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Design of superposition polar coding for binary-input less-noisy broadcast channels☆

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Abstract

In this study, we design a superposition polar code that can achieve the capacity region of binary-input *less-noisy* broadcast channels asymptotically. Simulation results show that a better rate region is achievable by superposition polar coding than by time sharing between two point-to-point channel polar codes.

Keywords: Broadcast channel; Superposition coding; Polar coding

1. Introduction

The broadcast channel (BC), introduced by T.M. Cover [1], consists of a single transmitter and K receivers, where the transmitter transmits K independent messages to receivers. For the general broadcast channel, the capacity region remains unknown. Superposition coding proposed by T.M. Cover is known to be optimal in classes of *degraded*, *less noisy*, *more capable* broadcast channels, and in a broadcast channel with *degraded message sets* [2].

Polar codes, invented by Arikan [3], is the first known code to achieve the capacity of the binary-input memoryless symmetric-output channels with low decoding complexity. In polar codes, a codeword is a transformed version of a message sequence via multiplication through a specifically designed matrix. A polar code decodes the message sequence successively from the received output sequence. Arikan proved that conditioned on the output sequence and its previous bits, the distribution of each bit of the message sequence either converges to a constant or a uniform distribution as the blocklength goes to infinity. Furthermore, he showed that the fraction of the message bits converging to a constant converges to the mutual information of the channel. This design is further generalized to achieve the capacity region of various channels, such as the binary-input asymmetric channel [4] and the binary-input multiple access channel [5]. Recently, N. Goela et al. [6] proved that polar codes can be designed to achieve the superposition coding bound and Marton's binning region for the binary input broadcast channel under the condition of *degrade-ness* between two channels $p(y_1|v)$ and $p(y_2|v)$ where V is given as an auxiliary random variable. Later, Mondelli et al. [7] designed polar codes for the general binary-input broadcast channels.

The scheme proposed by Goela et al. shows lower construction and decoding complexity compared to the scheme on a general broadcast channel when $p(y_2|v)$ is *degraded* from $p(y_1|v)$. Therefore, to apply the simpler broadcast channel polar coding scheme into practical broadcast channels, we have to check the *degrade-ness* between the channels induced by the auxiliary random variable V. Such a procedure is exhaustive as there is no simple formula to check the *degrade-ness* between two channels.

In this study, we extend the *degrade-ness* condition between $p(y_1|v)$ and $p(y_2|v)$ to the classes of *less noisy* broadcast channels to alleviate such problems. In addition, we provide the simulation result, which is in our knowledge, the first simulation result of polar coding for *less noisy* broadcast channels. From the result, we show that the proposed polar codes achieve

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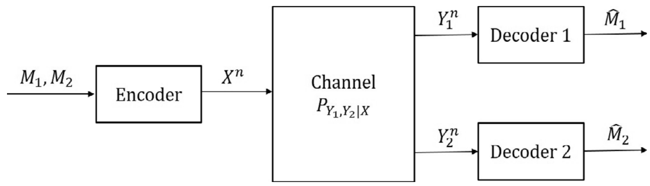

Fig. 1. Two-user broadcast channel.

a better rate region than time sharing between two point-to-point channel polar codes.

2. Preliminaries

2.1. Superposition coding

Let us consider the private message communication system with a discrete, memoryless channel $p(y_1, y_2|x)$ depicted in Fig. 1. Given the messages $M_1 \in \{0,1\}^{NR_1}$ and $M_2 \in \{0,1\}^{NR_2}$, the encoder determines the input sequence $X^N \in \{0,1\}^N$. Decoders 1 and 2 decode messages 1 and 2 from the received output sequences Y_1^N and Y_2^N, respectively. Then, the superposition coding bound is characterized as follows.

Theorem 1 (*Superposition coding bound [1]*). *Rate pair (R_1, R_2) is achievable for the broadcast channel $p(y_1, y_2|x)$ if*

$$R_1 < I(X; Y_1|V)$$
$$R_2 < I(Y_2; V)$$
$$R_1 + R_2 < I(X; Y_1)$$

for some pmf $p(v, x)$.

The key idea of superposition coding is to treat the message for decoder 2 as a common message that can be decoded by both receivers. Naturally, superposition coding is optimal in case where a strictly dominating relationship exists between the two channels, such as *degraded*, *less noisy*, and *more capable* condition.

Definition 1. Given a broadcast channel $P_{Y_1,Y_2|X}$, let S: $p(y_1|x)$ and T: $p(y_2|x)$. Then,

- Channel $p(y_1, y_2|x)$ is degraded (denoted as $p(y_1|x) \succ p(y_2|x)$) when $X \to Y_1 \to Y_2$ forms a Markov chain.
- S is *less noisy* than T when $I(U; Y_1) \geq I(U; Y_2)$ for all $p(u, x)$ such that $U \to X \to (Y_1, Y_2)$.
- S is *more capable* than T when $I(X; Y_1) \geq I(X; Y_2)$ for all $p(x)$.

For such channels, superposition coding achieves the capacity region of the broadcast channel as shown below.

Theorem 2 (*Capacity region of degraded, less noisy, or more capable broadcast channel [2]*). *The private-message capacity region of the* degraded, less noisy, *or* morecapable *broadcast channel is the set of rate pairs (R_1, R_2) such that*

$$R_1 < I(X; Y_1|V)$$
$$R_2 < I(V; Y_2)$$

for some pmf $p(v, x)$.

2.2. Construction of superposition polar codes

In this subsection, we introduce the superposition coding scheme proposed by Goela et al. for broadcast channels where $p(y_2|v)$ is *degraded* with respect to $p(y_1|v)$. However, we need certain conditions on the broadcast channel to make the scheme achieve every superposition coding bound of the broadcast channel.

We first review the main theorem in [6].

Theorem 3 (*[6]*). *Assume a two-user broadcast channel with binary input X and outputs Y_1 and Y_2. Let V denote a binary random variable satisfying the following.*

- *$V \to X \to (Y_1, Y_2)$ forms a Markov chain.*
- *$p(y_1|v) \succ p(y_2|v)$.*

Then, there exists a polar broadcast code that achieves the following rate pair

$$(I(X; Y_1|V), I(V; Y_2)).$$

For the remaining subsection, we summarize the polar coding scheme in [6]. Suppose we send $N = 2^n$ bits through the channel $p(y_1, y_2|x)$. Then, we set binary sequences U_1^N, U_2^N as $U_1^N = X^N G_N$, $U_2^N = V^N G_N$, where G_N is the multiplication of the nth Kronecker product of polarization matrix $F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and a *bit-reversal* matrix B_N. Then, we construct the *polarization sets* for each channel as follows for $\delta = 2^{-N^\beta}$ where $0 < \beta < \frac{1}{2}$.

$$H_{X|V}^{(N)}(\delta) \triangleq \{i \in [1:N] : Z(U_1(i) \mid U_1^{1:i-1}, V^N) \geq 1 - \delta\},$$
$$L_{X|VY_1}^{(N)}(\delta) \triangleq \{i \in [1:N] : Z(U_1(i) \mid U_1^{1:i-1}, V^N, Y_1^N) \leq \delta\},$$
$$H_V^{(N)}(\delta) \triangleq \{i \in [1:N] : Z(U_2(i) \mid U_2^{1:i-1}) \geq 1 - \delta\},$$
$$L_{V|Y_j}^{(N)}(\delta) \triangleq \{i \in [1:N] : Z(U_2(i) \mid U_2^{1:i-1}, Y_j^N) \leq \delta\} \quad (j = 1, 2).$$

Then, we select the *message sets* $\mathcal{M}_1^{(N)}$ and $\mathcal{M}_2^{(N)}$ by the selected *polarization sets*.

$$\mathcal{M}_1^{(N)} \triangleq H_{X|V}^{(N)}(\delta) \cap L_{X|VY_1}^{(N)}(\delta) \quad (1)$$

$$\mathcal{M}_2^{(N)} \triangleq H_V^{(N)}(\delta) \cap L_{V|Y_2}^{(N)}(\delta). \quad (2)$$

Here, $Z(W) = \sum_{y \in Y} P_Y(y) \sqrt{P_{X|Y}(0|y) P_{X|Y}(1|y)}$ is defined as the Bhattacharyya parameter of the channel W: $X \to Y$. Besides, we can interpret the set $H_{X|Y}$ and $L_{X|Y}$, for example, as the sets of deterministic indices and random indices of X given the information of the previous indices and output Y^N. From the channel polarization theorem on binary-input asymmetric channels, we can assure that the cardinalities of the *message sets* divided by the blocklength N converge to the superposition coding bound $I(X; Y_1|V)$ and $I(V; Y_2)$ [6], respectively.

Now we start the transmission via the selected *message sets*. The block diagram of the encoding and decoding process of the two-user superposition polar code is illustrated in Fig. 2.

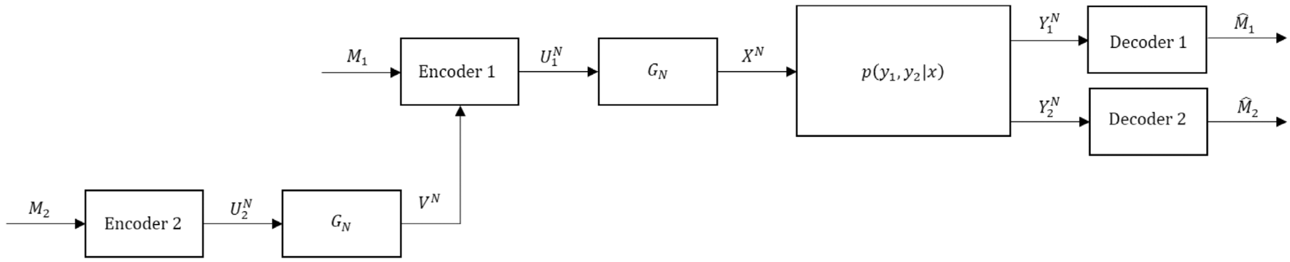

Fig. 2. Block diagram of the superposition polar codes.

Suppose we transmit messages $M_1 \in \{1, 2, \ldots, 2^{NR_1}\}$ and $M_2 \in \{1, 2, \ldots, 2^{NR_2}\}$. At the encoder side, we first encode U_{2i} successively from the message M_2.

$$U_{2i} = \begin{cases} \text{message for the 2nd user} & \text{if } i \in \mathcal{M}_2^{(N)} \\ \underset{u_{2i} \in \{0,1\}}{\operatorname{argmax}} P\left(U_{2i} = u_{2i} | U_2^{1:i-1}\right) & \text{if } i \notin \mathcal{M}_2^{(N)}. \end{cases} \quad (3)$$

Then, by applying the binary polar transform $V^N = U_2^N G_N$, we can obtain V^N. Next, we encode U_{1i} successively from the message M_1 and sequence V^N using the following rule.

$$U_{1i} = \begin{cases} \text{message for the 1st user} & \text{if } i \in \mathcal{M}_1^{(N)} \\ \underset{u_{1i} \in \{0,1\}}{\operatorname{argmax}} P\left(U_{1i} = u_{1i} | U_1^{1:i-1} V^N\right) & \text{if } i \notin \mathcal{M}_1^{(N)}. \end{cases} \quad (4)$$

Then, we take the binary polar transform $X^N = U_1^N G_N$ and send the sequence X^N to channel $p(y_1, y_2|x)$. At the first decoder, we first decode $\hat{U}_{2i}$ successively from the received sequence Y_1^N as follows.

$$\hat{U}_{2i} = \begin{cases} \underset{u_{2i} \in \{0,1\}}{\operatorname{argmax}} P\left(\hat{U}_{,i} = u_{2i} | \hat{U}_2^{1:i-1} Y_2^N\right) & \text{if } i \in \mathcal{M}_2^{(N)} \\ \underset{u_{2i} \in \{0,1\}}{\operatorname{argmax}} P\left(\hat{U}_{2i} = u_{2i} | \hat{U}_2^{1:i-1}\right) & \text{if } i \notin \mathcal{M}_2^{(N)}. \end{cases} \quad (5)$$

By applying the binary polar transform $\hat{V}^N = \hat{U}_2^N G_N$, we construct $\hat{V}^N$. Finally, we decode $\hat{U}_{1i}$ successively from the received sequence Y_1^N and $\hat{V}^N$.

$$\hat{U}_{1i} = \begin{cases} \underset{u_{1i} \in \{0,1\}}{\operatorname{argmax}} P\left(\hat{U}_{1i} = u_{1i} | \hat{U}_1^{1:i-1} Y_1^N \hat{V}^N\right) & \text{if } i \in \mathcal{M}_1^{(N)} \\ \underset{u_{1i} \in \{0,1\}}{\operatorname{argmax}} P\left(\hat{U}_{1i} = u_{1,i} | \hat{U}_1^{1:i-1} V^N\right) & \text{if } i \notin \mathcal{M}_1^{(N)}. \end{cases} \quad (6)$$

The same procedure is repeated for the second decoder to decode $\hat{U}_{2,i}$, except that we use Y_2^N instead of Y_1^N. Finally, we reconstruct the message sequences $\hat{M}_1$ and $\hat{M}_2$ via $\hat{U}_1^N$ and $\hat{U}_2^N$.

3. Extended sufficient condition for *alignment*

The proposed scheme requires the broadcast channel $p(y_1, y_2|v)$ to be *degraded* as a sufficient condition to ensure that the message decodable from the second decoder can also be decoded by the first decoder, i.e., $H_V(\delta) \cap L_{V|Y_1}(\delta) \supseteq H_V(\delta) \cap L_{V|Y_2}(\delta)$. This relationship is called an *alignment* between *polarization sets* $L_{V|Y_1}$ and $L_{V|Y_2}$. From two independent works [8,9], it was also proven that *less noisy* condition also ensures *alignment* as follows.

Lemma 4 (*Alignment between* polarization sets *with* less noisy *condition [8,9]*)**.** *Suppose $p(y_1|v)$ is* less noisy *than $p(y_2|v)$ for discrete memoryless channels $p(y_1|v)$ and $p(y_2|v)$. Then, for any $\delta \in (0, 1)$, we have*

$$L_{V|Y_1}(\delta) \supseteq L_{V|Y_2}(\delta). \quad (7)$$

Applying the results of Lemma 4 into Theorem 3, we get the proposition below.

Proposition 5. *Assume a two-user broadcast channel with binary input X and outputs Y_1 and Y_2. Let V denote a binary random variable satisfying the following.*

- *$V \to X \to (Y_1, Y_2)$ forms a Markov chain.*
- *$p(y_1|v)$ is* less noisy *than $p(y_2|v)$.*

Then, there exists a polar broadcast code that achieves the following rate pair

$$(I(X; Y_1|V), I(V; Y_2)).$$

By Proposition 5, we can extend the sufficient condition for *alignment* from *degraded-ness* to *less noisy* relationship between channels $p(y_1|v)$ and $p(y_2|v)$. Furthermore, we provide a sufficient condition for channel $p(y_1, y_2|x)$ such that *alignment* holds for any auxiliary binary random variable V.

Proposition 6 (*Alignment between* polarization sets *for the* less noisy *broadcast channel*)**.** *Suppose $p(y_1|x)$ is less noisy than $p(y_2|x)$, where X is binary and $V \to X \to (Y_1, Y_2)$ forms a Markov chain for a binary random variable V. Then for any $\delta \in (0, 1)$,*

$$L_{V|Y_1}(\delta) \supseteq L_{V|Y_2}(\delta). \quad (8)$$

This *proposition* can be easily proved by the definition of *less noisy* channels.

4. Simulation results

We performed numerical simulations for superposition polar codes for binary-input *degraded* and *less noisy* broadcast channels. Figs. 3 and 4 demonstrate achievable rate regions of superposition polar coding for *degraded* and *less noisy* broadcast channels. Each point on the achievable region is constructed by simulating codes with different auxiliary random variables to find maximal transmission rate under a specific target frame error rate (FER). Channel construction for selecting message bits of each decoder is done by the

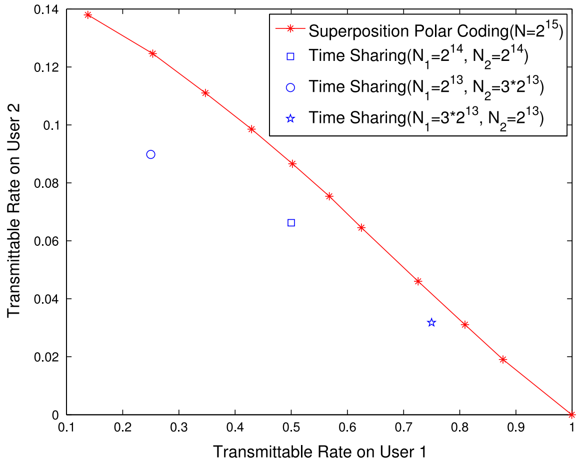

Fig. 3. Achievable region of the *degraded* broadcast channel under superposition polar coding and achievable points under time sharing between point-to-point channel polar codes, both with blocklength of 2^{15}.

quantization method described in [10]. The rate of each user is chosen by selecting δ to minimize the difference between the assumed transmission sum rate $R_1 + R_2$ and the sum rate of the selected *message set* $\frac{\left|\mathcal{M}_1^{(N)}\right|+\left|\mathcal{M}_2^{(N)}\right|}{N}$. In addition, as a baseline, we construct achievable points under time sharing between point-to-point channel polar codes with error rates equal to the given FER. The time-sharing scheme divides the whole blocklength of N bits into N_1 and N_2 bits to transmit point-to-point channel polar codes, each designed for channel $p(y_1|x)$ and channel $p(y_2|x)$.

First, we show simulation results for *degraded* broadcast channels. In Fig. 3, we assume $p(y_1|x)$ is the binary symmetric channel (BSC) with transition probability of 10^{-5} and $p(y_2|x)$ is BSC(0.25), where $p(y_2|x)$ is *degraded* with respect to $p(y_1|x)$. Then, we can find in [2] that the optimal auxiliary random variable V for achieving the capacity region of the broadcast channel $p(y_1, y_2|x)$ is given as

$$V \sim \text{Bern}\left(\frac{1}{2}\right), P\left(X - V = x - v|V = v\right) \sim \text{Bern}(\alpha) \quad (9)$$

where $\alpha \in [0, \frac{1}{2}]$.

For each simulation in Fig. 3, the number of simulations is 10^4 and the blocklength is 2^{15}. The target frame error rate is set as 0.1. This corresponds to the default target block error rate in the LTE standard [11].

Next, we perform simulation for *less noisy* broadcast channels. In Fig. 4, we assume $p(y_1|x)$ is the binary erasure channel (BEC) with erasure probability of 0.51 and $p(y_2|x)$ is BSC(0.25). Owing to the well-known results on the relationship between BSC and BEC [2], we can check that $p(y_1|x)$ is *less noisy* than $p(y_2|x)$, but not *degraded*. We use the same design criteria and auxiliary random variable V as before and compare the results to that obtained with time sharing, except that simulation is performed with a blocklength of 2^{17}.

In Figs. 3 and 4, superposition polar codes achieve bigger rate regions than time-sharing point-to-point channel polar codes.

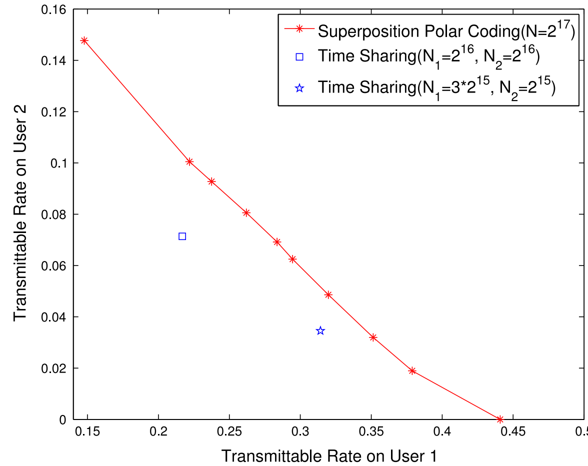

Fig. 4. Achievable region of the *less noisy* broadcast channel under superposition polar coding and achievable points under time sharing between point-to-point channel polar codes, both with blocklength of 2^{17}.

5. Conclusion

We have studied polar code that achieves the superposition coding region of the two-user binary-input broadcast channel. Our scheme is optimal if the channel is *less noisy* and the capacity-achieving auxiliary random variable is binary. Simulation results show the advantage of the superposition polar codes over the point-to-point channel polar codes. Furthermore, we note that our approach can be extended to more than two users.

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