilon \in N_\delta(\mathcal{Z}^{10d}_\varepsilon)$ of (3),

$$\|u_\varepsilon\|_{L^\infty} \leq K \quad \text{for all small } \varepsilon > 0. \tag{9}$$

The constant K depends only on the constants $a, b > 0$ in (f2), p in (f2) and n (refer to [5, Proposition 3.5]). Then we can find $\tilde{f} \in C^1(\mathbb{R})$ such that $\tilde{f}(t) = f(t)$ for $t \leq 2K$, $\tilde{f}(t) = bt^p$ for $t \geq 3K$ and that $\tilde{f}$ satisfies the conditions (f1),(f2) and (f3) with the same constants. Thus for small $\varepsilon > 0$, any solution $u_\varepsilon \in N_\delta(\mathcal{Z}^{10d}_\varepsilon)$ of (3) with $\tilde{f}$ replacing f satisfies the original equation (3). Thus, we can assume without loss of generality that the nonlinear function f satisfies further that for some $C > 0$, $|f'(t)t| \le C(t + t^p), t \ge 0$.

2.3. **Coordinate transforms.** By (H3), there exist linearly independent C^3 tangent vector fields $\{e_1, \cdots, e_{n-1}\}$ of $\partial\Omega$ on $\mathcal{N}^{10d} \setminus \mathcal{N}$. By the Gram-Schmidt process, we may assume that the tangent vector fields $\{e_1, \cdots, e_{n-1}\}$ are orthonormal. Let e_n be the inward normal unit vector of $\partial\Omega$. Now we have a C^3 smooth orthonormal frame $(e_1^X, \cdots, e_n^X)$ on $\mathcal{N}^{10d} \setminus \mathcal{N}$ such that $e_1^X, \cdots, e_{n-1}^X \in T_X(\partial\Omega)$ and e_n^X is the inward normal unit vector of $\partial\Omega$ at X. We consider

$$E(X) = \begin{pmatrix} e_1^X & \cdots & e_n^X \end{pmatrix} = \begin{pmatrix} a_{11}^X & \cdots & a_{1n}^X \\ \vdots & \ddots & \vdots \\ a_{n1}^X & \cdots & a_{nn}^X \end{pmatrix}$$

where $e_i^X = (a_{1i}^X, \cdots, a_{ni}^X)$; for convenience's sake, we will sometimes write e_i^X as a column vector. We may assume that $\det(E(X)) = 1$; thus $E(X) \in SO(n)$. This also gives us an orthonormal frame E^ε on $\mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon \equiv \frac{1}{\varepsilon}\mathcal{N}^{10d} \setminus \frac{1}{\varepsilon}\mathcal{N}$ by $E^\varepsilon(X) = E(\varepsilon X), X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$. ; for simplicity of notation, we will sometimes denote E^ε without superscript. For each $X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$, we consider a new coordinate system $Y = Y_X = (y_1, \cdots, y_n)$ on $T_X\mathbb{R}^n$, which represents the point

$$E(X)Y + X = E(X)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + X$$

in the original coordinate system. We define the first $(n-1)$ components of y by y'. In other words, $y' = (y_1, \cdots, y_{n-1})$. Then a neighborhood of X in $\partial\Omega$ is expressed as a graph $y_n = \psi_X(y')$ in the coordinate system for $T_X\mathbb{R}^n$. We want to get some dependence of the map ψ_X with respect to X. We take any point $X_0 \in \mathcal{N}^{10d} \setminus \mathcal{N}$. By a translation and a rotation, we may assume $X_0 = 0$ and $E_{X_0} = I$. For $X \in \partial\Omega$ with any small $|X|$, the function $\psi_X(y_1, \cdots, y_{n-1}) = y_n$ is implicitly defined by

$$G(y_1, \cdots, y_n, X) \equiv \psi_0\left(\begin{pmatrix} a_{11}^X & \cdots & a_{1n}^X \\ \vdots & \ddots & \vdots \\ a_{(n-1)1}^X & \cdots & a_{(n-1)n}^X \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}\right) - \begin{pmatrix} a_{n1}^X & \cdots & a_{nn}^X \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} - x_n = 0.$$

Note that for $i = 1, \cdots, n$,

$$\frac{\partial G}{\partial y_i} = \nabla\psi_0(V)\begin{pmatrix} a_{1i}^X \\ \vdots \\ a_{(n-1)i}^X \end{pmatrix} - a_{ni}^X,$$

where $V = \begin{pmatrix} a_{11}^X & \cdots & a_{1n}^X \\ \vdots & \ddots & \vdots \\ a_{(n-1)1}^X & \cdots & a_{(n-1)n}^X \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$.

Since $a_{nn}^0 = 1$, there exists $s_1 > 0$ such that $\frac{\partial G}{\partial y_n}(0, X) \ne 0$ for $X \in \partial\Omega \cap B(0, s_1)$. By the implicit function theorem, there exists $s_2 > 0$ such that for $X \in \partial\Omega \cap B(0, s_1)$, there exists a C^3-function ψ_X on $\{(y_1, \cdots, y_{n-1}) \mid |(y_1, \cdots, y_{n-1})| < s_2\}$ satisfying

$$G(y_1, \cdots, y_{n-1}, \psi_X(y_1, \cdots, y_{n-1}), X) = 0 \text{ for } |X| < s_1, |(y_1, \cdots, y_{n-1})| < s_2.$$

Then, a graph $y_n = \psi_X(y')$ for $|y'| < s_2$ represents $\partial\Omega$ near X with respect to the coordinate system $\{e_1^X, \cdots, e_{n-1}^X, e_n^X\}$ for $T_X\mathbb{R}^n$. By the implicit function theorem, for $i = 1, \cdots, n-1$,

$$\frac{\partial \psi_X}{\partial y_i} = -\frac{\sum_{j=1}^{n-1} D_j\psi_0(V)a_{ji}^X - a_{ni}^X}{\sum_{j=1}^{n-1} D_j\psi_0(V)a_{jn}^X - a_{nn}^X}. \tag{10}$$

Taking a sufficiently small $r > 0$, we may assume that for some $a \in (0,1)$ and $M > 0$,

$$|\nabla\psi_X(y')| \le a, |D^2\psi_X(y')| \le M, |D^3\psi_X(y')| \le M \text{ if } |X| < r, y' \in B^{n-1}(0,r). \tag{11}$$

Since $\frac{\partial \psi_X(0)}{\partial y_i} = 0$ for each $X \in B^n(0,r) \cap \partial\Omega$, we see that for $i \in \{1, \cdots, n-1\}$,

$$\sum_{j=1}^{n-1} D_j\psi_0(x_1, \cdots, x_{n-1})a_{ji}^X - a_{ni}^X = 0. \tag{12}$$

We may assume that the radius $r > 0$ and the constants a, M are independent of $X_0 \in \mathcal{N}^{10d} \setminus \mathcal{N}$. We define a coordinate transform, $\Psi_X : \mathbb{R}_+^n \cap B^n(0,r) \to \Omega$ by

$$\Psi_X(y_1, \cdots, y_n) = E(X)\begin{pmatrix} y_1 \\ \vdots \\ y_n + \psi_X(y') \end{pmatrix} + X.$$

The transform $z = \Psi_X(y)$ satisfies

$$D_y\Psi_X(y) = E(X)\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ D_{y_1}\psi_X(y') & \cdot & \cdots & D_{y_{n-1}}\psi_X(y') & 1 \end{pmatrix}; \tag{13}$$

thus

$$\det D_y\Psi_X(y) = 1 \text{ for any } X \in \mathcal{N}^{10d} \setminus \mathcal{N}, y \in \mathbb{R}_+^n \cap B^n(0,r). \tag{14}$$

We note that Ψ_X^{-1} is expressed by the relation

$$\Psi_X^{-1}((z_1, \cdots, z_n)) = E(X)^{-1}\begin{pmatrix} z_1 - x_1 \\ \vdots \\ z_n - x_n \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ \psi_X(y') \end{pmatrix},$$

where y' is the first $(n-1)$ components of $\Psi_X^{-1}(z_1, \cdots, z_n)$. From the bound of $|\nabla\psi_X|$, we see that

$$|\Psi_X(y) - X| \le (1+a)|y| \quad \text{for} \quad y \in \mathbb{R}_+^n \cap B^n(0,r)$$

and

$$|\Psi_X^{-1}(z)| \le \frac{1}{1-a}|z - X| \quad \text{for} \quad z \in \Psi_X(\mathbb{R}_+^n \cap B^n(0,r)).$$

Thus, for $0 < R' < R < R'' \le r$ satisfying $R''/R \ge 1 + a$ and $R/R' \ge 1/(1-a)$,

$$B^n(X, R') \cap \Omega \subset \Psi_X(B^n(0,R) \cap \mathbb{R}_+^n) \subset B^n(X, R'') \cap \Omega. \tag{15}$$

Now we will see Lipschitz continuity of $\psi_X(y_1, \cdots, y_{n-1})$ with respect to X. We consider $\psi_X(y_1, \cdots, y_{n-1})$ as a function of a variable X for a fixed $y' = (y_1, \cdots, y_{n-1}) \in B^{n-1}(0, r)$. Consider a smooth curve $X(t)$ on $\partial\Omega \cap B(0, r)$ and $|\frac{dX}{dt}(t)| = |(\dot{x}_1(t), \cdots, \dot{x}_n(t))| = 1$. We define

$$\frac{d}{dt} a_{ij}^{X(t)} \equiv \dot{a}_{ij}^{X(t)} = \dot{a}_{ij}^{X} \quad \text{for } 1 \leq i, j \leq n.$$

Since $\psi_0(x_1, \cdots, x_{n-1}) = x_n$, it follows that

$$\nabla\psi_0(x_1(t), \cdots, x_{n-1}(t)) \cdot (\dot{x}_1(t), \cdots, \dot{x}_{n-1}(t)) - \dot{x}_n(t) = 0.$$

Now, we see from above identity that

$$\begin{aligned}
0 =& \frac{d}{dt} G(y_1, \cdots, y_n, X(t)) \\
=& \nabla\psi_0(V) \left(\begin{pmatrix} \dot{a}_{11}^X & \cdots & \dot{a}_{1n}^X \\ \vdots & \ddots & \vdots \\ \dot{a}_{(n-1)1}^X & \cdots & \dot{a}_{(n-1)n}^X \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ \psi_X(y') \end{pmatrix} + \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{n-1}(t) \end{pmatrix} \right) \\
&+ \nabla\psi_0(V) \left(\begin{pmatrix} a_{11}^X & \cdots & a_{1n}^X \\ \vdots & \ddots & \vdots \\ a_{(n-1)1}^X & \cdots & a_{(n-1)n}^X \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{d}{dt}\psi_{X(t)}(y') \end{pmatrix} \right) \\
&- \begin{pmatrix} \dot{a}_{n1}^X & \cdots & \dot{a}_{n(n-1)}^X \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} - \dot{a}_{nn}^X \psi_{X(t)}(y') - a_{nn}^X \frac{d}{dt}\psi_{X(t)}(y') - \dot{x}_n
\end{aligned}$$

$$\begin{aligned}
=& \nabla\psi_0(V) \left(\begin{pmatrix} \dot{a}_{11}^X & \cdots & \dot{a}_{1n}^X \\ \vdots & \ddots & \vdots \\ \dot{a}_{(n-1)1}^X & \cdots & \dot{a}_{(n-1)n}^X \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ \psi_X(y') \end{pmatrix} \right) \\
&+ \Big(\nabla\psi_0(V) - \nabla\psi_0(x_1, \cdots, x_{n-1}) \Big) \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{n-1}(t) \end{pmatrix} \\
&- \begin{pmatrix} \dot{a}_{n1}^X & \cdots & \dot{a}_{n(n-1)}^X \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} - \dot{a}_{nn}^X \psi_{X(t)}(y') \\
&+ \left(\nabla\psi_0(V) \begin{pmatrix} a_{1n}^X \\ \vdots \\ a_{(n-1)n}^X \end{pmatrix} - a_{nn}^X \right) \frac{d}{dt}\psi_{X(t)}(y') \equiv I + II + III + IV.
\end{aligned}$$

Taking a smaller $r > 0$ if it is necessary, we see that $\left|\nabla\psi_0(V)\begin{pmatrix} a^X_{1n} \\ \vdots \\ a^X_{(n-1)n}\end{pmatrix} - a^X_{nn}\right| \geq$ 1/2. Since $I, II, III = O(|y'|)$, we conclude that

$$\left|\frac{d\psi_{X(t)}(y')}{dt}\right| = O(|y'|). \tag{16}$$

Taking a smaller $r > 0$ further if it is necessary, we see that for $|y'| < r$,

$$|\nabla_X \psi_X(y')| \leq 1. \tag{17}$$

Since $\overline{\mathcal{N}^{10d} \setminus \mathcal{N}}$ is compact, we may assume that above inequality holds for any $X \in \overline{\mathcal{N}^{10d} \setminus \mathcal{N}}$ and $y' \in B^{n-1}(0, r)$.

We define

$$G_i(X, y') \equiv \frac{\partial \psi_X}{\partial y_i}(y') \quad \text{on} \quad (\overline{B^n(0,r)} \cap \partial\Omega) \times \overline{B^{n-1}(0,r)} \quad \text{for } i = 1, \cdots, n-1.$$

Then, we can see from (10) that $G_i(X, y')$ is a C^2-smooth function with respect to y' and X. Note that

$$\frac{\partial \psi_X}{\partial y_i}(y') = \sum_{k=1}^{n-1} \frac{\partial G_i}{\partial y_k}(X, 0) y_k + R_X(y') \equiv -A^i_X y' + R_X(y'),$$

where A^i_X is the i-th column of the second fundamental form of $\partial\Omega$ at $X \in \partial\Omega$ and $R_X(y') = O(|y'|^2)$ is the remainder term. Let $X(t)$ be a curve in $B^n(0, r) \cap \partial\Omega$ satisfying $|\frac{dX}{dt}(t)| = |(\dot{x}_1(t), \cdots, \dot{x}_n(t))| = 1$. From (10) and (12), we see that

$$\frac{\partial \psi_X}{\partial y_i} = -\frac{\sum_{j=1}^{n-1} \big(D_j\psi_0(V) - D_j\psi_0(x_1, \cdots, x_{n-1})\big) a^X_{ji}}{\sum_{j=1}^{n-1} D_j\psi_0(V) a^X_{jn} - a^X_{nn}} \equiv \frac{E}{F}. \tag{18}$$

We define

$$E_{ji}(X, y') \equiv \big(D_j\psi_0(V) - D_j\psi_0(x_1, \cdots, x_{n-1})\big) a^X_{ji}.$$

Then, we see that

$$\begin{aligned}
&\frac{dE_{ji}(X(t), y')}{dt} \\
=&\nabla(\frac{\partial \psi_0}{\partial y_j})(V)\left(\begin{pmatrix} \dot{a}^X_{11} & \cdots & \dot{a}^X_{1n} \\ \vdots & & \vdots \\ \dot{a}^X_{(n-1)1} & \cdots & \dot{a}^X_{(n-1)n}\end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ \psi_X(y')\end{pmatrix}\right) a^X_{ji} \\
&+ \nabla(\frac{\partial \psi_0}{\partial y_j})(V)\left(\begin{pmatrix} a^X_{11} & \cdots & a^X_{1n} \\ \vdots & & \vdots \\ a^X_{(n-1)1} & \cdots & a^X_{(n-1)n}\end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{d}{dt}\psi_{X(t)}(y')\end{pmatrix}\right) a^X_{ji} \\
&+ \Big(\frac{\partial \psi_0}{\partial y_j}(V) - \frac{\partial \psi_0}{\partial y_j}(x_1, \cdots, x_{n-1})\Big)\dot{a}^X_{ji} \\
&+ \Big(\nabla(\frac{\partial \psi_0}{\partial y_j})(V) - \nabla(\frac{\partial \psi_0}{\partial y_j})(x_1, \cdots, x_{n-1})\Big)\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1}\end{pmatrix} a^X_{ji}
\end{aligned}$$

$$\equiv I + II + III + IV.$$

It is obvious that $I, III, IV = O(|y'|)$ and

$$\sum_{j=1}^{n-1} \big(D_j\psi_0(V) - D_j\psi_0(x_1, \cdots, x_{n-1})\big)a_{ji}^X = O(|y'|).$$

We see from (16) that $II = O(|y'|)$. Since $\frac{d}{dt}G_i(X(t), y') = \frac{\dot{E}F - E\dot{F}}{F^2}$, it follows that

$$\Big|\frac{dG_i(X(t), y')}{dt}\Big| = O(|y'|);\ \text{thus, for } i = 1, \cdots, (n-1),\quad \Big|\nabla_X \frac{\partial \psi_X}{\partial y_i}(y')\Big| = O(|y'|), \tag{19}$$

As for the remainder term, we can change the order of differentiation and we see that

$$\frac{d}{dt}R_{X(t)}(y') = \frac{d}{dt}\Big[G_i(X(t), y') - \sum_{k=1}^{n-1} \frac{\partial G_i}{\partial y_k}(X(t), 0)y_k\Big] = o(|y'|).$$

Thus, it follows that

$$|\nabla_X R_X(y')| = o(|y'|). \tag{20}$$

From the compactness of $\overline{\mathcal{N}^{10d} \setminus \mathcal{N}}$, we see that (11), (16) (19) and (20) hold uniformly for $X \in \overline{\mathcal{N}^{10d} \setminus \mathcal{N}^d}$.

Now, we consider $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. As far as there are no confusions, for small $\varepsilon > 0$, we use the same letters for the variables with the case $\varepsilon = 1$. For small $\varepsilon > 0$ and $X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$, the boundary points of Ω_ε near X is expressed by the graph $\psi_{\varepsilon,X} : B^{n-1}(0, r/\varepsilon) \to \mathbb{R}$ defined by

$$\psi_{\varepsilon,X}(y_1, \cdots, y_{n-1}) = \frac{1}{\varepsilon}\psi_{\varepsilon X}(\varepsilon y_1, \cdots, \varepsilon y_{n-1}). \tag{21}$$

We define a coordinate transform $\Psi_{\varepsilon,X} : \mathbb{R}_+^n \cap B^n(0, r/\varepsilon) \to \Omega_\varepsilon$ by

$$\Psi_{\varepsilon,X}(y_1, \cdots, y_n) = \frac{1}{\varepsilon}\Psi_{\varepsilon X}(\varepsilon y_1, \cdots, \varepsilon y_n). \tag{22}$$

Then we see that

$$D_y\Psi_{\varepsilon,X}(y) = E(\varepsilon X)\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ D_1\psi_{\varepsilon X}(\varepsilon y') & \cdot & \cdots & D_{n-1}\psi_{\varepsilon X}(\varepsilon y') & 1 \end{pmatrix}, \tag{23}$$

$$|\nabla\psi_{\varepsilon,X}| \le \varepsilon O(|y'|) \le a,\ |D^2\psi_{\varepsilon,X}| \le \varepsilon M,\ |D^3\psi_{\varepsilon,X}| \le \varepsilon^2 M \text{ on } B^{n-1}(0, r/\varepsilon), \tag{24}$$

where M is the bound in (11) and a can be chosen arbitrary small for small $r > 0$. Estimates (16) and (17) imply that for $|y'| < r/\varepsilon$,

$$|\nabla_X\psi_{\varepsilon,X}(y')| = \varepsilon O(|y'|). \tag{25}$$

and

$$|\nabla_X\psi_{\varepsilon,X}(y')| \le 1. \tag{26}$$

Estimate (19) is transformed to

$$\Big|\nabla_X \frac{\partial\psi_{\varepsilon,X}}{\partial y_i}(y')\Big| = \varepsilon^2 O(|y'|),\quad i = 1, \cdots, n-1. \tag{27}$$

The results for Ψ_X are transformed to the following results for $\Psi_{\varepsilon,X}$:

$$\det D_y\Psi_{\varepsilon,X}(y) = 1 \text{ for any } X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon,\ y \in \mathbb{R}_+^n \cap B^n(0, r/\varepsilon). \tag{28}$$

$$|\Psi_{\varepsilon,X}(y) - X| \le (1+a)|y| \quad \text{for } X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon,\ y \in \mathbb{R}^n_+ \cap B^n(0, r/\varepsilon) \tag{29}$$

and

$$|\Psi^{-1}_{\varepsilon,X}(x)| \le \frac{1}{1-a}|x - X| \quad \text{for} \quad x \in \Psi_{\varepsilon,X}(\mathbb{R}^n_+ \cap B^n(0, r/\varepsilon)). \tag{30}$$

Thus, for $0 < R' < R < R'' \le r$ satisfying $R''/R \ge 1 + a$ and $R/R' \ge 1/(1-a)$,

$$B^n(X, R'/\varepsilon) \cap \Omega_\varepsilon \subset \Psi_{\varepsilon,X}(B^n(0, R/\varepsilon) \cap \mathbb{R}^n_+) \subset B^n(X, R''/\varepsilon) \cap \Omega_\varepsilon. \tag{31}$$

We write

$$\frac{\partial \psi_{\varepsilon,X}}{\partial y_i}(y') = \sum_{k=1}^{n-1} \frac{\partial^2 \psi_{\varepsilon,X}(0)}{\partial y_i \partial y_k} y_k + R_{\varepsilon,X}(y') \equiv -A^i_{\varepsilon,X} y' + R_{\varepsilon,X}(y'),$$

where $A^i_{\varepsilon,X}$ is the i-th column of the second fundamental form of $\partial\Omega_\varepsilon$ at $X \in \partial\Omega_\varepsilon$ and $R_{\varepsilon,X}$ is the remainder term. Then, it follows that

$$|R_{\varepsilon,X}(y')| = \varepsilon^2 O(|y'|^2) \text{ and } |\nabla_X R_{\varepsilon,X}(y')| = \varepsilon o(\varepsilon|y'|). \tag{32}$$

Summarizing what we have proved, we get the following proposition.

Proposition 3. *Suppose that condition (H3) holds. Then for a small $q > 0$ given in (H2) and each small $d \in (0, q)$, there exist constants of $r, M > 0$ such that for a C^3 smooth orthonormal frame $X \in \mathcal{N}_\varepsilon^{d_0} \setminus \mathcal{N}_\varepsilon \mapsto E(X) = (e_1^X, \cdots, e_n^X) \in SO(n)$ satisfying that $e_1^X, \cdots, e_{n-1}^X \in T_X(\partial\Omega_\varepsilon)$ and e_n^X is the inner-normal unit vector of Ω_ε at X, there is a function $\psi_{\varepsilon,X}$ representing $\partial\Omega_\varepsilon$ as a graph on $B^{n-1}(0, r/\varepsilon) \subset T_X\partial\Omega_\varepsilon$ such that for each $X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$, a function $\psi_{\varepsilon X}$ given by (21) represents $\partial\Omega$ as a graph on $B^{n-1}(0, r) \subset T_{\varepsilon X}\partial\Omega$ and the coordinate transform map $\Psi_{\varepsilon,X}$: $\mathbb{R}^n_+ \cap B^n(0, r/\varepsilon) \to \Omega_\varepsilon$ given by*

$$\Psi_{\varepsilon,X}(y_1, \cdots, y_n) = E(X) \begin{pmatrix} y_1 \\ \vdots \\ y_n + \psi_{\varepsilon,X}(y') \end{pmatrix} + X$$

satisfying the following properties
(i) the Jacobian determinant of $\Psi_{\varepsilon,X}$ is 1;
(ii) $|\nabla \psi_{\varepsilon,X}| < a < 1/1000$, $|D^2\psi_{\varepsilon,X}| \le \varepsilon M, |D^3\psi_{\varepsilon,X}| \le \varepsilon^2 M$ in $B^{n-1}(0, r/\varepsilon)$,
$|\nabla_X \psi_{\varepsilon,X}| \le \min\{1, \varepsilon O(|y'|)\}$, $\left|\nabla_X \frac{\partial \psi_{\varepsilon,X}}{\partial y_i}(y')\right| = \varepsilon^2 O(|y'|), \quad i = 1, \cdots, n-1;$
(iii)

$$|\Psi_{\varepsilon,X}(y) - X| \le (1+a)|y| \text{ for } y \in \mathbb{R}^n_+ \cap B^n(0, r/\varepsilon),$$
$$|\Psi^{-1}_{\varepsilon,X}(x)| \le \frac{1}{1-a}|x - X| \text{ for } x \in \Psi_{\varepsilon,X}(\mathbb{R}^n_+ \cap B^n(0, r/\varepsilon))$$

and for $0 < R' < R < R'' \le r$ with $R/R' \ge 1/(1-a), R''/R \ge 1 + a$,

$$B^n(X, R'/\varepsilon) \cap \Omega_\varepsilon \subset \Psi_{\varepsilon,X}(\mathbb{R}^n_+ \cap B^n(0, R/\varepsilon)) \subset B^n(X, R''/\varepsilon) \cap \Omega_\varepsilon;$$

(iv) $\frac{\partial \psi_{\varepsilon,X}}{\partial y_i}(y') = -A^i_{\varepsilon,X} y' + R_{\varepsilon,X}(y')$ where $A^i_{\varepsilon,X}$ is the i-th column of the second fundamental form of $\partial\Omega_\varepsilon$ at X and $R_{\varepsilon,X}(y')$ is a remainder term satisfying $|R_{\varepsilon,X}(y')| = \varepsilon^2 O(|y'|^2)$ and $|\nabla_X R_{\varepsilon,X}(y')| = \varepsilon o(\varepsilon|y'|)$.

Remark 1. Note that $\partial\Omega$ admits locally smooth orthonormal frame on $T\partial\Omega$. Thus, since $\partial\Omega$ is compact, there exists a constant $r_0 > 0$ such that for each $z_0 \in \partial\Omega$, there exist a C^3 smooth orthonormal frame $X \in B^n(z_0/\varepsilon, r_0/\varepsilon) \cap \Omega_\varepsilon \mapsto E(X) = (e_1^X, \cdots, e_n^X) \in SO(n)$ satisfying that $e_1^X, \cdots, e_{n-1}^X \in T_X(\partial\Omega_\varepsilon)$ and e_n^X is the inner-normal unit vector of Ω_ε at X. Then, by the same arguments with the proof of

Proposition 3, there exists a constant $r > 0$ such that the properties (i)-(iv) of Proposition 3 with maps $\psi_{\varepsilon,X}(y')$, $\Psi_{\varepsilon,X}(y)$ defined for $X \in B(z_0/\varepsilon, r_0/\varepsilon) \cap \partial\Omega_\varepsilon$, $y' \in B^{n-1}(0, r/\varepsilon)$ and $y \in B^n(0, r/\varepsilon) \cap \mathbb{R}^n_+$. This will be used to characterize a concentration point of a spike layer solution in Proposition 19.

Remark 2. We claim that for a continuous curve $p : [0,1] \to \partial\Omega$, there there exists an orthonormal frame $(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)})$ on $TN_{q(s)}$ which is continuous with respect to $s \in [0,1]$.

In fact, any point $X_0 \in \partial\Omega$ has a small neighborhood N_{X_0} such that condition (H3) holds, that is, there exists a C^3 smooth orthonormal frame $(e_1^X, \cdots, e_{n-1}^X)$ on TN_{X_0} for $X \in N_{X_0}$. This implies that there exists a positive integer $k > 0$ such that for each $i \in \{0, \cdots, k-1\}$, there exists a continuous orthonormal frame $(e_{1,i}^{q(s)}, \cdots, e_{n-1,i}^{q(s)})$ on $TN_{q(s)}$ for $s \in [i/k, (i+1)/k]$.

For $s \in [0, 1/k]$, define $(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)}) = (e_{1,0}^{q(1/k)}, \cdots, e_{n-1,0}^{q(1/k)})$. Let $A_1(1/k)$ be an orthogonal transformation on $T_{q(1/k)}(\partial\Omega)$ such that

$$(e_1^{q(1/k)}, \cdots, e_{n-1}^{q(1/k)}) = A_1(1/k)(e_{1,1}^{q(1/k)}, \cdots, e_{n-1,1}^{q(1/k)}).$$

Let $A_1 \in O(n-1)$ be the matrix representation of the transformation $A_1(1/k)$ with respect to a basis $(e_{1,1}^{q(1/k)}, \cdots, e_{n-1,1}^{q(1/k)})$. Define $A_1(s)$ as the orthogonal transformation on $T_{q(s)}(\partial\Omega)$ which is represented by the matrix A_1 with respect to a basis $(e_{1,1}^{q(s)}, \cdots, e_{n-1,1}^{q(s)})$, $s \in [1/k, 2/k]$. Now we define an orthonormal frame $(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)})$ on $TN_{q(s)}$ for $s \in [1/k, 2/k]$ as follows:

$$(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)}) = A_1(s)(e_{1,1}^{q(s)}, \cdots, e_{n-1,1}^{q(s)}), \quad s \in [1/k, 2/k].$$

For $i = 2, ..., (k-1)$, we define $(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)})$ on $TN_{q(s)}$ for $s \in [i/k, (i+1)/k]$, inductively. Then, we get an orthonormal frame $(e_1^{q(s)}, \cdots, e_{n-1}^{q(s)})$ on $TN_{q(s)}$ which is continuous with respect to $s \in [0,1]$.

Now let $e_n^{q(s)}$ be the inward normal unit vector of $\partial\Omega$ at $q(s)$ and define

$$E(q(s)) = (e_1^{q(s)}, \cdots, e_n^{q(s)}).$$

Then, we have maps $\psi_{\varepsilon,q(s)}$ and $\Psi_{\varepsilon,q(s)}$ satisfying properties (i)-(iv) in Proposition 3 except the properties involving the derivatives with respect to $X = q(s)$. We will use these properties to get a lower bound in Proposition 18.

Lemma 2.1. *Suppose that condition (H3) holds. For $X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$, let $z \in \partial\Omega_\varepsilon$ with $|z - X| < s$ for some constant $s > 0$. If $|y - \Psi_{\varepsilon,X}^{-1}(z)| \le c/\sqrt[3]{\varepsilon}$, then for small $\varepsilon > 0$,*

$$\left|(\Psi_{\varepsilon,X}(y) - z) - E(X)(y - \Psi_{\varepsilon,X}^{-1}(z))\right| \le O(\sqrt[3]{\varepsilon}).$$

Proof. By (iii) in Proposition 3, $|\Psi_{\varepsilon,X}^{-1}(z)| \le \frac{s}{1-a}$. Then, if $|y - \Psi_{\varepsilon,X}^{-1}(z)| \le c/\sqrt[3]{\varepsilon}$, it holds that for small $\varepsilon > 0$, $|y| \le 2c/\sqrt[3]{\varepsilon}$. We write

$$\Psi_{\varepsilon,X}^{-1}(z) = ((\Psi_{\varepsilon,X}^{-1}(z))', \Psi_{\varepsilon,X}^{-1}(z)_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Then, we see that

$$\Psi_{\varepsilon,X}(y) - z = E(X) \begin{pmatrix} y_1 - (\Psi_{\varepsilon,X}^{-1}(z))_1 \\ \vdots \\ y_{n-1} - (\Psi_{\varepsilon,X}^{-1}(z))_{n-1} \\ y_n - (\Psi_{\varepsilon,X}^{-1}(z))_n + \psi_{\varepsilon,X}(y') - \psi_{\varepsilon,X}((\Psi_{\varepsilon,X}^{-1}(z))') \end{pmatrix}$$

Therefore, for small $\varepsilon > 0$,

$$
\begin{aligned}
&\Big|(\Psi_{\varepsilon,X}(y) - z) - E(X)(y - \Psi_{\varepsilon,X}^{-1}(z))\Big| \\
\le& |\psi_{\varepsilon,X}(y')| + |\psi_{\varepsilon,X}((\Psi_{\varepsilon,X}^{-1}(z))')| \\
\le& \frac{1}{\varepsilon}[M(\varepsilon|y'|)^2 + M(\varepsilon\frac{s}{1-a})^2] \le 5Mc^2\sqrt[3]{\varepsilon}.
\end{aligned}
$$

□

Proposition 4. *Suppose that condition (H3) holds. Let $U \in \mathcal{S}$ and ϕ_ε be the cut-off function in the definition of approximate solutions. Let $X \in \mathcal{N}_\varepsilon^{10d} \setminus \mathcal{N}_\varepsilon$ and $|z - X| < s$ for some constant $s > 0$. Then*

$$\|(\phi_\varepsilon U)(\cdot - z) \circ \Psi_{\varepsilon,X} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))\|_{H^1(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \le O(\sqrt[3]{\varepsilon}).$$

Proof. Because $\mathrm{supp}(\phi_\varepsilon) \subset B(0, 1/\varepsilon^{1/3})$, by (iii) in Proposition 3, the support of $(\phi_\varepsilon U)(\cdot - z) \circ \Psi_{\varepsilon,X}$ and the support of $(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))$ are contained in $B(0, 2/\varepsilon^{1/3}) \cap \mathbb{R}^n_+$ for small $\varepsilon > 0$. By Lemma 2.1, for $|y| \le 2/\sqrt[3]{\varepsilon}$,

$$\Big|(\Psi_{\varepsilon,X}(y) - z) - E(X)(y - \Psi_{\varepsilon,X}^{-1}(z))\Big| = O(\sqrt[3]{\varepsilon}),$$

We note that $\phi_\varepsilon U$ is radially symmetric. By the mean value theorem, there exists $\tilde{y} = t(\Psi_{\varepsilon,X}(y) - z) + (1-t)E(X)(y - \Psi_{\varepsilon,X}^{-1}(z))$ with $t \in [0,1]$ such that

$$\Big|(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(y) - z) - (\phi_\varepsilon U)(y - \Psi_{\varepsilon,X}^{-1}(z))\Big| \le |\nabla(\phi_\varepsilon U)(\tilde{y})| O(\sqrt[3]{\varepsilon}).$$

There exists $\hat{y} = t'(\Psi_{\varepsilon,X}(y) - z) + (1-t')E(X)(y - \Psi_{\varepsilon,X}^{-1}(z))$ with $t' \in [0,1]$ such that

$$\Big|\nabla(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(y) - z) - \nabla(\phi_\varepsilon U)(E(X)(y - \Psi_{\varepsilon,X}^{-1}(z)))\Big| \le |D^2(\phi_\varepsilon U)(\hat{y})| O(\sqrt[3]{\varepsilon}).$$

On the other hand, we see that in the support of $(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z)$,

$$D\Psi_{\varepsilon,X}(y) = E(X)(I + O(\varepsilon|y|)), \qquad |D\Psi_{\varepsilon,X} - E(X)| = O(\sqrt[3]{\varepsilon^2}).$$

We note that for a radially symmetric function h, $(\nabla h)(y)(E(X))^T = (\nabla h)(E(X)y)$. Therefore, by the decaying property of $D^\alpha U$ ($|\alpha| \le 2$), we can see that

$$
\begin{aligned}
&\|(\phi_\varepsilon U)(\cdot - z) \circ \Psi_{\varepsilon,X} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))\|_{H^1(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
=& \|(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z) - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
&+ \|\nabla(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z)E(X) - \nabla(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
&+ \|\nabla(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z)(D\Psi_{\varepsilon,X} - E(X))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
=& \|(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z) - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X}^{-1}(z))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
&+ \|\nabla(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z) - \nabla(\phi_\varepsilon U)(E(X)(\cdot - \Psi_{\varepsilon,X}^{-1}(z)))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
&+ \|\nabla(\phi_\varepsilon U)(\Psi_{\varepsilon,X}(\cdot) - z)(D\Psi_{\varepsilon,X} - E(X))\|_{L^2(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \\
\le& O(\sqrt[3]{\varepsilon}).
\end{aligned}
$$

This completes the proof. □

3. **Center of mass.** For $X \in \partial\Omega_\varepsilon$, we define

$$S(X) \equiv \{U_X \in C^2(\Omega_\varepsilon) \mid U_X(x) \equiv (\phi_\varepsilon U)(x-X), U \in \mathcal{S}\}.$$

We see that $\mathcal{Z}_\varepsilon^{10d} = \bigcup_{X \in \mathcal{N}_\varepsilon^{10d}} S(X)$ where $\mathcal{N}_\varepsilon^{10d} \equiv \frac{1}{\varepsilon}\mathcal{N}^{10d}$. For $\delta > 0$, we define

$$N_\delta(\mathcal{Z}_\varepsilon^{10d}) \equiv \{u \in H^1(\Omega_\varepsilon) \mid \inf_{v \in \mathcal{Z}_\varepsilon^{10d}} \|u - v\|_\varepsilon \le \delta\}.$$

When ε is sufficiently small, by the uniform exponential decay of least energy solutions, there exists $\xi > 0$ and $R_0 > 0$ such that, for all $u \in S(X), X \in \partial\Omega_\varepsilon$,

$$\|u\|_{H^1(B(X,R_0)\cap\Omega_\varepsilon)} \ge \xi, \quad \|u\|_{H^1(\Omega_\varepsilon \setminus B(X,R_0))} \le \xi/4. \tag{33}$$

Here, ξ and R_0 are independent of small ε. We take a small positive constant $\gamma > 0$ such that

$$\gamma < \xi/20 \tag{34}$$

and a function $\theta \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\theta(x) = 1$ for $|x| \le 2\gamma$ and $\theta(x) = 0$ for $|x| \ge 3\gamma$. We take a constant

$$\delta_0 \in (0, \gamma). \tag{35}$$

For $u \in H^1(\Omega_\varepsilon)$ and $X \in \partial\Omega_\varepsilon$, we define

$$d_\varepsilon(u, X) = \theta\left(\inf_{v \in S(X)} \|u - v\|_{H^1(B(X,R_0)\cap\Omega_\varepsilon)}\right).$$

The function d_ε is continuous with respect to X since the Lebesgue measure of a set $B(X, R_0) \triangle B(X', R_0)$ and $\|(\phi_\varepsilon U)(x - X) - (\phi_\varepsilon U)(x - X')\|_\varepsilon$ converge to zero as $X' \to X$.

Let $X, Y \in \partial\Omega_\varepsilon$ with $|X - Y| \ge 2R_0$. If $u \in N_{\delta_0}(S(Y))$, then

$$\inf_{v \in S(X)} \|u - v\|_{H^1(B(X,R_0)\cap\Omega_\varepsilon)} \ge \xi - \xi/4 - \delta_0 > 10\gamma; \tag{36}$$

thus $d_\varepsilon(u, X) = 0$ for $u \in N_{\delta_0}(S(Y))$. Then, for $u \in N_{\delta_0}(S(Y))$,

$$\frac{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u, z) z}{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u, z)}$$

is well defined and contained in the convex hull of $\partial\Omega_\varepsilon \cap B(Y, 2R_0)$.

For $u \in N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$, we define a center of mass $\Upsilon_\varepsilon(u)$ by

$$\Upsilon_\varepsilon(u) \equiv \pi_{\partial\Omega_\varepsilon}\left(\frac{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u, z) z}{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u, z)}\right), \tag{37}$$

where $\pi_{\partial\Omega_\varepsilon}(x)$ is the point on $\partial\Omega_\varepsilon$ which is closest to x among points of $\partial\Omega_\varepsilon$. For small $\varepsilon > 0$, $\pi_{\partial\Omega_\varepsilon}$ is well defined in $2R_0$-neighborhood of $\partial\Omega_\varepsilon$ and center of mass is well defined. Then Υ_ε is a continuous function in $N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$ and we have the following properties for small $\varepsilon > 0$.

Proposition 5. *(i) For $u = U_z + w \in N_{\delta_0}(S(z))$ with $U_z \in S(z)$ and $\|w\|_\varepsilon \le \delta_0$,*

$$|\Upsilon_\varepsilon(u) - z| \le 3R_0 \text{ if } \varepsilon > 0 \text{ is small.}$$

(ii) If $u, v \in N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$ satisfy

$$u(x) = v(x) \text{ in } \Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u), 8R_0),$$

then $\Upsilon_\varepsilon(u) = \Upsilon_\varepsilon(v)$.

Proof. Since we have proved that for $u \in N_{\delta_0}(S(Y))$,

$$\frac{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u,z)z}{\int_{\partial\Omega_\varepsilon} d_\varepsilon(u,z)}$$

is contained in the convex hull of $\partial\Omega_\varepsilon \cap B(Y, 2R_0)$, (i) follows.

For the proof of (ii), let $u \in N_{\delta_0}(S(X)), v \in N_{\delta_0}(S(Y))$. Then $|X - \Upsilon_\varepsilon(u)| \le 3R_0$ and $\|u\|_{H^1(B(\Upsilon_\varepsilon(u),4R_0)\cap\Omega_\varepsilon)} \ge \xi$. Thus it follows that $|Y - \Upsilon_\varepsilon(u)| \le 5R_0$. The support of $d_\varepsilon(v,\cdot)$ is contained in $B(Y, 2R_0)$. Moreover $d_\varepsilon(v,z) = d_\varepsilon(\tilde{v},z)$ if $v \equiv \tilde{v}$ in $\Omega_\varepsilon \cap B(z, R_0)$. Since $u \equiv v$ in $\Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u), 8R_0) \supset \Omega_\varepsilon \cap B(Y, 3R_0)$, $d_\varepsilon(v,\cdot) = d_\varepsilon(u,\cdot)$ and $\Upsilon_\varepsilon(u) = \Upsilon_\varepsilon(v)$. □

4. **Invariant neighborhoods.** For $\delta \in (0, \delta_0)$, $u \in N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$ and $v \in H^1(\Omega_\varepsilon)$, we define

$$|v|_{\delta,u} \equiv \int_{\Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla v|^2 + v^2 dx.$$

For $\delta \in (0, \delta_0)$ and $r \in (0, \delta)$, we define

$$G_r^\delta(\mathcal{Z}_\varepsilon^{10d}) \equiv \{u \in N_\delta(\mathcal{Z}_\varepsilon^{10d}) \mid |u - v_z|_{\delta,u} \le r^2/2 \text{ for some } z \in \mathcal{N}_\varepsilon^{10d}, v_z \in S(z)$$
$$\text{and } \int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 - 2F(u)dx \le r^2/2\}.$$

Proposition 6. *There exists a constant $q = q(\delta) > 0$ such that $\lim_{\delta\to 0} q(\delta) = 0$ uniformly for small $\varepsilon > 0$, and that for small $\varepsilon > 0$ and $u \in N_\delta(\mathcal{Z}_\varepsilon^{10d})$,*

$$(1-q)\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 dx \le \int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 - 2F(u)dx \tag{38}$$

$$(1+q)\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 dx \ge \int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 - 2F(u)dx. \tag{39}$$

Proof. Note that for any small $c > 0$, there exists $C > 0$ such that $|F(t)| \le ct^2 + Ct^{p+1}$. Thus, we see that

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |F(u)| dx \le \int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} cu^2 + Cu^{p+1} dx.$$

From the Sobolev embedding theorem, there exists $C' > 0$ such that

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} u^{p+1} dx \le C' \left(\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 dx \right)^{(p+1)/2}.$$

The constant C' is independent of small δ and small $\varepsilon > 0$ (refer to [1, Theorem 4.1]). Then, since above constant C' is independent of small δ and $\varepsilon > 0$, the inequalities (38) and (39) hold with $\lim_{\delta\to 0} q(\delta) = 0$. □

We choose $\delta > 0$ so small that $q(\delta)$ in Proposition 6 satisfies $q(\delta) < \sqrt{2} - 1$.

Proposition 7. *Let $c \in (0, 1]$, $c' \in (0, 1/2)$. If δ is small enough, then for small $\varepsilon > 0$,*

$$G_{(1-q)c\delta}^\delta(\mathcal{Z}_\varepsilon^{10d}) \subset N_{c\delta}(\mathcal{Z}_\varepsilon^{10d}) \quad \text{and} \quad N_{c'\delta}(\mathcal{Z}_\varepsilon^{10d}) \subset G_{(1+q)\sqrt{2}c'\delta}^\delta(\mathcal{Z}_\varepsilon^{10d}).$$

Proof. If $u \in G^{\delta}_{(1-q)c\delta}(\mathcal{Z}^{10d}_{\varepsilon})$, it follows from (38) that

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 dx \leq (1-q)(c\delta)^2/2. \tag{40}$$

Note that for some $z \in \mathcal{N}^{10d}_{\varepsilon}, v_z \in S(z)$,

$$\int_{\Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla(u-v_z)|^2 + (u-v_z)^2 dx \leq ((1-q)c\delta)^2/2. \tag{41}$$

We note that $u \in N_\delta(S(\tilde{z}))$ for some $\tilde{z} \in \mathcal{N}^{10d}_{\varepsilon}$. Then we see from Proposition 5 that $|\tilde{z} - \Upsilon_\varepsilon(u)| \leq 3R_0$. If $|z-\tilde{z}| \geq 2R_0$, it follows from (36) and (41) that for $1/\sqrt{\delta} \geq 4R_0$,

$$\|u - v_{\tilde{z}}\|_{H^1(\Omega_\varepsilon \cap B(\tilde{z},R_0))} \geq \|v_z - v_{\tilde{z}}\|_{H^1(\Omega_\varepsilon \cap B(\tilde{z},R_0))} - \|u - v_z\|_{H^1(\Omega_\varepsilon \cap B(\tilde{z},R_0))} > 2\delta;$$

this contradicts the fact $u \in N_\delta(S(\tilde{z}))$. Thus $|z-\tilde{z}| \leq 2R_0$. Then, it follows that $|\Upsilon_\varepsilon(u) - z| \leq 5R_0$. Now we see from (40), (41) and the decay property of v_z that for some $\tilde{C}, \tilde{c} > 0$,

$$\begin{aligned}
&\|u - v_z\|^2_\varepsilon \\
\leq &\|u - v_z\|^2_{\Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} + \int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla(u-v_z)|^2 + (u-v_z)^2 dx \\
\leq &(1-q)(c\delta)^2/2 + ((1-q)c\delta)^2/2 + \tilde{C}\exp(-\tilde{c}/\sqrt{\delta}) \leq (c\delta)^2.
\end{aligned}$$

Thus for small $\delta > 0$, $u \in N_{c\delta}(\mathcal{Z}^{10d}_{\varepsilon})$. This proves the first inclusion.

If $u \in N_{c'\delta}(\mathcal{Z}^{10d}_{\varepsilon})$, it follows that

$$\|u - (\phi_\varepsilon U)(\cdot - z)\|^2_\varepsilon \leq (c'\delta)^2 \text{ for some } z \in \mathcal{N}^{10d}_{\varepsilon}, U \in \mathcal{S}.$$

From Proposition 5 and the exponential decay property of U, we see that for some $\tilde{C}, \tilde{c} > 0$,

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla(\phi_\varepsilon U(\cdot - z))|^2 + ((\phi_\varepsilon U)(\cdot - z))^2 dx \leq \tilde{C}\exp(-\tilde{c}/\sqrt{\delta}).$$

Then it follows that for small $\delta > 0$,

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 dx \leq (1+q)(c'\delta)^2.$$

From (39), we see that

$$\int_{\Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u),1/\sqrt{\delta})} |\nabla u|^2 + u^2 - 2F(u) dx \leq ((1+q)c'\delta)^2.$$

This implies that $u \in G^{\delta}_{(1+q)\sqrt{2}c'\delta}(\mathcal{Z}^{10d}_{\varepsilon})$; thus the second inclusion follows. □

5. **An operator minimizing a tail.** For $b > 0$ and $u \in H^1(\Omega_\varepsilon)$ satisfying

$$|u|^*_{\delta,y} \equiv \int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} |\nabla u|^2 + u^2 \leq b^2/2,$$

we consider the following minimization problem:

$$I^\delta_{y,b}(u) \equiv \inf\left\{ \int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} \frac{1}{2}(|\nabla v|^2 + v^2) - F(v) dx \mid v \in H^\delta_{y,b}(u) \right\},$$

where

$$H^\delta_{y,b}(u) \equiv \left\{ v \in H^1(\Omega_\varepsilon \setminus B(y, 1/\sqrt{\delta})) \Big| \ |v|^*_{\delta,y} \le b^2, v = u \text{ on } B(y, 1/\sqrt{\delta}) \right\}.$$

A minimizer of $I^\delta_{y,b}(u)$ is a solution of this equation.

$$\begin{aligned} \Delta v - v + f(v) &= 0 \quad \text{on} \quad \Omega_\varepsilon \setminus B(y, 1/\sqrt{\delta}), \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega_\varepsilon \setminus B(y, 1/\sqrt{\delta}) \\ v &= u \quad \text{on} \quad \Omega_\varepsilon \cap \partial B(y, 1/\sqrt{\delta}). \end{aligned} \tag{42}$$

Then, we have the following result.

Proposition 8. *There exists $b_0 > 0$, independent of small ε, δ and $y \in \partial\Omega_\varepsilon$, such that if*

$$\int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} |\nabla u|^2 + (u)^2 dx \le b^2/2 \le b_0^2/2,$$

then for small $\varepsilon > 0$, there exists a unique minimizer $v_\varepsilon = v_\varepsilon^{y,\delta,u}$ of $I^\delta_{y,b}(u)$ in $H^\delta_{y,b}(u)$ satisfying equation (42). Moreover, there exist $c_1, C_1 > 0$, independent of small $\delta, \varepsilon > 0$, such that

$$v_\varepsilon(x) \le C_1 \exp(-c_1(|x-y| - \frac{1}{\sqrt{\delta}})) \qquad in \quad \{x \in \Omega_\varepsilon \mid \frac{1}{\sqrt{\delta}} + 1 \le |x-y|\}.$$

Proof. Because the condition (f1) and (f2) hold, if $b > 0$ is sufficiently small, then

$$\int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} \frac{1}{2}(|\nabla u|^2 + u^2) - F(u) dx < 3b^2/8.$$

On the other hand, for $v \in H^\delta_{y,b}(u)$ with $\int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} |\nabla v|^2 + v^2 dx = b^2$,

$$\int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} \frac{1}{2}(|\nabla v|^2 + v^2) - F(v) dx > 3b^2/8.$$

Thus, $I^\delta_{y,b}(u)$ is attained by a minimizer v_ε with $\int_{\Omega_\varepsilon \setminus B(y,1/\sqrt{\delta})} |\nabla v_\varepsilon|^2 + (v_\varepsilon)^2 dx < b^2$. Uniqueness and decay property can be proved by similar arguments in Proposition 2.3 of [6]. □

We see from Proposition 5 that for $u \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$, there exists a $z \in \mathcal{N}^{10d}_\varepsilon$ satisfying

$$|z - \Upsilon_\varepsilon(u)| \le 3R_0 \text{ and } \|u - (\phi_\varepsilon U)(\cdot - z)\|_\varepsilon \le \delta.$$

When δ is sufficiently small, we see that for any $u \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$,

$$\int_{|x - \Upsilon_\varepsilon(u)| \ge 1/\sqrt{\delta}} |\nabla u|^2 + u^2 dx \le \delta^2.$$

For small $\delta > 0$, Proposition 8 holds for $b = 2\delta$. We define $\tau_\varepsilon(u)$ by

$$\tau_\varepsilon(u)(x) = \begin{cases} u(x) & \text{for } \ x \in \Omega_\varepsilon \cap B(\Upsilon_\varepsilon(u), \frac{1}{\sqrt{\delta}}) \\ v_\varepsilon^{\Upsilon_\varepsilon(u),\delta,u}(x) & \text{for } \ x \in \Omega_\varepsilon \setminus B(\Upsilon_\varepsilon(u), \frac{1}{\sqrt{\delta}}) \end{cases} \tag{43}$$

By (ii) in Proposition 5, we can see that for small $\delta > 0$, $\Upsilon_\varepsilon(\tau_\varepsilon(u)) = \Upsilon_\varepsilon(u)$ and $\tau_\varepsilon(u) \in G^\delta_{c\delta}(\mathcal{Z}^{10d}_\varepsilon)$ for $u \in G^\delta_{c\delta}(\mathcal{Z}^{10d}_\varepsilon)$, $c \in (0, 1]$. From the uniqueness of tail-minimizing operator, we can see that τ_ε is a continuous map from $G^\delta_{c\delta}(\mathcal{Z}^{10d}_\varepsilon)$ to $G^\delta_{c\delta}(\mathcal{Z}^{10d}_\varepsilon)$ for each $c \in (0, 1]$.

By standard elliptic estimates, we can see that the operator τ_ε satisfies the following decay estimates.

Proposition 9. *There exists constant $C_1, c_1 > 0$ independent of small $\varepsilon, \delta > 0$ and $u \in G^{\delta}_{\delta}(\mathcal{Z}^{10d}_{\varepsilon})$ such that*

$$|\nabla \tau_\varepsilon(u)|, |D^2\tau_\varepsilon(u)| \le C_1 \exp(-c_1(|x - \Upsilon_\varepsilon(u)| - \frac{1}{\sqrt{\delta}} - 1))$$

in $\{x \in \Omega_\varepsilon \mid \frac{1}{\sqrt{\delta}} + 1 \le |x - \Upsilon_\varepsilon(u)|\}$.

Proof. We will estimate $W^{2,q}-$norm of $\tau_\varepsilon(u)$ in a small neighborhood $B(z, s/2) \cap \Omega_\varepsilon$ for a point $z \in \Omega_\varepsilon$ with $\frac{1}{\sqrt{\delta}} + 1 \le |z - \Upsilon_\varepsilon(u)|$. If $B(z, s) \subset \{x \in \Omega_\varepsilon \mid \frac{1}{\sqrt{\delta}} + 1 \le |x - \Upsilon_\varepsilon(u)|\}$, the elliptic estimates in [20] implies the our conclusion. Thus it suffices to consider a case $z \in \partial\Omega_\varepsilon$. By a translation and a rotation, we may assume that $z = 0$. Then for small $s > 0$, there is a C^4 function ψ defined on $B(0, s) \cap \partial\mathbb{R}^n_+$ with $\psi(0) = \nabla\psi(0) = 0$ and such that near 0, $\partial\Omega_\varepsilon$ is given by

$$\{(y', \psi(y')) \mid y' \in B(0, s) \cap \partial\mathbb{R}^n_+\}.$$

Then, define $\Psi = (\Psi_1, \cdots, \Psi_n) : B(0, s) \cap \bar{\mathbb{R}}^n_+ \to \bar{\Omega}_\varepsilon$ by

$$\Psi_j(y) = \begin{cases} y_j - y_n \dfrac{\partial \psi}{\partial y_j}(y') & \text{for } \ j = 1, \cdots, n-1, \\ y_n + \psi(y') & \text{for } \ j = n. \end{cases}$$

Then Ψ is a C^3 map with $\Psi(0) = 0$ and $D\Psi(0) = I$. Hence for small $s > 0$, Ψ is a diffeomorphism between $\mathbb{R}^n_+ \cap B(0, s)$ with its image $N \subset \Omega_\varepsilon$. Note that

$$\frac{\partial \Psi}{\partial y_n}(y', 0) = \big(- \frac{\partial \psi}{\partial y_1}(y'), \cdots, - \frac{\partial \psi}{\partial y_{n-1}}(y'), 1\big)$$

is an inward normal vector of $\partial\Omega_\varepsilon$ and

$$D\Psi(y) = \begin{pmatrix} 1 - y_n \dfrac{\partial^2 \psi}{\partial y_1^2}(y') & -y_n \dfrac{\partial^2 \psi}{\partial y_2 \partial y_1}(y') & \cdots & -\dfrac{\partial \psi}{\partial y_1}(y') \\ -y_n \dfrac{\partial^2 \psi}{\partial y_1 \partial y_2} & 1 - y_n \dfrac{\partial^2 \psi}{\partial y_1 \partial y_2}(y') & \cdots & -\dfrac{\partial \psi}{\partial y_2}(y') \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \psi}{\partial y_1} & \dfrac{\partial \psi}{\partial y_2} & \cdots & 1 \end{pmatrix}.$$

Let $\Phi : N \to \mathbb{R}^n_+ \cap B(0, s) \equiv B_+(0, s)$ be the inverse of Ψ. By setting $w \equiv \tau_\varepsilon(u) \circ \Psi$, we can transform equation (42) in $B(0, s) \cap \Omega_\varepsilon$ to the following equation

$$\sum_{1 \le i, j \le n} a_{ij}(y) \frac{\partial^2 w}{\partial y_i \partial y_j} - \sum_{i=1}^{n} b_i(y) \frac{\partial w}{\partial y_i} - w + f(w) = 0 \text{ in } B_+(0, s),$$

where

$$a_{ij}(y) = \nabla\Phi_i(\Psi(y)) \cdot \nabla\Phi_j(\Psi(y));$$
$$b_i(y) = (\Delta\Phi_i)(\Psi(y)).$$

Next we will use a reflection and make an equation in $B(0, s)$. We see that $\nabla\Phi_n(\Psi(y', 0))$ is orthogonal to $\partial\Omega_\varepsilon$ because $\Phi_n = 0$ on $\partial\Omega_\varepsilon$. For $i = 1, \cdots, (n-1)$, $\nabla\Phi_i(\Psi(y', 0))$ is orthogonal to the surface $\{(y' - y_n\nabla\psi(y'), y_n + \psi(y')) \mid y' \in B(0, s) \cap \partial\mathbb{R}^n_+, y_n \ge 0 \text{ and fixed } y_i\}$ at $(y', \psi(y'))$. This implies that for $i = 1, \cdots, n-1$, $\nabla\Phi_i(\Psi(y', 0))$ is in the tangent space of $\partial\Omega_\varepsilon$ at $(y', \psi(y'))$. Thus for $i = 1, \cdots, n-1$,

$$\nabla\Phi_i(\Psi(y', 0)) \cdot \nabla\Phi_n(\Psi(y', 0)) = 0.$$

Hence $a_{in}(y) = a_{ni}(y) = 0$ for $y_n = 0$, $i = 1, \cdots, (n-1)$. Now for $(y', y_n) \in B(0,s), y_n < 0$, we define $\bar{a}_{ij}(y', y_n) = a_{ij}(y', |y_n|)$ if $i, j \le n-1$; $\bar{a}_{in}(y', y_n) = -a_{in}(y', |y_n|)$ if $i \le n-1$; $\bar{a}_{nn}(y', y_n) = a_{nn}(y', |y_n|)$. We also define $\bar{b}_i(y', y_n) = b_i(y', |y_n|)$ if $i \le n-1$; $\bar{b}_n(y', y_n) = -b_n(y', |y_n|)$. For the solution w, we define $\bar{w}$ on $B(0,s)$ by $\bar{w}(y', y_n) = w(y', |y_n|)$. Then $\bar{w}$ is a solution of

$$\sum_{1\le i,j\le n} \bar{a}_{ij}(y)\frac{\partial^2 \bar{w}}{\partial y_i \partial y_j} - \sum_{i=1}^{n} \bar{b}_i(y)\frac{\partial \bar{w}}{\partial y_i} - \bar{w} + f(\bar{w}) = 0 \text{ in } B(0,s).$$

Since $\bar{a}_{ij}$ is continuous and $\bar{b}_i, f(\bar{w}) \in L^\infty$, we see from [20, Theorem 9.11] that for any $q > 1$, $\bar{w} \in W^{2,q}(B(0,s/2))$ and

$$\|\bar{w}\|_{W^{2,q}(B(0,s/2))} \le C(\|\bar{w}\|_{L^q(B(0,s))} + \|f(\bar{w})\|_{L^q(B(0,s))}),$$

where the constant C depends only on $n, q, s, \lambda, \Lambda$. Here Λ is the L^∞ bound of $\bar{a}_{ij}, \bar{b}_i$ and $\lambda \equiv \inf\{\sum_{i,j=1}^n \bar{a}_{ij}\mu_i\mu_j \mid \sum_{i=1}^n \mu_i^2 = 1\}$. This implies $\bar{w} \in C^{1,\alpha}(B(0,s/2))$ and $u \in C^{1,\alpha}(\bar{\Omega})$ and

$$\|\bar{w}\|_{C^{1,\alpha}(B(0,s/2))} \le C(\|\bar{w}\|_{L^p(B(0,s))} + \|f(\bar{w})\|_{L^p(B(0,s))}).$$

From exponential decay property in Proposition 8, we see that

$$\|\bar{w}\|_{L^p(B(0,s))} + \|f(\bar{w})\|_{L^p(B(0,s))} \le C_1' \exp(-c_1'(|z - \Upsilon_\varepsilon(u)| - \frac{1}{\sqrt{\delta}} - 1)).$$

for some constants C_1', c_1' independent of δ, ε, z. This implies the exponential decay of $\nabla\tau_\varepsilon(u)$.

Now, we will estimate C^2 norm of $\bar{w}$ by Schauder estimate. Note that $\bar{a}_{ij}$ is Lipschitz continuous and $\bar{b}_n(y)\frac{\partial \bar{w}}{\partial y_n}$ is Hölder continuous in $B(0,s/2)$ because $\bar{w} \in C^{1,\alpha}(B(0,s/2))$ and $\frac{\partial \bar{w}}{\partial y_n} = 0$ on $y_n = 0$. $\bar{w}$ is a solution of

$$\sum_{1\le i,j\le n} \bar{a}_{ij}(y)\frac{\partial^2 \bar{w}}{\partial y_i \partial y_j} - \sum_{i=1}^{n-1} \bar{b}_i(y)\frac{\partial \bar{w}}{\partial y_i} - \bar{w} = -f(\bar{w}) + \bar{b}_n(y)\frac{\partial \bar{w}}{\partial y_n} \text{ in } B(0,s).$$

By Schauder estimate [20, Theorem 6.2],

$$|\bar{w}|_{C^{2,\alpha}(B(0,s/3))} \le C(|\bar{w}|_{C^0(B(0,s/2))} + |f(\bar{w})|_{C^{0,\alpha}(B(0,s/2))} + |\bar{b}_n(y)\frac{\partial \bar{w}}{\partial y_n}|_{C^{0,\alpha}(B(0,s/2))}),$$

where constant C depends on n, α, λ, s and C^α-norm of coefficients. Since C^1 norm of $\bar{a}_{ij}, \bar{b}_i$ is uniform with respect to z, the constant in Schauder estimate is uniformly bounded with respect to z. Since $\|\bar{w}\|_{C^{1,\alpha}(B(z,s/2))} \le C\exp(-c(|z - \Upsilon_\varepsilon(u)| - \frac{1}{\sqrt{\delta}}))$, we get the decay estimate for $|D^2\tau_\varepsilon(u)|$. □

6. An initial surface and its energy estimate.

By condition (H1),

$$H(z) \le m \text{ for } z \in L. \tag{44}$$

Using a deformation through the gradient flow in (4) and Proposition 1, we may assume that for some $\alpha_0 > 0$,

$$H(z) \le m - \alpha_0 \text{ for } z \in L \setminus \mathcal{N}^d. \tag{45}$$

We take an element $U \in \mathcal{S}$ satisfying

$$\int_{\mathbb{R}^n} |\nabla U(y)|^2 |y| dy = \begin{cases} \min\limits_{\tilde{U}\in\mathcal{S}} \int_{\mathbb{R}^n} |\nabla \tilde{U}(y)|^2 |y| dy, \text{ if } m \ge 0, \\ \max\limits_{\tilde{U}\in\mathcal{S}} \int_{\mathbb{R}^n} |\nabla \tilde{U}(y)|^2 |y| dy, \text{ if } m < 0, \end{cases} \tag{46}$$

and define an initial surface $A_\varepsilon : (0,\infty)\times L \to H^1(\Omega_\varepsilon)$ by

$$A_\varepsilon(t,z)(x) = \phi_\varepsilon(x/t - z/t\varepsilon)U(x/t - z/t\varepsilon)$$

for some $U \in \mathcal{S}$. We define $A_\varepsilon(0,z) = 0$.

Note that there is a large $T > 1$, independent of small $\varepsilon > 0$, such that, for all $z \in L$, $\Gamma_\varepsilon(A_\varepsilon(T,z)) < -1$.

Proposition 10. *It holds that*

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(A_\varepsilon(t,z)) = \frac{1}{4}(t^{n-2} - \frac{n-2}{n}t^n)\int_{\mathbb{R}^n}|\nabla U|^2 \quad \textit{uniformly for} \quad t\in[0,T], z\in L,$$

and that

$$c_\varepsilon \equiv \max_{t\in[0,T],z\in L}\Gamma_\varepsilon(A_\varepsilon(t,z)) = \frac{1}{2}\Gamma(U) + \varepsilon\frac{m}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U(y)|^2|y|dy + o(\varepsilon).$$

Moreover, for any small $\sigma > 0$,

$$\limsup_{\varepsilon\to 0}\max\{\Gamma_\varepsilon(A_\varepsilon(t,z)) \mid t\in[0,T]\setminus(1-\sigma,1+\sigma), z\in L\} < \frac{\Gamma(U)}{2},$$

and for small $\nu > 0$

$$\max\{\Gamma_\varepsilon(A_\varepsilon(t,z)) \mid t\in[0,T], z\in L, z\notin \mathcal{N}^d\} < c_\varepsilon - \nu\varepsilon$$

if ε is sufficiently small.

Proof. For the estimates for $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(A_\varepsilon(t,z))$ and c_ε, refer to [6, Proposition 3.1]. The third estimate comes from the estimate for $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(A_\varepsilon(t,z))$. Combining the estimate for c_ε and (45), we get the last estimate. □

Now we take $\delta_1 \in (0,\delta_0)$ so that Proposition 7, Proposition 8, Proposition 9 and all previous results hold for $\delta\in(0,\delta_1)$. By Proposition 10, for small $\gamma > 0$, there exists $\sigma = \sigma(\gamma) > 0$ such that for small $\varepsilon > 0$, then

$$\max\{\Gamma_\varepsilon(A_\varepsilon(t,z)) \mid t\in[0,T]\setminus(1-\sigma,1+\sigma), z\in L\} < \frac{\Gamma(U)}{2} - \gamma. \tag{47}$$

We can take $\sigma(\gamma) > 0$ so that $\sigma(\gamma)\to 0$ as $\gamma,\varepsilon\to 0$.

Note that for $z\in L$ and $t\in[0,T]$,

$$\mathrm{supp}(A_\varepsilon(t,z)) \subset B(z/\varepsilon, t/\varepsilon^{1/3}) \subset B(z/\varepsilon, T/\varepsilon^{1/3}).$$

We take a sufficiently small $T_0\in(0,1)$ so that $\Gamma_\varepsilon(A_\varepsilon(T_0,z)) \le \Gamma(U)/4$ for $z\in L$. We may assume that δ_0 is small and that definition (37) and Proposition 5 hold for $N_{2\delta_0}(\mathcal{Z}_\varepsilon^{10d})$. Now, we extend the center of mass Υ_ε on $N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$ to a continuous function on

$$N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d}) \cup \{A_\varepsilon(t,z) \mid t\in[T_0,T], z\in L\}$$

so that for $A_\varepsilon(t,z)\in N_{2\delta_0}(\mathcal{Z}_\varepsilon^{10d})$, Υ_ε is defined by (37) and for any $t\in[T_0,T]$ and $z\in L$,

$$|\Upsilon_\varepsilon(A_\varepsilon(t,z)) - z/\varepsilon| \le 4R_0,$$

and that

$$\Upsilon_\varepsilon(A_\varepsilon(t,z)) = z/\varepsilon \;\text{ in a neighborhood } N \text{ of } \partial([T_0,T]\times L).$$

Then we can take $\delta_2 \in (0, \min\{\delta_1, 1/(8R_0)^2\})$ so that for $\delta\in(0,\delta_2]$, $t\in[0,T]$ and $z\in L$,

$$\int_{\Omega_\varepsilon\setminus B(\Upsilon_\varepsilon(A_\varepsilon(t,z)),1/\sqrt{\delta})}|\nabla A_\varepsilon(t,z)|^2 + |A_\varepsilon(t,z)|^2dx \le \delta^2 \le b_0^2/2.$$

Then we apply the tail minimizing operator τ_ε in (43) for $A_\varepsilon(t,z)$ with the ball $B(\Upsilon_\varepsilon(A_\varepsilon(t,z)), 1/\sqrt{\delta})$ and define

$$\tilde{A}_\varepsilon(t,z) \equiv \tau_\varepsilon(A_\varepsilon(t,u)).$$

If $\tilde{A}_\varepsilon(t,z) \in N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d})$, then $A_\varepsilon(t,z) \in N_{2\delta_0}(\mathcal{Z}_\varepsilon^{10d})$. In this case, we see from (ii) of Proposition 5 that

$$\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) = \Upsilon_\varepsilon(A_\varepsilon(t,z)).$$

Then, defining the center of mass for $\tilde{A}_\varepsilon(t,z)$ by

$$\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) = \begin{cases} \Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)), & \text{if } \tilde{A}_\varepsilon(t,z) \in N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d}), \\ \Upsilon_\varepsilon(A_\varepsilon(t,z)), & \text{if } \tilde{A}_\varepsilon(t,z) \notin N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d}), \end{cases} \tag{48}$$

the center of mass Υ_ε is continuous on

$$N_{\delta_0}(\mathcal{Z}_\varepsilon^{10d}) \cup \{\tilde{A}_\varepsilon(t,z) \mid t \in [T_0,T], z \in L\}$$

since $\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) = \Upsilon_\varepsilon(A_\varepsilon(t,z))$ for all $t \in [T_0,T], z \in L$. Thus we see that for any $t \in [T_0,T]$ and $z \in L$,

$$|\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) - z/\varepsilon| \leq 4R_0, \tag{49}$$

and that

$$\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) = z/\varepsilon \ \text{ in a neighborhood } N \text{ of } \partial([T_0,T] \times L). \tag{50}$$

Moreover, when we apply τ_ε for $\tilde{A}_\varepsilon(t,z)$ with $B(\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)), 1/\sqrt{\delta})$,

$$\tilde{A}_\varepsilon(t,z) = \tau_\varepsilon(\tilde{A}_\varepsilon(t,z)).$$

We define

$$b_\varepsilon \equiv \max_{t\in[0,T], z\in L} \Gamma_\varepsilon(\tilde{A}_\varepsilon(t,z)). \tag{51}$$

We see from the energy decreasing property of the tail minimizer operator τ_ε that

$$b_\varepsilon \leq \frac{1}{2}\Gamma(U) + \varepsilon \frac{m}{n+1} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} |\nabla U|^2 |y| dy + o(\varepsilon). \tag{52}$$

7. **Gradient estimate.** We define

$$\Gamma_\varepsilon^c \equiv (\Gamma_\varepsilon)^c = \{u \in H^1(\Omega_\varepsilon) \mid \Gamma_\varepsilon(u) \leq c\}.$$

Then we can prove the following gradient estimate.

Proposition 11. *There exists $\delta_3 \in (0,\delta_2)$ such that for $\delta \in (0,\delta_3)$ and $0 < r_1 < r_2 < \delta$, there exist $\mu = \mu(\delta, r_1, r_2) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta, r_1, r_2) > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,*

$$\inf\{\|\Gamma_\varepsilon'(u)\|_\varepsilon^* \mid u \in \Gamma_\varepsilon^{b_\varepsilon} \cap \left(G_{r_2}^\delta(\mathcal{Z}_\varepsilon^{10d}) \setminus G_{r_1}^\delta(\mathcal{Z}_\varepsilon^{10d})\right), \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{9d}\} \geq \mu,$$

where $\|\Gamma_\varepsilon'(u)\|_\varepsilon^*$ is the operator norm of $\Gamma_\varepsilon'(u)$ on $H^{1,2}(\Omega_\varepsilon)$.

Proof. To the contrary, suppose that for any $\delta_3 \in (0,\delta_2)$, there exist $\delta \in (0,\delta_3)$ and $0 < r_1 < r_2 < \delta$ such that there exists a sequence of elements $u_\varepsilon \in G_{r_2}^\delta(\mathcal{Z}_\varepsilon^{10d}) \setminus G_{r_1}^\delta(\mathcal{Z}_\varepsilon^{10d})$ with $\Upsilon_\varepsilon(u_\varepsilon) \in \mathcal{N}_\varepsilon^{9d}$ such that $\liminf_{\varepsilon\to 0} |\Gamma_\varepsilon'(u_\varepsilon)| = 0$. Without loss of generality, we may assume that $\lim_{\varepsilon\to 0} |\Gamma_\varepsilon'(u_\varepsilon)| = 0$. Since $u_\varepsilon \in G_{r_2}^\delta(\mathcal{Z}_\varepsilon^{10d})$ and $\Upsilon_\varepsilon(u_\varepsilon) \in \mathcal{N}_\varepsilon^{9d}$, there exists $x_\varepsilon \in \partial\Omega_\varepsilon$ with $\text{dist}(x_\varepsilon, \mathcal{N}_\varepsilon^{9d}) \leq 3R_0$ such that $u_\varepsilon = (\phi_\varepsilon U_\varepsilon)(\cdot - x_\varepsilon) + w_\varepsilon$ for some $U_\varepsilon \in \mathcal{S}$ and w_ε with $\|w_\varepsilon\|_\varepsilon \leq \delta$.

Suppose that there exist $z_\varepsilon \in B(x_\varepsilon, 1/\varepsilon^{1/3}) \setminus B(x_\varepsilon, 1/2\varepsilon^{1/3})$ and $R > 0$

$$\limsup_{\varepsilon\to 0} \int_{\Omega_\varepsilon \cap B(z_\varepsilon, R)} (u_\varepsilon)^2 dx > 0.$$

We can assume that, up to a subsequence, $\varepsilon z_\varepsilon \to z_0$ as $\varepsilon \to 0$. Then one of the following two cases holds :
(case 1) $\limsup_{\varepsilon\to 0} \operatorname{dist}(z_\varepsilon, \partial\Omega_\varepsilon) = \infty$; (case 2) $\liminf_{\varepsilon\to 0} \operatorname{dist}(z_\varepsilon, \partial\Omega_\varepsilon) < \infty$.
In (case 1), we see that $u_\varepsilon(\cdot + z_\varepsilon)$ converges weakly, up to a subsequence, to some $w \in H^1(\mathbb{R}^n) \setminus \{0\}$ which satisfies

$$\Delta w(y) - w(y) + f(w(y)) = 0 \text{ for } y \in \mathbb{R}^n.$$

In (case 2), we find $z'_\varepsilon \in \partial\Omega_\varepsilon$ with $|z_\varepsilon - z'_\varepsilon| = \operatorname{dist}(z_\varepsilon, \partial\Omega_\varepsilon)$ and define $y_\varepsilon = \varepsilon z'_\varepsilon$. We may assume that $\lim_{\varepsilon\to 0} y_\varepsilon = y_0 \in \partial\Omega$. Then, for each small $\varepsilon > 0$, as in the proof of Proposition 9, we can find a diffeomorphic map $\tilde{\Psi}_\varepsilon : B(0,r)\cap\mathbb{R}^n_+ \to \Omega$ such that $\tilde{\Psi}_\varepsilon(0) = y_\varepsilon$, $\nabla\tilde{\Psi}_\varepsilon(0) \in SO(n)$, $\tilde{\Psi}_\varepsilon(B(0,r)\cap\partial\mathbb{R}^n_+) \subset \partial\Omega$ and $\frac{\partial\tilde{\Psi}_\varepsilon}{\partial y_n}(y',0) \perp \partial\Omega$ for $(y',0) \in \mathbb{R}^{n-1}\times\mathbb{R}$ with $|y'| < r$. We define $\Psi_\varepsilon : B(0, r/\varepsilon) \to \Omega_\varepsilon$ by $\Psi_\varepsilon(y) = \frac{1}{\varepsilon}\tilde{\Psi}_\varepsilon(\varepsilon y)$. We find a function $\chi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_\varepsilon(x) = 1$ for $|x| \le 1/\sqrt{\varepsilon}$, $\chi_\varepsilon(x) = 0$ for $|x| \ge 2/\sqrt{\varepsilon}$ and $|\nabla\chi_\varepsilon| \le 2\sqrt{\varepsilon}$. Then we see that $(\chi_\varepsilon(\cdot + z'_\varepsilon)u_\varepsilon) \circ \Psi_\varepsilon$ converges weakly to $w \in H^1(\mathbb{R}^n_+) \setminus \{0\}$ satisfying

$$\Delta w - w + f(w) = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial w}{\partial y_n} = 0 \text{ on } \partial\mathbb{R}^n_+.$$

In both cases, it follows from the Pohozaev identity that for $U \in \mathcal{S}$,

$$\frac{1}{n}\int_{\mathbb{R}^n_+} |\nabla w|^2 dy = \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla w|^2 + w^2 dy - \int_{\mathbb{R}^n_+} F(w)dy \ge \frac{\Gamma(U)}{2}.$$

From the weak convergence of $(\chi_\varepsilon(\cdot + z'_\varepsilon)u_\varepsilon) \circ \Psi_\varepsilon$, we see that for large $R' > R$,

$$\liminf_{\varepsilon\to 0} \int_{B(z'_\varepsilon, R')\cap\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \ge \frac{n}{4}\Gamma(U).$$

For $\delta_3 \in (0, \frac{1}{2}\sqrt{\frac{n}{4}\Gamma(U)})$, we get a contradiction. Thus we conclude that for each $R > 0$,

$$\lim_{\varepsilon\to 0} \sup_{z\in B(x_\varepsilon, 1/\varepsilon^{1/3})\setminus B(x_\varepsilon, 1/2\varepsilon^{1/3})} \int_{\Omega_\varepsilon\cap B(z,R)} (u_\varepsilon)^2 dx = 0.$$

Then we see from [27, Lemma 1.1] by P.L. Lions that

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon\cap \operatorname{supp}(|\nabla\phi_\varepsilon(\cdot - x_\varepsilon)|)} |u_\varepsilon|^{p+1} dx = 0.$$

This implies that for $u^1_\varepsilon(x) \equiv \phi_\varepsilon(x - x_\varepsilon)u_\varepsilon(x)$ and $u^2_\varepsilon \equiv u_\varepsilon - u^1_\varepsilon$,

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} F(u_\varepsilon) - F(u^1_\varepsilon) - F(u^2_\varepsilon) dx = 0.$$

Here the function $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ satisfies (7). Note that

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u^1_\varepsilon) + \Gamma_\varepsilon(u^2_\varepsilon) + \int_{\Omega_\varepsilon} (\phi_\varepsilon u_\varepsilon)((1-\phi_\varepsilon)u_\varepsilon)dx \\ &\quad + \int_{\Omega_\varepsilon} \nabla(\phi_\varepsilon u_\varepsilon)\cdot\nabla((1-\phi_\varepsilon)u_\varepsilon)dx - \int_{\Omega_\varepsilon} F(u_\varepsilon) - F(u^1_\varepsilon) - F(u^2_\varepsilon)dx.\end{aligned}$$

Since $|\nabla\phi_\varepsilon| \le 3\varepsilon^{1/3}$, we deduce that as $\varepsilon \to 0$,

$$\Gamma_\varepsilon(u_\varepsilon) \ge \Gamma_\varepsilon(u^1_\varepsilon) + \Gamma_\varepsilon(u^2_\varepsilon) + o(1).$$

We see from (f1),(f2) and Sobolev's inequality that for some $C, c > 0$,

$$\Gamma_\varepsilon(u^2_\varepsilon) \ge \frac{1}{4}\int_{\Omega_\varepsilon} |\nabla u^2_\varepsilon|^2 + |u^2_\varepsilon|^2 dx - C\int_{\Omega_\varepsilon} |u^2_\varepsilon|^{\frac{2n}{n-2}} dx$$

$$\geq \frac{1}{4}\|u_\varepsilon^2\|_\varepsilon^2 - Cc\|u_\varepsilon^2\|_\varepsilon^{2n/(n-2)}.$$

Thus, taking a small $\delta_3 > 0$, we see that for $\delta \in (0, \delta_3)$,

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \frac{1}{8}\|u_\varepsilon^2\|_\varepsilon^2 + o(1). \tag{53}$$

We recall a flattening map $\Psi_{\varepsilon,x_\varepsilon} : \mathbb{R}^n_+ \cap B^n(0, r/\varepsilon) \to \Omega_\varepsilon$ in (22). Then, $w_\varepsilon^1 \equiv u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon}$ converges weakly to a nonzero function $W \in H^1(\mathbb{R}^n_+) \setminus \{0\}$. As before, we see that for $\delta \in (0, \delta_3)$ and $R > 0$, $\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon \cap B(q_\varepsilon, R)} (u_\varepsilon)^2 dx = 0$ if $\lim_{\varepsilon\to 0} |x_\varepsilon - q_\varepsilon| = \infty$. Using again [27, Lemma 1.1], we see that $\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} F(u_\varepsilon^1) = \int_{\mathbb{R}^n_+} F(W)$. Since $\Gamma'_\varepsilon(u_\varepsilon) \to 0$ as $\varepsilon \to 0$, we see from (23) and (ii) of Proposition 3 that

$$\Delta W - W + f(W) = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial W}{\partial y_n} = 0 \text{ on } \partial\mathbb{R}^n_+.$$

Then, from the weak convergence of $u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon}$ to W, we see that

$$\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^1) \geq \frac{1}{2}\Gamma(\tilde{W}),$$

where $\tilde{W} \in H^1(\mathbb{R}^n)$ is the even reflection of W. Thus we see from (53) that $\lim_{\varepsilon\to 0} \|u_\varepsilon^2\|_\varepsilon = 0$ and $\tilde{W} = U(\cdot - y_0)$ for some $U \in \mathcal{S}$ and $y_0 \in \partial\mathbb{R}^n_+$. Since $\mathrm{supp}(u_\varepsilon^1) \subset B^n(x_\varepsilon, 2/\varepsilon^{1/3}) \cap \Omega_\varepsilon$, it follows from (23) and (24) that

$$\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon}|^2 + (u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon})^2 dx = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon^1|^2 + (u_\varepsilon^1)^2 dx + o(1) \text{ for small } \varepsilon > 0.$$

By Proposition 4,

$$\|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) \circ \Psi_{\varepsilon,x_\varepsilon} - (\phi_\varepsilon U)(\cdot - y_0)\|_{H^1(\mathbb{R}^n_+)} \leq O(\varepsilon^{1/3}).$$

Since supp $\phi_\varepsilon \subset B^n(x_\varepsilon, 2/\varepsilon^{1/3}) \cap \Omega_\varepsilon$, it follows from (23) and (24) that for some $C > 0$,

$$\|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) - u_\varepsilon^1\|_\varepsilon \leq \|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) \circ \Psi_{\varepsilon,x_\varepsilon} - u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon}\|_{H^1(\mathbb{R}^n_+)} + C\varepsilon^{1/3}.$$

Then, as $\varepsilon \to 0$,

$$\begin{aligned}
&\|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) - u_\varepsilon^1\|_\varepsilon \\
\leq&\|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) \circ \Psi_{\varepsilon,x_\varepsilon} - u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon}\|_{H^1(\mathbb{R}^n_+)} + O(\varepsilon^{1/3}) \\
\leq&\|(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,x_\varepsilon}(y_0)) \circ \Psi_{\varepsilon,x_\varepsilon} - (\phi_\varepsilon U)(\cdot - y_0)\|_{H^1(\mathbb{R}^n_+)} \\
&\quad + \|u_\varepsilon^1 \circ \Psi_{\varepsilon,x_\varepsilon} - (\phi_\varepsilon U)(\cdot - y_0)\|_{H^1(\mathbb{R}^n_+)} + O(\varepsilon^{1/3}) \\
=&o(1).
\end{aligned}$$

Since $\mathrm{dist}(x_\varepsilon, \mathcal{N}_\varepsilon^{9d}) \leq 3R_0$, this implies that for any small $s \in (0, \delta)$, $u_\varepsilon \in G_s^\delta(\mathcal{Z}_\varepsilon^{10d})$. This contradicts to the fact $u_\varepsilon \notin G_{r_1}^\delta(\mathcal{Z}_\varepsilon^{10d})$ and completes the proof. □

8. **Transplantation flow.** Define

$$\omega_1 \equiv \sup\left\{|D_y \Psi_X(y)| \;\middle|\; X \in \mathcal{N}^{10d} \setminus \mathcal{N}^d, y \in B(0,r) \cap \mathbb{R}^n_+\right\},$$

$$\omega_2 \equiv \sup\left\{|D_y (\Psi_X)^{-1}(y)| \;\middle|\; X \in \mathcal{N}^{10d} \setminus \mathcal{N}^d, y \in \Psi_X(B(0,r) \cap \mathbb{R}^n_+)\right\}$$

and $\omega \equiv \max\{\omega_1\omega_2, 1\}$.

Here, $|A|$ is the operator norm for a matrix A. Taking a smaller $r > 0$, we may assume that

$$\omega < 2.$$

Note that

$$\omega_1 = \sup\left\{|D_y\Psi_{\varepsilon,X}(y)| \;\middle|\; X\in\mathcal{N}_\varepsilon^{10d}\setminus\mathcal{N}_\varepsilon^{d}, y\in B(0,r/\varepsilon)\cap\mathbb{R}^n_+\right\},$$
$$\omega_2 = \sup\left\{|D_y(\Psi_{\varepsilon,X})^{-1}(y)| \;\middle|\; X\in\mathcal{N}_\varepsilon^{10d}\setminus\mathcal{N}_\varepsilon^{d}, y\in \Psi_{\varepsilon,X}(B(0,r/\varepsilon)\cap\mathbb{R}^n_+)\right\}.$$

We find a function $\rho_1\in C_0^2(\partial\Omega,[0,1])$ such that $\rho_1(X)=1$ for $X\in\mathcal{N}^{7d}\setminus\mathcal{N}^{3d}$ and $\rho_1(X)=0$ for $X\in\mathcal{N}^{2d}\cup(\partial\Omega\setminus\mathcal{N}^{8d})$. For $\delta\in(0,\delta_3)$, we also find a function $\rho_2^\delta\in C^2(G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d}),[0,1])$ such that $\rho_2^\delta(u)=1$ for $u\in G^\delta_{\delta/6\omega}(\mathcal{Z}_\varepsilon^{10d})$ and $\rho_2^\delta(u)=0$ for $u\notin G^\delta_{\delta/4\omega}(\mathcal{Z}_\varepsilon^{10d})$. We fix $l_t\in(0,1)$ so that for any $l\in[0,l_t]$ and $X\in\partial\Omega$,

$$\mathrm{dist}_{\partial\Omega}(\Phi(l,X),X)+|\Phi(l,X)-X|\le\min\{r/10,d/10\},\tag{54}$$

where $\mathrm{dist}_{\partial\Omega}(X_1,X_2)$ is the distance between X_1 and X_2 in $\partial\Omega$. We define a flow function $\tilde\Phi:[0,l_t]\times\partial\Omega\times H^1(\Omega_\varepsilon)\to\partial\Omega$ by

$$\tilde\Phi(l,X,u)\equiv\Phi(\rho_1(X)\rho_2^\delta(u)l,X).$$

We choose a radially symmetric function $\varphi_\varepsilon\in C_0^\infty(\mathbb{R}^n,[0,1])$ such that

$$\varphi_\varepsilon(x)=0 \text{ for } |x|\ge r/2\varepsilon, \varphi_\varepsilon(x)=1 \text{ for } |x|\le r/3\varepsilon \text{ and } |\nabla\varphi_\varepsilon|\le 10\varepsilon/r.\tag{55}$$

Now, we define a transplantation operator

$$P_\varepsilon:[0,l_t]\times G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d})\to H^1(\Omega_\varepsilon)$$

by $P_\varepsilon(l,u)=u$ when $\Upsilon_\varepsilon(u)\notin\mathcal{N}_\varepsilon^{8d}\setminus\mathcal{N}_\varepsilon^{2d}$ and

$$\begin{aligned}P_\varepsilon(l,u)(x) \;&=\; (\varphi_\varepsilon(\cdot-\Upsilon_\varepsilon(u))u)\circ\Psi_{\varepsilon,\Upsilon_\varepsilon(u)}\circ\Psi^{-1}_{\varepsilon,\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon}(x)\\ &\quad+(1-\varphi_\varepsilon(x-\Upsilon_\varepsilon(u)))u(x)\end{aligned}\tag{56}$$

when $\Upsilon_\varepsilon(u)\in\mathcal{N}_\varepsilon^{8d}\setminus\mathcal{N}_\varepsilon^{2d}$. Here, we define

$$(\varphi_\varepsilon(\cdot-\Upsilon_\varepsilon(u))u)\circ\Psi_{\varepsilon,\Upsilon_\varepsilon(u)}\circ\Psi^{-1}_{\varepsilon,\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon}(x)=0$$

if $x\notin\Psi_{\varepsilon,\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon}(B(0,r/\varepsilon))$. We note that $\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon=\Upsilon_\varepsilon(u)$ if $\Upsilon_\varepsilon(u)\in\partial(\mathcal{N}_\varepsilon^{8d}\setminus\mathcal{N}_\varepsilon^{2d})$. By Proposition 3, we see that for $a\in(0,1/1000)$,

$$B^n(\Upsilon_\varepsilon(u),(1-a)r/\varepsilon)\cap\Omega_\varepsilon\subset\Psi_{\varepsilon,\Upsilon_\varepsilon(u)}(B^n(0,r/\varepsilon)\cap\mathbb{R}^n_+).$$

Thus, the operator P_ε is continuous.

Proposition 12. *There exists a constant $R_1>0$, independent of small $\varepsilon>0$ such that for $\delta\in(0,\delta_3)$ and $u\in G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d})$,*

$$|\Upsilon_\varepsilon(P_\varepsilon(l,u))-\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon|\le R_1.$$

Proof. By the definition of cut-off functions ρ_1,ρ_2^δ, we may assume $u\in G^\delta_{\delta/4\omega}(\mathcal{Z}_\varepsilon^{10d})$ and $\Upsilon_\varepsilon(u)\in\mathcal{N}_\varepsilon^{8d}$. Let $X_\varepsilon(l,u)\equiv\tilde\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)/\varepsilon$ and $X_\varepsilon\equiv\Upsilon_\varepsilon(u)$. By Proposition 5 and Proposition 7,

$$\|u-(\phi_\varepsilon U)(\cdot-z)\|_\varepsilon\le\delta/4(1-q)\omega\tag{57}$$

for some $U\in\mathcal{S}$ and $z\in\partial\Omega_\varepsilon$ with $|z-X_\varepsilon|\le 3R_0$. We claim that

$$\|P_\varepsilon(l,u)-(\phi_\varepsilon U)(\cdot-\Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon\le\delta.\tag{58}$$

Suppose that the claim is true. Then, we see from Proposition 5 that

$$|\Upsilon_\varepsilon(P_\varepsilon(l,u))-\Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z)|\le 3R_0.$$

By (iii) in Proposition 3,

$$|X_\varepsilon(l,u)-\Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z)|\le(1+a)|\Psi^{-1}_{\varepsilon,X_\varepsilon}(z)|\le\frac{1+a}{1-a}|X_\varepsilon-z|\le 3R_0\frac{1+a}{1-a}.$$

Then, taking $R_1 = 3R_0\frac{2}{1-a}$, the conclusion follows by (58).

To complete the proof, we prove the claim (58). From definition of $P_\varepsilon(l,u)$, we see that

$$\begin{aligned}&\|P_\varepsilon(l,u) - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)} \circ \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon\\ \le&\|(1-\varphi_\varepsilon)u\|_\varepsilon + \|(\varphi_\varepsilon u)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\|_\varepsilon\\ &+ \|(\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon.\end{aligned}$$

By (57) and the bound of $\nabla\varphi_\varepsilon$,

$$\|(1-\varphi_\varepsilon)u\|_\varepsilon \le (1+\varepsilon/r)\frac{\delta}{4(1-q)\omega}. \tag{59}$$

Since the Jacobian determinants of coordinate transforms $\Psi_{\varepsilon,X_\varepsilon}$, $\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}$ are 1 and $|D\Psi_{\varepsilon,X_\varepsilon}||D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}| \le \omega$, we see from a change of variables that for some $C > 0$, independent of small $\varepsilon > 0$,

$$\begin{aligned}&\|(\varphi_\varepsilon u)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\|_\varepsilon\\ \le&\|\nabla(\varphi_\varepsilon u - (\phi_\varepsilon U)(\cdot - z))D\Psi_{\varepsilon,X_\varepsilon}D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\|_{L^2(\Omega_\varepsilon)}\\ &+ \|(\varphi_\varepsilon u) - (\phi_\varepsilon U)(\cdot - z)\|_{L^2(\Omega_\varepsilon)}\\ \le&\frac{\delta}{4(1-q)} + C\varepsilon.\end{aligned} \tag{60}$$

By the triangle inequality, it follows that

$$\begin{aligned}&\|(\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon\\ \le&\|(\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\|_\varepsilon\\ &+ \|(\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon.\end{aligned} \tag{61}$$

By Proposition 4, it follows that

$$\|(\phi_\varepsilon U)(\cdot - z)\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\|_\varepsilon \le O(\varepsilon^{1/3}). \tag{62}$$

Since $|z - X_\varepsilon| < 3R_0$, it follows from (iii) in Proposition 3 that

$$|X_\varepsilon(l,u) - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z)| < 3\frac{(1+a)}{(1-a)}R_0. \tag{63}$$

Then, it follows from Lemma 2.1 that if $|\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z)| < C\varepsilon^{-1/3}$,

$$\left|(x - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z)) - E(X_\varepsilon(l,u))(\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\right| \le O(\varepsilon^{1/3}). \tag{64}$$

Then we see that for $E'_\varepsilon \equiv (E(X_\varepsilon(l,u)))^{-1}$, by the triangle inequality,

$$\begin{aligned}&\|(\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon\\ \le&\|\nabla(\phi_\varepsilon U)(\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(\cdot) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))E'_\varepsilon - \nabla(\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_{L^2(\Omega_\varepsilon)}\\ &+ \|\nabla(\phi_\varepsilon U)(\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(\cdot) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))(D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(\cdot) - (E(X_\varepsilon(l,u)))^{-1})\|_{L^2(\Omega_\varepsilon)}\\ &+ \|(\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_{L^2(\Omega_\varepsilon)}\\ \equiv&TI + TII + TIII.\end{aligned}$$

To estimate TII, we see that $D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) = (I + O(\varepsilon|x - X_\varepsilon(l,u)|))E'_\varepsilon$. Since $\mathrm{supp}(\phi_\varepsilon U) \subset \{x\big|\ |\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z)| < C\varepsilon^{-1/3}\}$, we see from (63) that for some $C' > 0$, $\mathrm{supp}(\phi_\varepsilon U) \subset \{x\big|\ |x - X_\varepsilon(l,u)| < C'\varepsilon^{-1/3}\}$. This implies that

$$\big|D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) - E(X_\varepsilon(l,u))^{-1}\big| = O(\varepsilon^{2/3})$$

for $x \in \mathrm{supp}(\phi_\varepsilon U)(\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(\cdot) - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))$. Then, we get an estimate that for small $\varepsilon > 0, TII = O(\varepsilon^{2/3})$. For estimates $TI, TIII$, we note that for a radially symmetric function h and $E \in O(n)$, $h(Ex) = h(x)$ and $\nabla h(x)(E)^{-1} = (\nabla h)(Ex)$. Then, using the mean value theorem with (64), we deduce that $TI, TIII = O(\varepsilon^{1/3})$. Thus, we get

$$\|(\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z)) \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,X_\varepsilon(l,u)} \circ \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_\varepsilon = O(\varepsilon^{1/3}). \quad (65)$$

By (59), (60) and (65), the claim (58) holds. This completes the proof. □

Proposition 13. *There exists $\delta_4 \in (0, \delta_3)$ such that if $\delta \in (0, \delta_4), c \in (0, 1]$ and $u \in G^\delta_{c\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon)$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}^{8d}_\varepsilon$, then $P_\varepsilon(l, \tau_\varepsilon(u)) \in G^\delta_{3c\delta/5}(\mathcal{Z}^{10d}_\varepsilon)$ for $l \in [0, l_t]$ and small $\varepsilon > 0$. Thus, for $\delta \in (0, \delta_4)$ and small $\varepsilon > 0$, $P_\varepsilon(l, \tau_\varepsilon(u)) \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ if $u \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon), l \in [0, l_t]$,*

Proof. In the proof, let $Y_\varepsilon \equiv \Upsilon_\varepsilon(\tau_\varepsilon(u))$, $Y_\varepsilon(l, u) \equiv \tilde\Phi(l, \varepsilon Y_\varepsilon, \tau_\varepsilon(u))/\varepsilon$ and $g_\varepsilon(l, u) \equiv \Upsilon_\varepsilon(P_\varepsilon(l, \tau_\varepsilon(u)))$. We note that $Y_\varepsilon = \Upsilon_\varepsilon(u)$. From Proposition 12 and the definition of l_t, we know that for $l \in [0, l_t]$ and small $\varepsilon > 0$,

$$g_\varepsilon(l, u) \in \mathcal{N}^{9d}_\varepsilon \ \text{ and } \ |Y_\varepsilon(l, u) - Y_\varepsilon| < r/5\varepsilon,\ |g_\varepsilon(l, u) - Y_\varepsilon| < r/5\varepsilon.$$

Since $u \in G^\delta_{c\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon)$, it follows that $\tau_\varepsilon(u) \in G^\delta_{c\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon) \subset N_{c\delta/4\omega(1-q)}(\mathcal{Z}^{10d}_\varepsilon)$ and

$$\int_{\Omega_\varepsilon\setminus B(Y_\varepsilon,1/\sqrt{\delta})} |\nabla\tau_\varepsilon(u)|^2 + \tau_\varepsilon(u)^2 - 2F(\tau_\varepsilon(u))dx \le \frac{1}{2}(\frac{c\delta}{4\omega})^2.$$

Thus, there exists $Y \in \mathcal{N}^{9d}_\varepsilon$ and $U \in \mathcal{S}$ such that $|Y_\varepsilon - Y| \le 3R_0$ and

$$\|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - Y)\|_\varepsilon \le c\delta/4\omega(1 - q) \quad (66)$$

Since $\varphi_\varepsilon(\cdot - Y_\varepsilon) \equiv 1$ in $B(Y_\varepsilon, r/3\varepsilon)$, $|g_\varepsilon(l, u) - Y_\varepsilon| < r/5\varepsilon$ and

$$\mathrm{dist}_{\partial\Omega}(\Phi(l, X), X) + |\Phi(l, X) - X| \le \min\{r/10, d/10\} \ \text{ for } \ l \in [0, l_t],$$

we see that for small $\varepsilon > 0$,

$$\begin{aligned} &|P_\varepsilon(l, \tau_\varepsilon(u)) - (\phi_\varepsilon U)(\cdot - Y) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)}|_{\delta,P_\varepsilon(l,\tau_\varepsilon(u))} \\ =&\int_{\Omega_\varepsilon\cap B(g_\varepsilon(l,u),1/\sqrt{\delta})} |\nabla((\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - Y)) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})|^2dx \\ &+ \int_{\Omega_\varepsilon\cap B(g_\varepsilon(l,u),1/\sqrt{\delta})} ((\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - Y)) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})^2dx. \end{aligned} \quad (67)$$

Then, using the same argument with that for the proof of (60), we deduce that

$$\begin{aligned} &|P_\varepsilon(l, \tau_\varepsilon(u)) - (\phi_\varepsilon U)(\cdot - Y) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)}|_{\delta,P_\varepsilon(l,\tau_\varepsilon(u))} \\ \le&\omega(\frac{c\delta}{4\omega(1 - q)})^2 \le \Big(\frac{c\delta}{4(1 - q)}\Big)^2. \end{aligned}$$

Moreover, following the arguments proving (61), (62) and (65) in the proof of Proposition 12, we deduce that

$$\|(\phi_\varepsilon U)(\cdot - Y) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)} - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,Y_\varepsilon(l,u)} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon}(Y))\| \le O(\varepsilon^{1/3}).$$

Therefore, when ε is sufficiently small,

$$|P_\varepsilon(l,\tau_\varepsilon(u)) - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon,Y_\varepsilon(l,u)} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon}(Y))|_{\delta,P_\varepsilon(l,\tau_\varepsilon(u))} < \Big(\frac{c\delta(1+q)}{4(1-q)}\Big)^2. \quad (68)$$

From now, we prove that

$$\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} |\nabla P_\varepsilon(l,\tau_\varepsilon(u))|^2 + P_\varepsilon(l,\tau_\varepsilon(u))^2 - 2F(P_\varepsilon(l,\tau_\varepsilon(u)))dx \le \frac{(c\delta)^2}{7}.$$

Note that, by Cauchy-Schwartz inequality,

$$\begin{aligned}
&\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} |\nabla P_\varepsilon(l,\tau_\varepsilon(u))|^2 + P_\varepsilon(l,\tau_\varepsilon(u))^2 dx \\
\le &\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} 2|\nabla(\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})|^2 dx \\
&+ \int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} 2|\nabla((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))|^2 dx \qquad (69)\\
&+ \int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} 2(\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})^2 dx \\
&+ \int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} 2((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))^2 dx.
\end{aligned}$$

By the decaying property of $\tau_\varepsilon(u)$, there exist some constants $c', C' > 0$, independent of ε, δ such that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))|^2 + ((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))^2 dx \\
=&\int_{\Omega_\varepsilon \setminus B(Y_\varepsilon, r/3\varepsilon)} |\nabla((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))|^2 + ((1-\varphi_\varepsilon(\cdot - Y_\varepsilon))\tau_\varepsilon(u))^2 dx \qquad (70)\\
\le& C' \exp(-c'/\varepsilon).
\end{aligned}$$

Since $|Y_\varepsilon(l,u) - g_\varepsilon(l,u)| \le R_1$ from Proposition 12, there exists $\delta_4' \in (0,\delta_3)$ such that for $\delta \in (0,\delta_4']$ and small $\varepsilon > 0$,

$$\Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)}(B(g_\varepsilon(l,u),1/\sqrt{\delta})) \supset B(Y_\varepsilon, 1/2\sqrt{\delta}).$$

Thus, we see that for $\delta \in (0,\delta_4')$ and small $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} |\nabla(\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})|^2 dx \\
&+ \int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u),1/\sqrt{\delta})} (\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u) \circ \Psi_{\varepsilon,Y_\varepsilon} \circ \Psi^{-1}_{\varepsilon,Y_\varepsilon(l,u)})^2 dx \\
\le& \omega^2 \int_{\Omega_\varepsilon \setminus B(Y_\varepsilon,1/2\sqrt{\delta})} |\nabla(\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u))|^2 + (\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u))^2 dx.
\end{aligned}$$

By decaying property of $\tau_\varepsilon(u)$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon \setminus B(Y_\varepsilon,1/2\sqrt{\delta})} |\nabla(\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u))|^2 + (\varphi_\varepsilon(\cdot - Y_\varepsilon)\tau_\varepsilon(u))^2 dx \\
=&\int_{\Omega_\varepsilon \setminus B(Y_\varepsilon,1/2\sqrt{\delta})} |\nabla(\tau_\varepsilon(u))|^2 + (\tau_\varepsilon(u))^2 dx + o(\varepsilon).
\end{aligned}$$

Since $|Y_\varepsilon - Y| \le 3R_0$ and $\phi_\varepsilon U$ has the exponential decay property, it follows from (66) that for small $\delta > 0$,

$$\int_{\Omega_\varepsilon \setminus B(Y_\varepsilon, 1/2\sqrt{\delta})} |\nabla \tau_\varepsilon(u)|^2 + \tau_\varepsilon(u)^2 dx \le \Big(\frac{c\delta}{4\omega(1-q)}\Big)^2 + o(\delta^2). \tag{71}$$

Then, combining (69), (70) and (71), we see that for small $\delta, \varepsilon > 0$,

$$\begin{aligned}&\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u), 1/\sqrt{\delta})} |\nabla P_\varepsilon(l, \tau_\varepsilon(u))|^2 + P_\varepsilon(l, \tau_\varepsilon(u))^2 dx \\ \le& 2\omega^2 (\frac{c\delta}{4\omega(1-q)})^2 + o(\delta^2) + o(\varepsilon)\end{aligned}$$

Then, it follows from (39) that for small $\delta, \varepsilon > 0$,

$$\begin{aligned}&\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u), 1/\sqrt{\delta})} |\nabla P_\varepsilon(l, \tau_\varepsilon(u))|^2 + P_\varepsilon(l, \tau_\varepsilon(u))^2 - 2F(P_\varepsilon(l, \tau_\varepsilon(u)))dx \\ \le& 2\omega^2(1+q)(c\delta/4\omega(1-q))^2 + o(\delta^2) + o(\varepsilon) \\ =&(1+q)(c\delta)^2/8(1-q)^2 + o(\delta^2) + o(\varepsilon).\end{aligned}$$

We take a small $\delta_4 \in (0, \delta_4')$ such that

$$(1+q)(c\delta)^2/8(1-q)^2 + o(\delta^2) \le (c\delta)^2/7, \quad \Big(\frac{c\delta(1+q)}{4(1-q)}\Big)^2 \le (c\delta)^2/7.$$

Thus, for $\delta \in (0, \delta_4)$ and small $\varepsilon > 0$,

$$\int_{\Omega_\varepsilon \setminus B(g_\varepsilon(l,u), 1/\sqrt{\delta})} |\nabla P_\varepsilon(l, \tau_\varepsilon(u))|^2 + P_\varepsilon(l, \tau_\varepsilon(u))^2 - 2F(P_\varepsilon(l, \tau_\varepsilon(u)))dx \le \frac{1}{2}\Big(\frac{3c\delta}{5}\Big)^2.$$

Also, we see from (68) that

$$|P_\varepsilon(l, \tau_\varepsilon(u)) - (\phi_\varepsilon U)(\cdot - \Psi_{\varepsilon, Y_\varepsilon(l,u)} \circ \Psi^{-1}_{\varepsilon, Y_\varepsilon}(Y))|_{\delta, P_\varepsilon(l, \tau_\varepsilon(u))} \le \frac{1}{2}\Big(\frac{3c\delta}{5}\Big)^2.$$

This implies that $P_\varepsilon(l, \tau_\varepsilon(u)) \in G^\delta_{3c\delta/5}(\mathcal{Z}^{10d}_\varepsilon)$ for $\delta \in (0, \delta_4)$, $l \in [0, l_t]$ and small $\varepsilon > 0$.

Lastly, the claim of the invariance of $G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ by the operator $P_\varepsilon(l, \tau_\varepsilon(\cdot))$ comes from the fact that for $\tau_\varepsilon(u) \notin G^\delta_{\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon)$, $\rho^\delta_2(\tau_\varepsilon(u)) = 0$ and $P_\varepsilon(l, \tau_\varepsilon(u)) = \tau_\varepsilon(u)$. This completes the proof. □

Proposition 14. *There exists $\delta_5 \in (0, \delta_4)$ such that for $\delta \in (0, \delta_5)$, the transplantation flow satisfies the energy decreasing property: for $u \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}^{8d}_\varepsilon$ and small $\varepsilon > 0$, the energy functional $\Gamma_\varepsilon(P_\varepsilon(l, \tau_\varepsilon(u)))$ is nonincreasing with respect to $l \in [0, l_t]$. Moreover, if $u \in G^\delta_{\delta/6\omega}(\mathcal{Z}^{10d}_\varepsilon)$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}^{5d}_\varepsilon \setminus \mathcal{N}^{4d}_\varepsilon$, there exists a constant $\mu_0 > 0$ independent of $\delta \in (0, \delta_5)$ such that for small $\varepsilon > 0$,*

$$\Gamma_\varepsilon(P_\varepsilon(l_t, \tau_\varepsilon(u))) - \Gamma_\varepsilon(P_\varepsilon(0, \tau_\varepsilon(u))) \le -\mu_0 \varepsilon.$$

Proof. Let $X_\varepsilon = \Upsilon_\varepsilon(\tau_\varepsilon(u))$, $X_\varepsilon(l,u) = \tilde{\Phi}(l, \varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u)), \tau_\varepsilon(u))/\varepsilon$ and $0 \le l < l + l' \le l_t$. Since the Jacobian determinants of $\Psi_{\varepsilon, X_\varepsilon}, \Psi^{-1}_{\varepsilon, X_\varepsilon(l,u)}$ are 1, it follows that

$$\begin{aligned}&\int_{\Omega_\varepsilon} ((\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \circ \Psi_{\varepsilon, X_\varepsilon} \circ \Psi^{-1}_{\varepsilon, X_\varepsilon(l+l',u)}(x))^2 dx \\ =&\int_{\Omega_\varepsilon} ((\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \circ \Psi_{\varepsilon, X_\varepsilon} \circ \Psi^{-1}_{\varepsilon, X_\varepsilon(l,u)}(x))^2 dx,\end{aligned}$$

and

$$\begin{aligned}
&\Gamma_\varepsilon(P_\varepsilon(l+l',\tau_\varepsilon(u))) - \Gamma_\varepsilon(P_\varepsilon(l,\tau_\varepsilon(u)))\\
&= \int_{\Omega_\varepsilon} \nabla((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u))\cdot\nabla\widetilde{\Psi}^{l+l'}_\varepsilon dx - \int_{\Omega_\varepsilon}\nabla((1-\varphi_\varepsilon(\cdot-X_\varepsilon))\tau_\varepsilon(u))\cdot\nabla\widetilde{\Psi}^l_\varepsilon dx\\
&\quad + \int_{\Omega_\varepsilon}((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u))\widetilde{\Psi}^{l+l'}_\varepsilon dx - \int_{\Omega_\varepsilon}((1-\varphi_\varepsilon(\cdot-X_\varepsilon))\tau_\varepsilon(u))\widetilde{\Psi}^l_\varepsilon dx\\
&\quad - \int_{\Omega_\varepsilon} F(P_\varepsilon(l+l',\tau_\varepsilon(u))) - F(P_\varepsilon(l,\tau_\varepsilon(u)))dx + \frac{1}{2}\int_{\Omega_\varepsilon}|\nabla\widetilde{\Psi}^{l+l'}_\varepsilon|^2 - |\nabla\widetilde{\Psi}^l_\varepsilon|^2 dx\\
&\equiv\ TI - TII + TIII - TIV + TV + TVI,
\end{aligned}$$

where $\widetilde{\Psi}^l_\varepsilon \equiv (\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u))\circ\Psi_{\varepsilon,X_\varepsilon}\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}$. We denote $w_\varepsilon \equiv (\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u))\circ\Psi_{\varepsilon,X_\varepsilon}$. Assume that $\rho_1(\varepsilon X_\varepsilon)\rho_2^\delta(\tau_\varepsilon(u)) > 0$ and let $h \equiv \rho_1(\varepsilon X_\varepsilon)\rho_2^\delta(\tau_\varepsilon(u))l'$. We may assume l' is small so that

$$\mathrm{dist}_{\partial\Omega}(\varepsilon X_\varepsilon(l,u), \varepsilon X_\varepsilon(l+l',u)) < \frac{r}{100} \text{ and } |E(\varepsilon X_\varepsilon(l,u)) - E(\varepsilon X_\varepsilon(l+l',u))| < \frac{1}{1000}. \tag{72}$$

If $h = 0$, $\Gamma_\varepsilon(P_\varepsilon(l+l',\tau_\varepsilon(u))) = \Gamma_\varepsilon(P_\varepsilon(l,\tau_\varepsilon(u)))$. Now, we assume that $h > 0$.

First, we estimate $|TIII - TIV|/h$.

$$\begin{aligned}
&|TIII - TIV|\\
&= \left|\int_{\Omega_\varepsilon}((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u))[w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} - w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}](x)dx\right|\\
&\le \int_{\Omega_\varepsilon}|((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u))|A_1(x)A_2(x)dx
\end{aligned}$$

where $A_2(x) = \left|\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)\right|$ and

$$A_1(x) = \frac{|w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x) - w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)|}{|\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)|}.$$

From (iii) in Proposition 3, we can see that for small $\varepsilon > 0$,

$$\mathrm{supp}(w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}) \subset \Omega_\varepsilon\cap B(X_\varepsilon(l,u), \frac{r}{2\varepsilon} + \frac{r}{50\varepsilon}) \subset \Omega_\varepsilon\cap B(X_\varepsilon, \frac{4r}{5\varepsilon}). \tag{73}$$

Recall that for $y = (y', y_n) = \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)$ and $\tilde{y} = (\tilde{y}', \tilde{y}_n) = \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x)$,

$$\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) = (E(\varepsilon X_\varepsilon(l,u)))^{-1}\begin{pmatrix} x_1 - (X_\varepsilon(l,u))_1\\ \vdots\\ x_n - (X_\varepsilon(l,u))_n\end{pmatrix} - \begin{pmatrix}0\\ \vdots\\ \psi_{\varepsilon,X_\varepsilon(l,u)}(y')\end{pmatrix}$$

and there is a similar identity for $\tilde{y} = \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x)$.

When $x \in \mathrm{supp}(w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}) \cup \mathrm{supp}(w_\varepsilon\circ\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)})$, it follows from (72), (73) and (ii) of Proposition 3 that

$$\begin{aligned}
&|y-\tilde{y}| = |\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)| \\
\le &|(E(\varepsilon X_\varepsilon(l+l',u)))^{-1} - (E(\varepsilon X_\varepsilon(l,u)))^{-1}||x - X_\varepsilon(l+l',u)| \\
&+ |X_\varepsilon(l+l',u) - X_\varepsilon(l,u)| + |\psi_{\varepsilon,X_\varepsilon(l,u)}(y') - \psi_{\varepsilon,X_\varepsilon(l+l',u)}(\tilde{y}')| \\
\le &|(E(\varepsilon X_\varepsilon(l+l',u)))^{-1} - (E(\varepsilon X_\varepsilon(l,u)))^{-1}||x - X_\varepsilon(l+l',u)| \\
&+ |X_\varepsilon(l+l',u) - X_\varepsilon(l,u)| + |\psi_{\varepsilon,X_\varepsilon(l,u)}(y') - \psi_{\varepsilon,X_\varepsilon(l+l',u)}(y')| \\
&+ |\psi_{\varepsilon,X_\varepsilon(l+l',u)}(y') - \psi_{\varepsilon,X_\varepsilon(l+l',u)}(\tilde{y}')| \\
\le &|(E(\varepsilon X_\varepsilon(l+l',u)))^{-1} - (E(\varepsilon X_\varepsilon(l,u)))^{-1}||x - X_\varepsilon(l+l',u)| \\
&+ 2\mathrm{dist}_{\partial\Omega_\varepsilon}(X_\varepsilon(l+l',u), X_\varepsilon(l,u)) + a|y-\tilde{y}|;
\end{aligned} \tag{74}$$

thus

$$\begin{aligned}
\Big|\frac{A_2(x)}{h}\Big| \le &\frac{r}{\varepsilon(1-a)} \frac{|(E(\varepsilon X_\varepsilon(l+l',u)))^{-1} - (E(\varepsilon X_\varepsilon(l,u)))^{-1}|}{h} \\
&+ \frac{2}{1-a}\frac{\mathrm{dist}_{\partial\Omega_\varepsilon}(X_\varepsilon(l+l',u), X_\varepsilon(l,u))}{h}.
\end{aligned} \tag{75}$$

Since the frame E and the transplantation flow Φ are smooth, we see from the notation $X_\varepsilon = \Upsilon_\varepsilon(\tau_\varepsilon(u))$, $X_\varepsilon(l,u) = \tilde{\Phi}(l, \varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u)), \tau_\varepsilon(u))/\varepsilon$ that for some constant $C>0$,

$$\Big|\frac{A_2(x)}{h}\Big| \le C/\varepsilon. \tag{76}$$

Since $\mathrm{supp}(1-\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \cap B(X_\varepsilon, r/3\varepsilon) = \emptyset$, using (76) and mean value theorem for A_1 with the decay properties of Propositions 8, 9, we deduce that for some $c, D > 0$, independent of small $\delta > 0, \varepsilon > 0$,

$$\Big|\frac{TIII - TIV}{h}\Big| \le \frac{D}{\varepsilon}\int_{\Omega_\varepsilon \setminus B(X_\varepsilon, r/3\varepsilon)} \exp(-c|x - X_\varepsilon|)dx = o(\varepsilon). \tag{77}$$

We can estimate $|TI - TII|/h$ in a similar way to $|TIII - TIV|/h$. We note that

$$\begin{aligned}
&|TI - TII| \\
=&\Big|\int_{\Omega_\varepsilon} \nabla((1-\varphi_\varepsilon)\tau_\varepsilon(u)) \cdot [\nabla(w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}) - \nabla(w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)})](x)dx\Big| \\
\le &\int_{\Omega_\varepsilon} |\nabla((1-\varphi_\varepsilon)\tau_\varepsilon(u))| \ \Big(B_1(x)B_2(x) + B_3(x)|\nabla w_\varepsilon \circ (\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)})|\Big)dx
\end{aligned}$$

where

$$B_1(x) = \frac{|\nabla w_\varepsilon(\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}) D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) - \nabla w_\varepsilon(\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}) D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)|}{|\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}(x) - \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)|},$$

$B_2(x) = \Big|\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} - \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\Big|$ and $B_3(x) = \Big|D\Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} - D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\Big|$.
We see from the mean value theorem and decay property of Propositions 8, 9 that B_1 is uniformly bounded on $\Omega_\varepsilon \setminus B(X_\varepsilon, r/3\varepsilon)$ for small $\varepsilon > 0$ and $\delta > 0$, by the same arguments with the estimate for $|TIII - TIV|/h$, we see that

$$\frac{|TI - TII|}{h} = o(\varepsilon).$$

We estimate $|TV|/h$. We denote also $w_\varepsilon \equiv (\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \circ \Psi_{\varepsilon,X_\varepsilon}$. Since we have $\det D\Psi_{\varepsilon,X_\varepsilon(l,u)} = 1$ for $l \in [0,l_t]$, we see that

$$\begin{aligned}
&|TV|\\
=&\Big|\int_{\Omega_\varepsilon} F((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + (\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \circ \Psi_{\varepsilon,X_\varepsilon} \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)})dx\\
&\qquad - \int_{\Omega_\varepsilon} F((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + (\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u)) \circ \Psi_{\varepsilon,X_\varepsilon} \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)})dx\Big|\\
=&\Big|\int_{\Omega_\varepsilon} F((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)}) - F(w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)})\\
&\qquad - \int_{\Omega_\varepsilon} F((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}) - F(w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)})dx\Big|\\
=&\Big|\int_{\Omega_\varepsilon} b(1) - b(0)dx\Big|,
\end{aligned}$$

where $b(t) = F((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + a(t)) - F(a(t))$ with

$$a(t) = tw_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} + (1-t)w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}.$$

Then, by the mean value theorem, there exist $t_0, t_1 \in [0,1]$ such that

$$\begin{aligned}
|TV| &= \Big|\int_{\Omega_\varepsilon} (f((1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + a(t_0)) - f(a(t_0)))a'(t_0)dx\Big|\\
&= \Big|\int_{\Omega_\varepsilon} f'((t_1(1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u) + a(t_0))(1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u)a'(t_0)dx\Big|.
\end{aligned}$$

Since

$$a'(t) = w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} - w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}$$

and $(1-\varphi_\varepsilon(\cdot - X_\varepsilon))$ vanishes on $B(X_\varepsilon, r/3\varepsilon)$, we see from the exponential decay property of $\tau_\varepsilon(u)$ that for some $C>0$, independent of small $\varepsilon>0$,

$$\begin{aligned}
|TV| \le& C\Big|\int_{\Omega_\varepsilon\setminus B(X_\varepsilon,r/3\varepsilon)} (1-\varphi_\varepsilon(\cdot - X_\varepsilon))\tau_\varepsilon(u)(w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l+l',u)} - w_\varepsilon \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)})dx\Big|\\
\le& C|TIII - TIV|.
\end{aligned}$$

Then, by the estimate (77), we see that $\frac{|TV|}{h} = o(\varepsilon)$.

Finally, we estimate TVI/h. For a change of variable $y = \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x)$, we see that

$$\begin{aligned}
&D\Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}(x) = (D\Psi_{\varepsilon,X_\varepsilon(l,u)}(y))^{-1}\\
=&\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0\\
0 & 1 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & \cdots & 0 & 1 & 0\\
-D_1\psi_{\varepsilon X_\varepsilon(l,u)}(\varepsilon y') & \cdot & \cdots & -D_{n-1}\psi_{\varepsilon X_\varepsilon(l,u)}(\varepsilon y') & 1
\end{pmatrix}(E(\varepsilon X_\varepsilon(l,u)))^{-1};
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\nabla\big((\varphi_\varepsilon\tau_\varepsilon(u)) \circ \Psi_{\varepsilon,X_\varepsilon} \circ \Psi^{-1}_{\varepsilon,X_\varepsilon(l,u)}\big)(x)\\
=&\Big(\nabla w_\varepsilon(y) - D_n w_\varepsilon(y)\left(D_1\psi_{\varepsilon X_\varepsilon(l,u)}(\varepsilon y'), \cdots, D_{n-1}\psi_{\varepsilon X_\varepsilon(l,u)}(\varepsilon y'), 0\right)\Big)\\
&\times (E(\varepsilon X_\varepsilon(l,u)))^{-1}.
\end{aligned}$$

Here, the right hand side is the multiplication of a row vector and a $n \times n$ matrix. Then, it follows that

$$TVI = \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n w_\varepsilon(y) \sum_{i=1}^{n-1} D_i w_\varepsilon(y)(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy$$
$$+ \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} |D_n w_\varepsilon(y)|^2 \sum_{i=1}^{n-1} (|D_i \psi_{\varepsilon X_\varepsilon(l+l',u)}|^2 - |D_i \psi_{\varepsilon X_\varepsilon(l,u)}|^2)(\varepsilon y')dy.$$

By (ii),(iv) in Proposition 3, there exists $M > 0$ such that for small $\varepsilon > 0$,

$$\frac{TVI}{h} \leq \frac{1}{h} \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n w_\varepsilon(y) \sum_{i=1}^{n-1} D_i w_\varepsilon(y)(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy$$
$$+ M\varepsilon^2 \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} |D_n w_\varepsilon(y)|^2 |(y_1, \cdots, y_{n-1})|^2 dy.$$

Since $\tau_\varepsilon(u) \in G^\delta_{\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon) \subset N_{\delta/4\omega(1-q)}(\mathcal{Z}^{10d}_\varepsilon)$, $\|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - z)\|_{H^1(\Omega_\varepsilon)} < \delta/4\omega(1-q)$ for some z with $|z - X_\varepsilon| < 3R_0$ and $U \in \mathcal{S}$. Then, for sufficiently small $\varepsilon > 0$,

$$\|\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - z)\|_{H^1(\Omega_\varepsilon)} < \delta,$$
$$\|(\varphi_\varepsilon(\cdot - X_\varepsilon)\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - z)) \circ \Psi_{\varepsilon,X_\varepsilon}\|_{H^1(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} < \delta.$$

Moreover, by Proposition 4,

$$\|(\phi_\varepsilon U)(\cdot - z) \circ \Psi_{\varepsilon,X_\varepsilon} - (\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_{H^1(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} \leq O(\varepsilon^{1/3}).$$

Therefore, for sufficiently small $\varepsilon > 0$,

$$\|w_\varepsilon - (\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))\|_{H^1(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))} < 2\delta. \tag{78}$$

Let $\tilde{w}_\varepsilon = (\phi_\varepsilon U)(\cdot - \Psi^{-1}_{\varepsilon,X_\varepsilon}(z))$. By (ii) in Proposition 3, the exponential decay properties of $w_\varepsilon, \tilde{w}_\varepsilon$, and (78), we see that for some $C > 0$, independent of small $\varepsilon > 0, \delta > 0$,

$$\int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n(w_\varepsilon - \tilde{w}_\varepsilon) \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y) \frac{1}{h}(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy$$
$$+ \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n w_\varepsilon \sum_{i=1}^{n-1} D_i(w_\varepsilon - \tilde{w}_\varepsilon)(y) \frac{1}{h}(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy$$
$$\leq C\varepsilon\delta.$$

This implies that

$$\begin{aligned} &\frac{1}{h} \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n w_\varepsilon(y) \sum_{i=1}^{n-1} D_i w_\varepsilon(y)(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy \\ \leq &\frac{1}{h} \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n \tilde{w}_\varepsilon(y) \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y)(D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y')dy \\ &+ C\varepsilon\delta. \end{aligned} \tag{79}$$

Since $\text{supp}(\tilde{w}_\varepsilon) \subset B(0, 2\varepsilon^{-1/3})$ for small $\varepsilon > 0$, we see from (iv) in Proposition 3 and the radial symmetry property of $\phi_\varepsilon U$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+ \cap B(0,2\varepsilon^{-1/3})} D_n \tilde{w}_\varepsilon \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y) \frac{1}{h} (D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y') dy \\
&= \int_{\mathbb{R}^n_+ \cap B(0,2\varepsilon^{-1/3})} D_n \tilde{w}_\varepsilon \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y) \frac{1}{h} (D_i \psi_{\varepsilon, X_\varepsilon(l,u)} - D_i \psi_{\varepsilon, X_\varepsilon(l+l',u)})(y') dy \\
&= - \int_{\mathbb{R}^n_+ \cap B(0,2\varepsilon^{-1/3})} D_n \tilde{w}_\varepsilon \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y) \frac{1}{h} (A^i_{\varepsilon, X_\varepsilon(l,u)} - A^i_{\varepsilon, X_\varepsilon(l+l',u)}) \cdot y' dy + o(\varepsilon) \\
&= - \int_{\mathbb{R}^n_+ \cap B(0,2\varepsilon^{-1/3})} D_n (\phi_\varepsilon U) \sum_{i=1}^{n-1} D_i (\phi_\varepsilon U) \frac{1}{h} (A^i_{\varepsilon, X_\varepsilon(l,u)} - A^i_{\varepsilon, X_\varepsilon(l+l',u)}) \cdot y' dy + o(\varepsilon) \\
&= - \frac{1}{h(n-1)} \sum_{i=1}^{n-1} \left(a^i_{\varepsilon, X_\varepsilon(l,u)} - a^i_{\varepsilon, X_\varepsilon(l+l',u)} \right) \int_{\mathbb{R}^n_+} |\nabla U(y)|^2 y_n |y'|^2 / |y|^2 dy + o(\varepsilon),
\end{aligned}
$$

where $a^i_{\varepsilon,X}$ is the i-th component of $A^i_{\varepsilon,X}$ for $X \in \partial\Omega_\varepsilon$. Since $\frac{1}{n-1} \sum_{i=1}^{n-1} a^i_{\varepsilon,X}$ is the mean curvature of Ω_ε at $X \in \partial\Omega_\varepsilon$, we see that

$$
\frac{1}{h(n-1)} \sum_{i=1}^{n-1} \left(a^i_{\varepsilon, X_\varepsilon(l,u)} - a^i_{\varepsilon, X_\varepsilon(l+l',u)} \right) = \varepsilon \frac{H(\varepsilon X_\varepsilon(l,u)) - H(\varepsilon X_\varepsilon(l+l',u))}{h},
$$

where $H(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$. We note that for any radially symmetric function G,

$$
\int_{\mathbb{R}^n_+} G(y) y_n dy = \frac{1}{n-1} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} G(x)|x| dx
$$

and

$$
\int_{\mathbb{R}^n_+} G(y)(y_n)^3 dy = \frac{2}{(n-1)(n+1)} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} G(x)|x|^3 dx.
$$

Using these identities, we get

$$
\int_{\mathbb{R}^n_+} |\nabla U(y)|^2 y_n |y'|^2 / |y|^2 dy = \frac{1}{n+1} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} |\nabla U|^2 |y| dy.
$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+ \cap B(0,2\varepsilon^{-1/3})} D_n \tilde{w}_\varepsilon \sum_{i=1}^{n-1} D_i \tilde{w}_\varepsilon(y) \frac{1}{h} (D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y') dy \\
=& \varepsilon \frac{H(\varepsilon X_\varepsilon(l+l',u)) - H(\varepsilon X_\varepsilon(l,u))}{h} \frac{1}{n+1} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} |\nabla U|^2 |y| dy + o(\varepsilon).
\end{aligned}
$$

Then, (79) implies that

$$
\begin{aligned}
&\frac{1}{h} \int_{\mathbb{R}^n_+ \cap B(0,r/\varepsilon)} D_n w_\varepsilon(y) \sum_{i=1}^{n-1} D_i w_\varepsilon(y) (D_i \psi_{\varepsilon X_\varepsilon(l,u)} - D_i \psi_{\varepsilon X_\varepsilon(l+l',u)})(\varepsilon y') dy \\
\leq& \varepsilon \frac{H(\varepsilon X_\varepsilon(l+l',u)) - H(\varepsilon X_\varepsilon(l,u))}{h} \frac{1}{n+1} \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\mathbb{R}^n} |\nabla U|^2 |y| dy + C\varepsilon\delta + o(\varepsilon).
\end{aligned}
$$

Since

$$\begin{aligned}&\frac{H(\varepsilon X_\varepsilon(l+l',u))-H(\varepsilon X_\varepsilon(l,u))}{h}\\ =&\frac{H(\Phi(\rho_1(\varepsilon X_\varepsilon)\rho_2^\delta(\tau_\varepsilon)l+h,\varepsilon X_\varepsilon))-H(\Phi(\rho_1(\varepsilon X_\varepsilon)\rho_2^\delta(\tau_\varepsilon)l,\varepsilon X_\varepsilon)}{h},\end{aligned}$$

$X_\varepsilon=\varepsilon\Upsilon(\tau_\varepsilon(u))\in\mathcal{N}^{8d}\setminus\mathcal{N}^{2d}$ if $h\neq 0$ and $|\Phi(l,\varepsilon X_\varepsilon)-\varepsilon X_\varepsilon|\le d/10$ for $l\in[0,l_t]$, we see from Proposition 1 that

$$\frac{H(\varepsilon X_\varepsilon(l+l',u))-H(\varepsilon X_\varepsilon(l,u))}{h}\le-\alpha.$$

Thus, taking a small $\delta_5\in(0,\delta_4)$, we see that for $\delta\in(0,\delta_5)$,

$$\frac{TVI}{h}\le-\varepsilon\frac{\alpha}{2(n+1)}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U|^2|y|dy \text{ if } h>0.$$

Then, combining the estimates for $TI, TII, TIII, TIV, TV$ and TVI, we see that for $\delta\in(0,\delta_5)$ and small $\varepsilon>0$, the energy nonincreasing property holds. Especially, if $u\in G^\delta_{\delta/6\omega}(\mathcal{Z}^{10d}_\varepsilon)$ with $\Upsilon_\varepsilon(u)\in\mathcal{N}^{5d}_\varepsilon\setminus\mathcal{N}^{4d}_\varepsilon$, we have $h=l'$. Thus, if $u\in G^\delta_{\delta/6\omega}(\mathcal{Z}^{10d}_\varepsilon)$ with $\Upsilon_\varepsilon(u)\in\mathcal{N}^{5d}_\varepsilon\setminus\mathcal{N}^{4d}_\varepsilon$, for sufficiently small $\varepsilon>0$,

$$\Gamma_\varepsilon(P_\varepsilon(l_t,\tau_\varepsilon(u)))-\Gamma_\varepsilon(P_\varepsilon(0,\tau_\varepsilon(u)))\le-\varepsilon\mu_0,$$

where $\mu_0=\inf_{U\in\mathcal{S}}\frac{l_t\alpha}{4(n+1)}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n_+}|\nabla U|^2|y|dy$. This completes the proof. □

9. **Proof of the main theorem.** In this section, we will prove that there exist small $\delta,\nu>0$ such that for small $\varepsilon>0$, Γ_ε has a critical point in

$$G(\varepsilon,\nu,\delta)\equiv\left(\Gamma_\varepsilon^{b_\varepsilon}\setminus\Gamma_\varepsilon^{b_\varepsilon-\varepsilon\nu}\right)\cap G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon).$$

Recall b_ε is the value coming from the upper energy estimate (51) given by

$$b_\varepsilon=\frac{1}{2}\Gamma(U)+\varepsilon\frac{m}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U|^2|y|dy+o(\varepsilon).$$

We fix $\delta\in(0,\delta_5)$ so that $q=q(\delta)<1/1000$. Then, by Proposition 7,

$$G^\delta_{3\delta/8}(\mathcal{Z}^{10d}_\varepsilon)\subset N_{5\delta/12}(\mathcal{Z}^{10d}_\varepsilon)\subset N_{\delta/2}(\mathcal{Z}^{10d}_\varepsilon)\subset G^\delta_{4\delta/5}(\mathcal{Z}^{10d}_\varepsilon)\subset N_\delta(\mathcal{Z}^{10d}_\varepsilon). \tag{80}$$

In the definition of the initial surface, we can see that $\tilde{A}_\varepsilon(1,z)\in G^\delta_{\delta/40\omega}(\mathcal{Z}^{10d}_\varepsilon)$ for small $\varepsilon>0$. By Proposition 10, there exist a constant $\nu>0$ such that for small $\varepsilon>0$,

$$\Gamma_\varepsilon(\tilde{A}_\varepsilon(t,z))\le b_\varepsilon-\varepsilon\nu \text{ if } \tilde{A}_\varepsilon(t,z)\notin G^\delta_{\delta/40\omega}(\mathcal{Z}^{10d}_\varepsilon) \text{ or } \Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z))\notin\mathcal{N}^{2d}_\varepsilon. \tag{81}$$

We will prove by contradiction. **We assume that for some small $\varepsilon>0$, Γ_ε has no critical points in $G(\varepsilon,\nu,\delta)$.**

Now, we will define a gradient flow. We choose a smooth function χ^ν_ε on $\mathbb{R}$ such that $\chi^\nu_\varepsilon(l)=1$ if $|l-b_\varepsilon|\le\varepsilon\nu/2$ and $\chi^\nu_\varepsilon(l)=0$ if $|l-b_\varepsilon|\ge\varepsilon\nu$. By (80), we can take a smooth Lipschitz function $\kappa^\delta_\varepsilon$ on $H^1(\Omega_\varepsilon)$ such that $\kappa^\delta_\varepsilon(u)=1$ for $u\in G^\delta_{3\delta/8}(\mathcal{Z}^{10d}_\varepsilon)$ and $\kappa^\delta_\varepsilon(u)=0$ for $u\notin G^\delta_{4\delta/5}(\mathcal{Z}^{10d}_\varepsilon)$. We define the gradient flow by the following ODE:

$$\frac{d\eta_\varepsilon(s,u)}{ds}=-\chi^\nu_\varepsilon(\Gamma_\varepsilon(\eta_\varepsilon))\kappa^\delta_\varepsilon(\eta_\varepsilon)\Gamma'_\varepsilon(\eta_\varepsilon)/\|\Gamma'_\varepsilon(\eta_\varepsilon)\|^*_\varepsilon,\quad \eta_\varepsilon(0,u)=u. \tag{82}$$

This ODE has a unique solution $\eta_\varepsilon=\eta_\varepsilon(s,u)$ for $s\in[0,\infty)$. By the definition of cut-off functions χ^ν_ε and $\kappa^\delta_\varepsilon$, we can see that the set $(\Gamma^{b_\varepsilon}_\varepsilon\setminus\Gamma^{b_\varepsilon-\varepsilon\nu}_\varepsilon)\cap G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ is invariant by the flow η_ε.

Proposition 15. *Let* $u \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ *and* $\Upsilon_\varepsilon(u) \in \mathcal{N}^{9d}_\varepsilon$. *Suppose that for* $0 \le s_1(\varepsilon) < s_2(\varepsilon)$ *and some constant* $c > 0$*, independent of* $\varepsilon > 0$,

$$|\Upsilon_\varepsilon(\eta_\varepsilon(s_1, u)) - \Upsilon_\varepsilon(\eta_\varepsilon(s_2, u))| \ge c/\varepsilon.$$

Then

$$\lim_{\varepsilon\to 0} |s_2(\varepsilon) - s_1(\varepsilon)| = \infty.$$

Proof. We consider the function $\Upsilon_\varepsilon(\eta_\varepsilon(s, u)), s \in [s_1, s_2]$. We take a partition $s_1 = s^0 < s^1 < \cdots < s^{k-1} < s^k = s_2$ of $[s_1, s_2]$ such that $|\Upsilon_\varepsilon(\eta_\varepsilon(s^i, u)) - \Upsilon_\varepsilon(\eta_\varepsilon(s^{i-1}, u))| \ge c/k\varepsilon \ge 8R_0$ for $i = 1, \cdots, k$. From Proposition 5, for each $i = 0, \cdots, k$, there exist $z_i \in \mathcal{N}^{10d}_\varepsilon$ and $W_i \in S(z_i)$ satisfying

$$\|\eta_\varepsilon(s^i, u) - W_i\| \le \delta, \quad |\Upsilon_\varepsilon(\eta_\varepsilon(s^i, u)) - z_i| \le 3R_0.$$

By (34), (35) and (36), it holds that for small $\varepsilon > 0$,

$$\begin{aligned}
&\|\eta_\varepsilon(s^{i+1}, u) - \eta_\varepsilon(s^i, u)\|_\varepsilon \\
\ge &\|W_i - W_{i+1}\|_\varepsilon - \|\eta_\varepsilon(s^i, u) - W_i\|_\varepsilon - \|\eta_\varepsilon(s^{i+1}, u) - W_{i+1}\|_\varepsilon \\
\ge &\xi/2 - 2\delta \ge 10\delta - 2\delta = 8\delta.
\end{aligned}$$

Note that $\|\partial\eta_\varepsilon(s, u)/\partial s\|^*_\varepsilon \le 1$. Then it is standard to see that for some $A > 0$, independent of $i = 1, \cdots, k$ and small $\varepsilon > 0$, $|s^i - s^{i-1}| \ge A$. This implies that for any $k \ge 1$, $|s_1 - s_2| \ge kA$ if $\varepsilon > 0$ is sufficiently small. This proves the claim. □

Since $q = q(\delta) < 1/1000$, it holds that for any $a, b \in (0, \delta)$,

$$G^\delta_a(\mathcal{Z}^{10d}_\varepsilon) \subset N_{1000a/999}(\mathcal{Z}^{10d}_\varepsilon), \quad N_{2b/3}(\mathcal{Z}^{10d}_\varepsilon) \subset G^\delta_b(\mathcal{Z}^{10d}_\varepsilon). \tag{83}$$

Proposition 16. *Suppose that* $u_\varepsilon \in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon) \cap \Gamma^{b_\varepsilon}$ *and for some* $l \in [0, \delta/120\omega]$, $\eta_\varepsilon(l, u_\varepsilon) \in G^\delta_{\delta/6\omega}(\mathcal{Z}^{10d}_\varepsilon) \setminus G^\delta_{\delta/40\omega}(\mathcal{Z}^{10d}_\varepsilon)$ *and* $\Upsilon_\varepsilon(\eta_\varepsilon(l, u_\varepsilon)) \in \mathcal{N}^{8d}_\varepsilon$. *Then,*

$$\Gamma_\varepsilon\big(\eta_\varepsilon\big(\frac{\delta}{120\omega}, u_\varepsilon\big)\big) \le b_\varepsilon - \nu\varepsilon/2$$

for sufficiently small $\varepsilon > 0$.

Proof. We note from (83) that

$$G^\delta_{\delta/6\omega}(\mathcal{Z}^{10d}_\varepsilon) \subset N_{5\delta/24\omega}(\mathcal{Z}^{10d}_\varepsilon) \subset N_{\delta/4\omega}(\mathcal{Z}^{10d}_\varepsilon) \subset G^\delta_{3\delta/8\omega}(\mathcal{Z}^{10d}_\varepsilon),$$

and that

$$G^\delta_{\delta/40\omega}(\mathcal{Z}^{10d}_\varepsilon) \supset N_{\delta/60\omega}(\mathcal{Z}^{10d}_\varepsilon) \supset N_{\delta/80\omega}(\mathcal{Z}^{10d}_\varepsilon) \supset G^\delta_{\delta/120\omega}(\mathcal{Z}^{10d}_\varepsilon).$$

We see that

$$\delta/4\omega - 5\delta/24\omega \ge \delta/240\omega, \quad \delta/60\omega - \delta/80\omega \ge \delta/240\omega.$$

Thus, if $l \in [\delta/240\omega, \delta/120\omega]$, we see from the fact $\|d\eta_\varepsilon/ds\|_\varepsilon \le 1$ that

$$\eta_\varepsilon(s, u_\varepsilon) \in G^\delta_{3\delta/8\omega}(\mathcal{Z}^{10d}_\varepsilon) \setminus G^\delta_{\delta/120\omega}(\mathcal{Z}^{10d}_\varepsilon) \text{ for } s \in [l - \delta/240\omega, l],$$

and from Proposition 15 that if $\varepsilon > 0$ is small,

$$\Upsilon_\varepsilon(\eta_\varepsilon(s, u_\varepsilon)) \in \mathcal{N}^{9d}_\varepsilon \text{ for } s \in [l - \delta/240\omega, l].$$

If $\Gamma_\varepsilon(\eta_\varepsilon(s, u_\varepsilon)) \ge b_\varepsilon - \varepsilon\nu/2$ for any $s \in [l - \delta/240\omega, l]$, then from the definition of cut-off functions $d\eta_\varepsilon/dt = -\Gamma'_\varepsilon(\eta_\varepsilon)/||\Gamma'_\varepsilon(\eta_\varepsilon)\|^*$ for small $\varepsilon > 0$. Since the flow η_ε decreases the energy Γ_ε, we see that for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\eta_\varepsilon(l, u)) - b_\varepsilon \le \Gamma_\varepsilon(\eta_\varepsilon(l, u_\varepsilon)) - \Gamma_\varepsilon(\eta_\varepsilon(l - \delta/240\omega, u_\varepsilon))$$

$$= \int_{l-\delta/240\omega}^{l} \Gamma'_\varepsilon(\eta_\varepsilon(s,u_\varepsilon))\frac{d\eta_\varepsilon(s,u_\varepsilon)}{ds}ds = -\int_{l-\delta/240\omega}^{l} \|\Gamma'_\varepsilon\|^*_\varepsilon ds$$
$$\le -\frac{\delta}{240\omega}\mu(3\delta/8\omega, \delta/120\omega) \le -\varepsilon\nu.$$

This contradicts that $\Gamma_\varepsilon(\eta_\varepsilon(s,u_\varepsilon)) \ge b_\varepsilon - \varepsilon\nu/2$ for any $s \in [l-\delta/240\omega, l]$; thus

$$\Gamma_\varepsilon(\eta_\varepsilon(\delta/120\omega, u_\varepsilon)) \le b_\varepsilon - \varepsilon\nu/2.$$

If $l \in [0, \delta/240\omega]$, by a similar argument with $s \in [l, l+\delta/240\omega]$, we can show that for small $\varepsilon > 0$, $\Gamma_\varepsilon(\eta_\varepsilon(\delta/120\omega, u_\varepsilon)) \le b_\varepsilon - \varepsilon\nu/2$. This completes the proof. □

9.1. **Iteration through a gradient flow and a transplantation flow.** For $l_g \equiv \frac{\delta}{120\omega}$, we define an operator

$$I(u) \equiv \tau_\varepsilon \circ P_\varepsilon(l_t,\cdot) \circ \tau_\varepsilon \circ \eta_\varepsilon(l_g, u),$$

where $l_t \in (0,1)$ is chosen so that (54) holds. Let I^i be the i-fold composition of I. Recall the notation

$$G(\varepsilon,\nu,\delta) \equiv \left(\Gamma_\varepsilon^{b_\varepsilon} \setminus \Gamma_\varepsilon^{b_\varepsilon-\varepsilon\nu}\right) \cap G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d}).$$

Since we assume that there exist no solutions in $G(\varepsilon,\nu,\delta)$ for small $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ such that

$$\|\Gamma'_\varepsilon(u)\|^*_\varepsilon \ge k(\varepsilon) \text{ for any } u \in G(\varepsilon,\nu,\delta).$$

We take a positive integer j_ε satisfying

$$j_\varepsilon \ge \frac{\varepsilon\nu}{k(\varepsilon)l_g}.$$

Proposition 17. *If there exists no critical points of Γ_ε in $G(\varepsilon,\nu,\delta)$, then*

$$\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) \le b_\varepsilon - \varepsilon\min\{\nu/2, \mu_0/2\} \text{ for any } t \in [0,T], z \in L,$$

where μ_0 is the constant in Proposition 14.

Proof. Note that Γ_ε is nonincreasing by $\eta_\varepsilon, \tau_\varepsilon, P_\varepsilon$. By (81), it suffices to consider only when $\tilde{A}_\varepsilon(t,z) \in G^\delta_{\delta/40\omega}$ and $\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) \in \mathcal{N}_\varepsilon^{2d}$. We recall the following properties.

(i) (Sections 4 and 5)

$$\tau_\varepsilon(G^\delta_{c\delta}(\mathcal{Z}_\varepsilon^{10d})) \subset G^\delta_{c\delta}(\mathcal{Z}_\varepsilon^{10d}) \text{ for each } c \in (0,1).$$
$$\Upsilon_\varepsilon(\tau_\varepsilon(u)) = \Upsilon_\varepsilon(u) \text{ for } u \in G^\delta_\delta(\mathcal{Z}_\varepsilon^{10d}).$$

(ii) (Proposition 12) For $u \in G^\delta_\delta(\mathcal{Z}_\varepsilon^{10d}), l \in [0, l_t]$,

$$|\Upsilon_\varepsilon(P_\varepsilon(l,u)) - \tilde{\Phi}(l, \varepsilon\Upsilon_\varepsilon(u), u)/\varepsilon| \le R_1.$$

(iii) (Proposition 15) For $u \in G^\delta_\delta(\mathcal{Z}_\varepsilon^{10d})$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{9d}$,

$$|\Upsilon_\varepsilon(\eta_\varepsilon(l_g,u)) - \Upsilon_\varepsilon(u)| \le o(1)/\varepsilon \text{ as } \varepsilon \to 0.$$

(iv) (Proposition 13) For $c \in (0,1]$ and $u \in G^\delta_{c\delta/4\omega}(\mathcal{Z}_\varepsilon^{10d})$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{8d}$, then $P_\varepsilon(l, \tau_\varepsilon(u)) \in G^\delta_{3c\delta/5}(\mathcal{Z}_\varepsilon^{10d})$ when $l \in [0, l_t]$ and $\varepsilon > 0$ is small. Thus,

$$P_\varepsilon(l, \tau_\varepsilon(\tilde{A}_\varepsilon(t,z))) = P_\varepsilon(l, \tilde{A}_\varepsilon(t,z)) \in G^\delta_{3\delta/50}(\mathcal{Z}_\varepsilon^{10d}) \text{ for all } l \in [0, l_t],$$

(v) (Definition of the gradient flow (82))

$$\{\eta_\varepsilon(l,u) \mid l \in [0, l_g]\} \subset G^\delta_\delta(\mathcal{Z}_\varepsilon^{10d}) \text{ for } u \in G^\delta_\delta(\mathcal{Z}_\varepsilon^{10d}).$$

These imply that for small $\varepsilon > 0$,

$$\{\Upsilon_\varepsilon(\eta_\varepsilon(l,u)) \mid l \in [0,l_g]\} \subset \mathcal{N}_\varepsilon^{cd+d/10} \text{ if } \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{cd}, c \in [1,9],$$

and that when $\varepsilon > 0$ is small enough, for $i = 0,1,\cdots,j_\varepsilon - 1$,

$$|\Upsilon_\varepsilon(\eta_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z)))) - \Upsilon_\varepsilon(I^{i+1}(\tilde{A}_\varepsilon(t,z)))| \le R_1 + d/10\varepsilon, \tag{84}$$

$$|\Upsilon_\varepsilon(\eta_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z)))) - \Upsilon_\varepsilon(I^{i}(\tilde{A}_\varepsilon(t,z)))| \le d/10\varepsilon. \tag{85}$$

Now, we see that when $\tilde{A}_\varepsilon(t,z) \in G^\delta_{\delta/40\omega}$ and $\Upsilon_\varepsilon(\tilde{A}_\varepsilon(t,z)) \in \mathcal{N}_\varepsilon^{2d}$, at least one of the following three cases occurs:

(Case A) for all $i = 0,1,\cdots,j_\varepsilon - 1$, $l \in [0,l_g]$

$$\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z))) \in G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d}) \cap \{u \mid \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{5d}\};$$

(Case B) for some $i = 0,1,\cdots,j_\varepsilon - 1$,

$$\eta_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z))) \in G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d}) \cap \{u \mid \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{5d-d/10} \setminus \mathcal{N}_\varepsilon^{4d+d/10}\};$$

(Case C) for some $i \in \{0,1,\cdots,j_\varepsilon - 1\}$ and some $l \in [0,l_g]$,

$$\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z))) \in \Big(G^\delta_{\delta/6\omega}(\mathcal{Z}_\varepsilon^{10d}) \setminus G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d})\Big) \cap \{u \mid \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{8d}\}.$$

If (Case C) does not occur,

$$\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z))) \notin (G^\delta_{\delta/6\omega}(\mathcal{Z}_\varepsilon^{10d}) \setminus G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d})) \cap \{u \mid \Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{8d}\}$$

for any $i = 0,1,\cdots,j_\varepsilon - 1$ and $l \in [0,l_g]$. If $\Upsilon_\varepsilon(\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z)))) \notin \mathcal{N}_\varepsilon^{5d}$ for some $i < j_\varepsilon, l \in [0,l_g]$, then there exists $0 \le i_0 \le j_\varepsilon - 1$ such that $\Upsilon_\varepsilon(\eta_\varepsilon(l_g, I^{i_0}(\tilde{A}_\varepsilon(t,z)))) \in \mathcal{N}_\varepsilon^{5d-d/10} \setminus \mathcal{N}_\varepsilon^{4d+d/10}$ by (84) and (85). We may assume $\Upsilon_\varepsilon(\eta_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z)))) \in \mathcal{N}_\varepsilon^{4d+d/10}$ for any $i < i_0$. Then $\Upsilon_\varepsilon(\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z)))) \in \mathcal{N}_\varepsilon^{5d}$ for any $i \le i_0$ and $l \in [0,l_g]$. By the property (iv), we see that for $u \in G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d})$ with $\Upsilon_\varepsilon(u) \in \mathcal{N}_\varepsilon^{8d}$,

$$P_\varepsilon(l,u) \subset G^\delta_{3\delta/50}(\mathcal{Z}_\varepsilon^{10d}) \subset G^\delta_{\delta/6\omega}(\mathcal{Z}_\varepsilon^{10d}), \quad l \in [0,l_t].$$

Since $G^\delta_{c\delta}(\mathcal{Z}_\varepsilon^{10d})$, $c \in (0,1]$ is invariant by the tail-minimizing operator τ_ε, this implies that, for $i \le i_0$ and $l \in [0,l_g]$,

$$\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z))) \in G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d}).$$

Therefore, it follows that (Case B) holds. When $\Upsilon_\varepsilon(\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z)))) \in \mathcal{N}_\varepsilon^{5d}$ for any $i < j_\varepsilon, l \in [0,l_g]$, we see that, for all $i < j_\varepsilon, l \in [0,l_g]$,

$$\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z))) \in G^\delta_{\delta/40\omega}(\mathcal{Z}_\varepsilon^{10d})$$

and (Case A) holds.

Now, if (Case C) occurs, by Proposition 16, for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) \le \Gamma_\varepsilon(\eta_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z)))) \le b_\varepsilon - \varepsilon\nu/2.$$

If (Case B) occurs, by Proposition 14, for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(P_\varepsilon(l_t, \tau_\varepsilon(l_g, I^i(\tilde{A}_\varepsilon(t,z))))) \le b_\varepsilon + o(\varepsilon) - \varepsilon\mu_0 \le b_\varepsilon - \varepsilon\mu_0/2.$$

If (Case A) occurs, we claim that

$$\Gamma_\varepsilon(\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z)))) \le b_\varepsilon - \varepsilon\nu/2$$

for some $i \in \{1,\cdots,j_\varepsilon - 1\}$ and some $l \in [0,l_g]$. If the claim does not hold,

$$\Gamma_\varepsilon(\eta_\varepsilon(l, I^i(\tilde{A}_\varepsilon(t,z)))) \ge b_\varepsilon - \varepsilon\nu/2$$

for any $i \in \{1, \cdots, j_\varepsilon - 1\}$ and $l \in [0, l_g]$. In this case, it follows that

$$\begin{aligned} &\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) - \Gamma_\varepsilon(\tilde{A}_\varepsilon(t,z)) \\ &= \sum_{j=1}^{j_\varepsilon} [\Gamma_\varepsilon(I^j(\tilde{A}_\varepsilon(t,z))) - \Gamma_\varepsilon(I^{j-1}(\tilde{A}_\varepsilon(t,z)))] \\ &\le \sum_{j=1}^{j_\varepsilon} [\Gamma_\varepsilon(\eta_\varepsilon(l_g, I^{j-1}(\tilde{A}_\varepsilon(t,z)))) - \Gamma_\varepsilon(I^{j-1}(\tilde{A}_\varepsilon(t,z)))]. \end{aligned} \tag{86}$$

Since for any $j = 1, \cdots, j_\varepsilon - 1$ and $l \in [0, l_g]$,

$$\Gamma_\varepsilon(\eta_\varepsilon(l, I^{j-1}(\tilde{A}_\varepsilon(t,z)))) \ge b_\varepsilon - \varepsilon\nu/2, \quad \Upsilon_\varepsilon(\eta_\varepsilon(l, I^{j-1}(\tilde{A}_\varepsilon(t,z)))) \in \mathcal{N}_\varepsilon^{5d},$$

from the definition of the gradient flow η_ε, we see that for any $j = 1, \cdots, j_\varepsilon - 1$ and $l \in [0, l_g]$,

$$\frac{d\eta_\varepsilon(l, I^{j-1}(\tilde{A}_\varepsilon(t,z)))}{dl} = -\Gamma'_\varepsilon(\eta_\varepsilon(l, I^{j-1}(\tilde{A}_\varepsilon(t,z))))/||\Gamma'_\varepsilon(\eta_\varepsilon(l, I^{j-1}(\tilde{A}_\varepsilon(t,z))))\|^*.$$

This implies that for any $j = 1, \cdots, j_\varepsilon - 1$,

$$\Gamma_\varepsilon(\eta_\varepsilon(l_g, I^{j-1}(\tilde{A}_\varepsilon(t,z)))) - \Gamma_\varepsilon(I^{j-1}(\tilde{A}_\varepsilon(t,z))) \le -l_g k(\varepsilon).$$

Therefore we get from (86) that

$$\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) - \Gamma_\varepsilon(\tilde{A}_\varepsilon(t,z)) \le -j_\varepsilon l_g k(\varepsilon) \le -\varepsilon\nu.$$

Thus, if (Case A) occurs, for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) \le b_\varepsilon - \varepsilon\nu/2.$$

Since $\Gamma_\varepsilon(I(u)) \le \Gamma_\varepsilon(u)$ for any $u \in G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d})$, we see from (81) that for any $t \in [0, T]$ and $z \in L$,

$$\Gamma_\varepsilon(I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z))) \le b_\varepsilon - \varepsilon \min\{\nu/2, \mu_0/2\}.$$

This completes the proof. □

Now, we define the deformed initial surface

$$B_\varepsilon(t,z) \equiv I^{j_\varepsilon}(\tilde{A}_\varepsilon(t,z)), \quad t \in [0,T], \ z \in L. \tag{87}$$

Since $\Gamma_\varepsilon(\tilde{A}_\varepsilon(t,z)) \le b_\varepsilon - \varepsilon\nu$ for $(t,z) \in \partial([T_0,T] \times L)$, the transplantation operator P_ε fixes any element u with $\Upsilon_\varepsilon(u) \notin \mathcal{N}_\varepsilon^{8d}$ and $\tau_\varepsilon(\tilde{A}_\varepsilon(t,z)) = \tilde{A}_\varepsilon(t,z)$, we see that

$$B_\varepsilon(t,z) = \tilde{A}_\varepsilon(t,z) \notin G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d}) \quad \text{for} \quad (t,z) \in \partial([T_0,T] \times L). \tag{88}$$

Moreover, since $\eta_\varepsilon, \tau_\varepsilon$ and P_ε are continuous map from $G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d})$ to $G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d})$ and $\tau_\varepsilon(\tilde{A}_\varepsilon(t,z)) = \tilde{A}_\varepsilon(t,z)$, we know that $B_\varepsilon : [T_0,T] \times L \to H^1(\Omega_\varepsilon)$ is continuous and

$$B_\varepsilon(t,z) = \tilde{A}_\varepsilon(t,z) \text{ if } B_\varepsilon(t,z) \notin G_\delta^\delta(\mathcal{Z}_\varepsilon^{10d}). \tag{89}$$

From Proposition 17, if ε is sufficiently small, for all $t \in [0,T]$ and $z \in L$

$$\Gamma_\varepsilon(B_\varepsilon(t,z)) \le b_\varepsilon - \varepsilon \min\{\nu/2, \mu_0/2\}. \tag{90}$$

We note from (49), (50) and (88) that for small $\varepsilon > 0$,

$$\Upsilon_\varepsilon(B_\varepsilon(t,z)) \in O \text{ for } (t,z) \in [T_0,T] \times L,$$

$$\Upsilon_\varepsilon(B_\varepsilon(t,z)) = z/\varepsilon \text{ for } (t,z) \text{ in a neighborhood of } \partial([T_0,T] \times L).$$

9.2. **Lower bound.** In this section, we will prove the following lower estimate.

Proposition 18. *There exists* $\hat{U} \in \mathcal{S}$ *such that for small* ε,

$$\max_{z\in L, t\in[0,T]} \Gamma_\varepsilon(B_\varepsilon(t,z)) \geq \frac{1}{2}\Gamma(\hat{U}) + \varepsilon\frac{m}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n} |\nabla\hat{U}(y)|^2|y|dy + o(\varepsilon).$$

Proof. By (H1), we have a continuous map

$$\Pi \circ (\varepsilon\Upsilon_\varepsilon) \circ B_\varepsilon : [T_0, T] \times L \to L.$$

It is well known that any C^1 manifold has a C^∞ differentiable structure which is C^1 diffeomorphic to the original differentiable structure. Thus, we may consider L as a C^∞ manifold. Then, we can approximate $\Pi \circ (\varepsilon\Upsilon_\varepsilon) \circ B_\varepsilon : [T_0, T] \times L \to L$ by a C^∞ map $\Xi^l_\varepsilon : [T_0, T] \times L \to L$ satisfying $\Xi^l_\varepsilon(t, z) = z$ in a neighborhood of $\partial([T_0, T] \times L)$ and $\|\Pi\circ(\varepsilon\Upsilon_\varepsilon)\circ B_\varepsilon - \Xi^l_\varepsilon\|_{L^\infty([T_0,T]\times L)} \leq 1/l$. By Sard's theorem, there exists a regular value $w^l_\varepsilon \in L$ of Ξ^l_ε near z_0 with $|z_0 - w^l_\varepsilon| \leq 1/l$. Then $(\Xi^l_\varepsilon)^{-1}(w^l_\varepsilon)$ contains a 1-dimensional manifold connecting (T_0, w^l_ε) and (T, w^l_ε) in $[T_0, T] \times L$. We find a diffeomorphism

$$p^l_\varepsilon = (t^l_\varepsilon, z^l_\varepsilon) : [0, 1] \to (\Xi^l_\varepsilon)^{-1}(w^l_\varepsilon) \subset [T_0, T] \times L$$

with $p^l_\varepsilon(0) = (T_0, w^l_\varepsilon)$ and $p^l_\varepsilon(1) = (T, w^l_\varepsilon)$. Then, we see that

$$\lim_{l\to\infty} \Pi \circ (\varepsilon\Upsilon_\varepsilon) \circ B_\varepsilon(p^l_\varepsilon(s)) = z_0 \text{ uniformly for } s \in [0, 1].$$

Then, defining $m_l \equiv \inf_{s\in[0,1]} H((\varepsilon\Upsilon_\varepsilon)\circ B_\varepsilon(p^l_\varepsilon(s)))$, we deduce that $\liminf_{l\to\infty} m_l \geq m$.

Now we consider $\phi_\varepsilon(x - \Upsilon_\varepsilon(B_\varepsilon(t, z)))B_\varepsilon(t, z)$ for $t \in [T_0, T], z \in L$, where ϕ_ε is a smooth function satisfying (7). Then, from the decaying property of the tail-minimizing operator, when $\varepsilon > 0$ is small,

$$\Gamma_\varepsilon(B_\varepsilon(t, z)) = \Gamma_\varepsilon(\phi_\varepsilon(x - \Upsilon_\varepsilon(B_\varepsilon(t, z)))B_\varepsilon(t, z)) + o(\varepsilon),$$

uniformly for $t \in [T_0, T], z \in L$. We define $\tilde{B}^l_\varepsilon(s) \equiv B_\varepsilon(p^l_\varepsilon(s)), s \in [0, 1]$. By Remark 2, we have maps $\psi_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}$ and $\Psi_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}$ satisfying properties (i)-(iv) in Proposition 3 except the properties involving the derivatives with respect to $X = \Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))$. Then, we define

$$w^l_\varepsilon(s)(y) \equiv (\phi_\varepsilon(\cdot - \Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s)))\tilde{B}^l_\varepsilon(s)) \circ \Psi_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}(y), \quad y \in \mathbb{R}^n_+,$$

$$W^l_\varepsilon(s)(y_1, ..., y_n) \equiv w^l_\varepsilon(s)(y_1, ..., y_{n-1}, |y_n|), \quad y = (y_1, \cdots, y_{n-1}, y_n) \in \mathbb{R}^n.$$

Then $w^l_\varepsilon(s), W^l_\varepsilon(s)$ are continuous in $[0, 1]$. Following the estimate for the term TVI in the proof of Proposition 14, we can get a lower bound for $\max_{s\in[0,1]} \Gamma_\varepsilon(\tilde{B}^l_\varepsilon(s))$. After a change of variable $y = \Psi^{-1}_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}(x)$, we can see that

$$\begin{aligned}\Gamma_\varepsilon(\tilde{B}^l_\varepsilon(s)) =& \frac{1}{4}\int_{\mathbb{R}^n} |\nabla W^l_\varepsilon(s)|^2 + |W^l_\varepsilon(s)|^2 - \frac{1}{2}\int_{\mathbb{R}^n} F(W^l_\varepsilon(s)) \\ &- \int_{\mathbb{R}^n_+} D_{y_n} w^l_\varepsilon(s)(y) \sum_{i=1}^{n-1} D_{y_i} w^l_\varepsilon(s)(y)(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))})(\varepsilon y_1, ..., \varepsilon y_{n-1})dy \\ &+ \frac{1}{2}\int_{\mathbb{R}^n_+} |D_{y_n} w^l_\varepsilon(s)(y)|^2 \sum_{i=1}^{n-1} |(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))})(\varepsilon y_1, ..., \varepsilon y_{n-1})|^2 dy + o(\varepsilon).\end{aligned} \tag{91}$$

Note that $|D\Psi_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}|$ is uniformly bounded in $L^\infty(\mathbb{R}^n_+ \cap B(0,r/\varepsilon))$; thus for some $C>0$, independent of l,s and small $\varepsilon>0$,

$$|\nabla w^l_\varepsilon(s)(y)| \le C|\nabla(\phi_\varepsilon(\cdot - \Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s)))\tilde{B}^l_\varepsilon(s))\circ \Psi_{\varepsilon,\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}|.$$

Since $B_\varepsilon(t,z)\in G^\delta_\delta(\mathcal{Z}^{10d}_\varepsilon)$ or $B_\varepsilon(t,z)=\tilde{A}_\varepsilon(t,z)$, there exists $D>0$, independent of l,s and small $\varepsilon>0$, such that $\|B_\varepsilon(t,z)\|_\varepsilon \le D$ for any $(t,z)\in[T_0,T]\times L$. Note from Proposition 3 that $|D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}(\varepsilon y')|\le O(\varepsilon|y'|)$. Then, we see from the uniform exponential decay property of $w^l_\varepsilon(s)$ for $l\ge 1$ and $s\in[0,1]$ that

$$\Big|\int_{\mathbb{R}^n_+} D_n w^l_\varepsilon(s)(y)\sum_{i=1}^{n-1} D_i w^l_\varepsilon(s) D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}(\varepsilon y_1,\dots,\varepsilon y_{n-1})dy\Big| = O(\varepsilon), \tag{92}$$

$$\int_{\mathbb{R}^n_+} |D_n w^l_\varepsilon(s)(y)|^2\sum_{i=1}^{n-1}|D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s))}(\varepsilon y_1,\dots,\varepsilon y_{n-1})|^2 dy = o(\varepsilon). \tag{93}$$

Since

$$\max_{s\in[0,1]}\Big(\frac14\int_{\mathbb{R}^n}|\nabla W^l_\varepsilon(s)(y)|^2+|W^l_\varepsilon(s)(y)|^2-\frac12\int_{\mathbb{R}^n}F(W^l_\varepsilon(s)(y))dy\Big)\ge\frac12\Gamma(U),\ \ U\in\mathcal{S},$$

by the upper estimate (90), we see that for each $l>0$,

$$\lim_{\varepsilon\to0}\max_{s\in[0,1]}\Big(\frac12\int_{\mathbb{R}^n}|\nabla W^l_\varepsilon(s)(y)|^2+|W^l_\varepsilon(s)(y)|^2-\int_{\mathbb{R}^n}F(W^l_\varepsilon(s)(y))dy\Big)=\Gamma(U).$$

Since $\{W^l_\varepsilon(s)\mid s\in[0,1],$ positive integer $l,$ small $\varepsilon>0\}$ is bounded, we see that there exist $s^l_\varepsilon\in[0,1]$, $q_l\in\partial\mathbb{R}^n_+$ and $U^l\in\mathcal{S}$ such that $\{q_l\}_l$ is bounded, $\Gamma(W^l_\varepsilon(s^l_\varepsilon)) = \max_{s\in[0,1]}\Gamma(W^l_\varepsilon(s))$, $\lim_{\varepsilon\to0}\Gamma(W^l_\varepsilon(s^l_\varepsilon))=\Gamma(U^l)$ and $\lim_{\varepsilon\to0}\|W^l_\varepsilon(s^l_\varepsilon)-U^l(\cdot-q_l)\|=0$. We see from (47) that $\lim_{\varepsilon\to0}t^l_\varepsilon(s^l_\varepsilon)=1$.

Now we see that

$$\max_{s\in[0,1]}\Gamma_\varepsilon(\tilde{B}^l_\varepsilon(s))\ge\Gamma_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon)).$$

From (91), we see that

$$\begin{aligned}
&\Gamma_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon))-\frac12\Gamma(U^l)\\
&\ge-\int_{\mathbb{R}^n_+}D_{y_n}w^l_\varepsilon(s^l_\varepsilon)(y)\sum_{i=1}^{n-1}D_{y_i}w^l_\varepsilon(s^l_\varepsilon)(y)(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon))})(\varepsilon y_1,\dots,\varepsilon y_{n-1})dy\\
&\quad+\frac12\int_{\mathbb{R}^n_+}|D_{y_n}w^l_\varepsilon(s^l_\varepsilon)(y)|^2\sum_{i=1}^{n-1}|(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon))})(\varepsilon y_1,\dots,\varepsilon y_{n-1})|^2dy+o(\varepsilon).
\end{aligned}\tag{94}$$

Then, by combining the argument proving (92) and the triangle inequality, we can see that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n_+}D_{y_n}w^l_\varepsilon(s^l_\varepsilon)(y)\sum_{i=1}^{n-1}D_{y_i}w^l_\varepsilon(s)(y)(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon))})(\varepsilon y')\\
&\qquad-D_nU^l(y-q_l)\sum_{i=1}^{n-1}D_iU^l(y-q_l)(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde{B}^l_\varepsilon(s^l_\varepsilon))})(\varepsilon y')dy\Big|=o(\varepsilon).
\end{aligned}\tag{95}$$

Moreover, since $\operatorname{supp}(\phi_\varepsilon U)\subset B(0,2\varepsilon^{-1/3})$, we see from (iv) in Proposition 3 and the exponential decay property of U^l that

$$\Big|\int_{\mathbb{R}^n_+} D_nU^l(y-q_l)\sum_{i=1}^{n-1} D_iU^l(y-q_l)(D_i\psi_{\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon))})(\varepsilon y') - D_nU^l(y-q_l)\sum_{i=1}^{n-1} D_iU^l(y-q_l)A^i_{\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon))}\cdot(\varepsilon y')dy\Big| = o(\varepsilon). \quad (96)$$

From radially symmetric property of U,

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} D_nU^l(y-q_l)\sum_{i=1}^{n-1} D_iU^l(y-q_l)A^i_{\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon))}\cdot(\varepsilon y')dy\\
=&\int_{\mathbb{R}^n_+} D_nU^l(y)\sum_{i=1}^{n-1} D_iU^l(y)A^i_{\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon))}\cdot(\varepsilon y')dy\\
=&\varepsilon H(\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon)))\int_{\mathbb{R}^n_+}|\nabla U^l(y)|^2 y_n(|y|^2-y_n^2)/|y|^2dy\\
=&\varepsilon\frac{H(\varepsilon\Upsilon_\varepsilon(\tilde B^l_\varepsilon(s^l_\varepsilon)))}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U^l(y)|^2|y|dy\\
\geq&\varepsilon\frac{m_l}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U^l(y)|^2|y|dy.
\end{aligned} \quad (97)$$

Now we may assume that $U^l\to\hat U\in\mathcal{S}$ up to a subsequence as $l\to\infty$. Then we see from (95), (96) and (97) that

$$\max_{z\in L,t\in[0,T]}\Gamma_\varepsilon(B_\varepsilon(t,z))\geq\frac{1}{2}\Gamma(U)+\varepsilon\frac{m}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla\hat U|^2|y|dy+o(\varepsilon) \quad (98)$$

This completes the proof. □

9.3. **Concentration points of one peak solutions.** The following result was proved in [37] when the nondegeneracy (f5) holds. Here, using the transplantation flow in section 8, we can prove the following.

Proposition 19. *Suppose that there exists a solution v_ε of (3) such that for some $z_\varepsilon\in\partial\Omega$ and $C,c>0$, $v_\varepsilon(x)\leq C\exp(-c|x-z_\varepsilon/\varepsilon|)$ and $w_\varepsilon(x)\equiv v_\varepsilon(\frac{1}{\varepsilon}\Psi_{z_\varepsilon}(\varepsilon x))$ converges uniformly, up to a subsequence, to $U\in\mathcal{S}$ for $x\in B(0,r/\varepsilon)\cap\mathbb{R}^n_+$ as $\varepsilon\to0$. Then, $\lim_{\varepsilon\to0}|\nabla H(z_\varepsilon)|=0$.*

Proof. Suppose that $\limsup_{\varepsilon\to0}|\nabla H(z_\varepsilon)|>0$. Taking a subsequence if it is necessary, we may assume that $\lim_{\varepsilon\to0}z_\varepsilon=z_0$ and $|\nabla H(z_0)|>0$. Then, by Remark 1, there exist constants $r_0,r>0$ such that the properties (i)-(iv) of Proposition 3 with maps $\psi_{\varepsilon,X}(y')$, $\Psi_{\varepsilon,X}(y)$ defined for $X\in B(z_0/\varepsilon,r_0/\varepsilon)\cap\partial\Omega_\varepsilon$, $y'\in B^{n-1}(0,r/\varepsilon)$ and $y\in B^n(0,r/\varepsilon)\cap\mathbb{R}^n_+$. Then, by the same arguments, Proposition 4 with $B(z_\varepsilon/\varepsilon,r_0/\varepsilon)$ replacing $\mathcal{N}^{10d}_\varepsilon\setminus\mathcal{N}^d_\varepsilon$ holds. As in (4), there exists a local solution of

$$\frac{d\Phi_\varepsilon}{dt}(t)=-\nabla H(\Phi_\varepsilon(t)),\qquad \Phi_\varepsilon(0)=z_\varepsilon. \quad (99)$$

Then, for the function $\varphi_\varepsilon\in C^\infty_0(\mathbb{R}^n,[0,1])$ satisfying (55), as in (56), we define

$$P_\varepsilon(l)(x)=(\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)\circ\Psi_{\varepsilon,z_\varepsilon/\varepsilon}\circ\Psi^{-1}_{\varepsilon,\Phi_\varepsilon(l)/\varepsilon}(x)+(1-\varphi_\varepsilon(x-z_\varepsilon/\varepsilon)v_\varepsilon(x). \quad (100)$$

We claim that $\lim_{l\to0}(\Gamma_\varepsilon(P_\varepsilon(l))-\Gamma_\varepsilon(v_\varepsilon))/l<0$ for sufficiently small $\varepsilon>0$. This is a contradiction since v_ε is a critical point of Γ_ε. To prove the claim, we note that since the Jacobian determinants of $\Psi_{\varepsilon,z_\varepsilon/\varepsilon}\circ\Psi^{-1}_{\varepsilon,\Phi_\varepsilon(l)/\varepsilon}(x)$ are 1,

$$\begin{aligned}&\int_{\Omega_\varepsilon}((\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)\circ\Psi_{\varepsilon,z_\varepsilon/\varepsilon}\circ\Psi^{-1}_{\varepsilon,\Phi_\varepsilon(l)/\varepsilon}(x))^2dx\\&=\int_{\Omega_\varepsilon}(\varphi_\varepsilon(x-z_\varepsilon/\varepsilon)v_\varepsilon(x))^2dx,\end{aligned}$$

and that

$$\begin{aligned}\Gamma_\varepsilon(P_\varepsilon(l))-\Gamma_\varepsilon(v_\varepsilon)=&\int_{\Omega_\varepsilon}\nabla((1-\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon))v_\varepsilon)\cdot\nabla\overline{\Psi}^l_\varepsilon dx\\&-\int_{\Omega_\varepsilon}\nabla((1-\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon))v_\varepsilon)\cdot\nabla(\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)dx\\&+\int_{\Omega_\varepsilon}((1-\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon))v_\varepsilon)\overline{\Psi}^l_\varepsilon dx\\&-\int_{\Omega_\varepsilon}((1-\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon))v_\varepsilon)(\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)dx\\&-\int_{\Omega_\varepsilon}F(P_\varepsilon(l))-F(v_\varepsilon)dx\\&+\frac12\int_{\Omega_\varepsilon}|\nabla\overline{\Psi}^l_\varepsilon|^2-|\nabla(\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)|^2dx\\\equiv&\ SI-SII+SIII-SIV+SV+SVI,\end{aligned}$$

where $\overline{\Psi}^l_\varepsilon\equiv(\varphi_\varepsilon(\cdot-z_\varepsilon/\varepsilon)v_\varepsilon)\circ\Psi_{\varepsilon,z_\varepsilon/\varepsilon}\circ\Psi^{-1}_{\varepsilon,\Phi_\varepsilon(l)/\varepsilon}$. From the elliptic estimate of Proposition 9, we see that for some $C',c'>0$, $|\nabla v_\varepsilon(x)|,|D^2v_\varepsilon(x)|\le C'\exp(-c'|x-z_\varepsilon/\varepsilon|)$. Then, by the same arguments with the proof of Proposition 14, we deduce that for small $\varepsilon,l>0$,

$$|SI-SII|/l=o(\varepsilon),|SIII-SIV|/l=o(\varepsilon),|SV|/l=o(\varepsilon). \tag{101}$$

Furthermore, following the almost same arguments with the estimate of $|TVI/h|$ in the proof of Proposition 14, we deduce that

$$SVI/l\le\varepsilon\frac{H(\Phi_\varepsilon(l))-H(z_\varepsilon)}{l}\frac{1}{n+1}\frac{|S^{n-2}|}{|S^{n-1}|}\Big(\int_{\mathbb{R}^n}|\nabla U|^2|y|dy+o(1)\Big)+o(\varepsilon),$$

where $o(1)\to0$ as $l\to0$. This implies that for small $\varepsilon>0$,

$$\lim_{l\to0}SVI/l\le-\varepsilon\frac{|\nabla H(z_0)|^2}{2(n+1)}\frac{|S^{n-2}|}{|S^{n-1}|}\int_{\mathbb{R}^n}|\nabla U|^2|y|dy. \tag{102}$$

Combining (101) and (102), we see that $\lim_{l\to0}(\Gamma_\varepsilon(P_\varepsilon(l)-\Gamma_\varepsilon(v_\varepsilon))/l<0$ for small $\varepsilon>0$. Since P_ε is a smooth function from a neighborhood of 0 in $\mathbb{R}$ to $H^1(\Omega_\varepsilon)$, this contradicts the fact $\Gamma'(v_\varepsilon)=0$. This completes the proof. □

9.4. **Completion of the proof of Theorem 1.1.** Since in the upper estimate (52), we take $U\in\mathcal{S}$ satisfying (46)

$$\int_{\mathbb{R}^n}|\nabla U|^2|y|dy=\begin{cases}\min_{\tilde U\in\mathcal{S}}\int_{\mathbb{R}^n}|\nabla\tilde U|^2|y|dy, & \text{if } m\ge0,\\ \max_{\tilde U\in\mathcal{S}}\int_{\mathbb{R}^n}|\nabla\tilde U|^2|y|dy, & \text{if } m<0,\end{cases}$$

the lower estimate (98) contradicts to (90) and (52). Therefore for each $\delta\in(0,\delta_5)$, there exists a critical point of Γ_ε in $G(\varepsilon,\nu,\delta)$ if $\varepsilon>0$ is sufficiently small. Let $v_\varepsilon\in G(\varepsilon,\nu,\delta)$ be a solution of (3) with $\Upsilon_\varepsilon(v_\varepsilon)=X_\varepsilon$. Then $u_\varepsilon(x)=v_\varepsilon(x/\varepsilon)$ is a solution of (1). Since $\lim_{\varepsilon\to0}b_\varepsilon=\frac12\Gamma(U)$, following the classical arguments in [34], we deduce that $v_\varepsilon(x) \le C\exp(-c|x-X_\varepsilon|)$ for C, c independent of ε, and that as $\varepsilon \to 0$, $v_\varepsilon \circ \Psi_{\varepsilon,X_\varepsilon}$ converges uniformly, up to a subsequence, to $U(\cdot - Q)$ on $B(0, r/\varepsilon) \cap \mathbb{R}^n_+$ for some $Q \in \partial\mathbb{R}^n_+$ and $U \in \mathcal{S}$. By applying the proof of Theorem 2.1 in [34], we can prove that, for small $\varepsilon > 0$, u_ε has a unique maximum point z_ε with $z_\varepsilon \in \partial\Omega$. Then, we deduce that $\sup_\varepsilon |z_\varepsilon/\varepsilon - X_\varepsilon| < \infty$ and $w_\varepsilon(x) \equiv v_\varepsilon \circ \Psi_{\varepsilon,z_\varepsilon/\varepsilon}$ converges uniformly, up to a subsequence, to U on $B(0, r/\varepsilon) \cap \mathbb{R}^n_+$ as $\varepsilon \to 0$. Since U is radially symmetric, we see that $u_\varepsilon(\Psi_{z_\varepsilon}(\varepsilon x))$ also converges uniformly, up to a subsequence, to U on $B(0, r/\varepsilon) \cap \mathbb{R}^n_+$ as $\varepsilon \to 0$. The property $\limsup_{\varepsilon\to 0} \operatorname{dist}(z_\varepsilon, \mathcal{N}) = 0$ comes from Proposition 19. Since for $q > 0$, we take a neighborhood $\mathcal{N} \subset \mathcal{M}^q$ of $\mathcal{M}$, this implies that $\limsup_{\varepsilon\to 0} \operatorname{dist}(z_\varepsilon, \mathcal{M}) = 0$.

This completes the proof of the main theorem, Theorem 1.1.

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