Weight Balancing on Boundaries and Skeletons

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ABSTRACT

Given a polygonal region containing a target point (which we assume is the origin), it is not hard to see that there are two points on the perimeter that are antipodal, i.e., whose midpoint is the origin. We prove three generalizations of this fact. (1) For any polygon (or any bounded closed region with connected boundary) containing the origin, it is possible to place a given set of weights on the boundary so that their barycenter (center of mass) coincides with the origin, provided that the largest weight does not exceed the sum of the other weights. (2) On the boundary of any 3-dimensional bounded polyhedron containing the origin, there exist three points that form an equilateral triangle centered at the origin. (3) On the 1-skeleton of any 3-dimensional bounded convex polyhedron containing the origin, there exist three points whose center of mass coincides with the origin.

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1. INTRODUCTION

We will discuss three generalizations of the following observation (in this paper, a polygon or a polyhedron is always closed and bounded).

THEOREM 0. *On the perimeter of any polygon containing the origin, there are two points that are* antipodal*, i.e., whose midpoint is the origin.*

In other words, we have

$$2P \subseteq \partial P \oplus \partial P$$

for any polygon P, where ∂P denotes its boundary, $A \oplus B = \{\, x + y \mid x \in A,\ y \in B \,\}$ is the Minkowski sum of regions

A and B, and $\alpha A = \{ \alpha x \mid x \in A \}$ is the copy of A scaled (about the origin) by a real number α.

PROOF OF THEOREM 0. Consider $-P$, the copy of the given polygon P reflected about the origin. Since P and $-P$ cannot be properly contained in the other (and they both contain the origin), their boundaries intersect at some point $q \in \partial P \cap (-\partial P)$. Then q and $-q$ form the desired pair of points. $\square$

Distinct weights

An interpretation of Theorem 0 is that we can put two equal weights on the perimeter and balance them about the origin. Generalizing this to different sets of weights, we prove the following in Section 2 (note that this subsumes Theorem 0).

THEOREM 1. *Suppose that k weights $w_1 \geq w_2 \geq \cdots \geq w_k$ satisfy $w_1 \leq w_2 + \cdots + w_k$. Then for any polygon (or any region enclosed by a Jordan curve) $P \subseteq \mathbb{R}^2$ containing the origin, the weights can be placed on the boundary ∂P so that their center of mass is the origin.*

In terms of the Minkowski sum, the theorem says that

$$(w_1 + \cdots + w_k)P \subseteq w_1 \partial P \oplus w_2 \partial P \oplus \cdots \oplus w_k \partial P$$

if none of the weights is bigger than the sum of the rest.

If P is the unit disk, Theorem 1 is related to a reachability problem of a chain of links (or a robot arm) of lengths w_1, w_2, ..., w_k where one end is placed at the origin, each link can be rotated around the joints, and the links are allowed to cross each other. In order to reach every point of the disk of radius $w_1 + \cdots + w_k$ centered at the origin, it is well known (e.g., [4]) that the condition $w_1 \leq w_2 + \cdots + w_k$ is sufficient (and necessary). Theorem 1 generalizes this to arbitrary P.

We will also give efficient algorithms to find such a location of points. On the other hand, if we drop the condition $w_1 \leq w_2 + \cdots + w_k$, then the conclusion does not hold in general (just let P be a disk centered at the origin), and it is NP-complete to decide whether it holds for a given polygon P.

Tripodal points

The other two results concern the 3-dimensional (or higher) setting, where instead of a polygon we are given a polyhedron. Generalizing the notion of antipodal points in Theorem 0, we prove the following in Section 3.

THEOREM 2. *On the boundary of any* 3*-dimensional polyhedron containing the origin, there are* tripodal *points, i.e., three points forming an equilateral triangle centered at the origin.*

A classical problem reminiscent of Theorem 2 is the square peg problem of Toeplitz. Given a closed curve in a plane, the problem asks for a location of four vertices of a square on it. It was conjectured by Otto Toeplitz in 1911 that every Jordan curve contains such four points. Although it is still open for general Jordan curves, it has been affirmatively solved for curves with some smoothness conditions. As a variant of this problem, Meyerson [8] and Kronheimer and Kronheimer [5] proved that for any triangle T and any Jordan curve C, we can find three points on C forming the vertices of a triangle similar to T (note the contrast to our Theorem 2 where we needed the triangle to be equilateral). See a recent survey of Matschke [7] on these problems.

Weights on the skeleton

By viewing Theorem 0 again as balancing of two equal weights, we can consider another generalization to polyhedra, asking whether we can put (equal) weights so that their barycenter is the origin. Note, however, that this is not very interesting if we are allowed to put them anywhere on the surface of the polyhedron: we can then cut the polyhedron by any plane through the origin and apply the 2-dimensional Theorem 1 to (a connected component of) the section, showing that this is possible for any number of equal weights. The question becomes interesting if we restrict the weights to lie on the *edges* of the polyhedron.

THEOREM 3. *On the edges of any* 3*-dimensional bounded* convex *polyhedron containing the origin, there exist three points whose barycenter coincides with the origin.*

In other words,

$$3P \subseteq S_1(P) \oplus S_1(P) \oplus S_1(P),$$

where $S_1(P)$ denotes the 1-skeleton of a convex polyhedron P.

In fact, we will prove in Section 4 that the same thing (with d weights) is true in dimension d when d is the product of a power of 2 and a power of 3. We conjecture that this is true for all d and for non-convex polyhedra, but this is left for future work. It may also be worth trying to combine this with our first question in this paper, asking whether we have something similar to Theorem 3 for distinct weights.

Related work

Bringing the center of mass to a desired point by putting counterweights is a common technique for reduction of vibrations in mechanical engineering [1]. There have been studies on Minkowski operations considering boundary of objects (e.g., Ghosh and Haralick [3]), but our paper seems to be the first to deal with the general question of covering the body with convex linear combinations of the boundary.

We note that another generalization of Theorem 0 is the Borsuk-Ulam theorem, which has many applications in discrete and computational geometry [6]. It states that every continuous function $f : \mathbb{S}^d \to \mathbb{R}^d$ on the d-dimensional sphere (i.e., the boundary of the $(d+1)$-dimensional ball) centered at the origin has a point x such that $f(x) = f(-x)$. When $d = 1$, this implies Theorem 0 for convex P if we define $f(x)$ as the distance from the origin to ∂P in the direction x. A variant of the Borsuk-Ulam theorem is used also in our proof of Theorem 3. We hope to discover more relations of our observations to the Borsuk-Ulam theorem.

2. DISTINCT WEIGHTS

We prove Theorem 1. Let P be a polygon. First, put the biggest weight w_1 at a nearest point p on ∂P from the origin, and all other weights at one point so that the barycenter of all weights is at the origin. This means putting the weights w_2, ..., w_k at the point $-p \cdot w_1/(w_2 + \ldots + w_k)$, which is inside P by the assumption $w_1 \leq w_2 + \cdots + w_k$.

Starting at this configuration, let w_1 run along the perimeter ∂P, while moving w_2 accordingly so that the barycenter of all weights remains at the origin (all other weights are held fixed). This means moving w_2 along a copy of ∂P magnified by $-w_1/w_2$. Since this is at least as big as ∂P, and w_2 initially lies inside P, it must come out at some point,

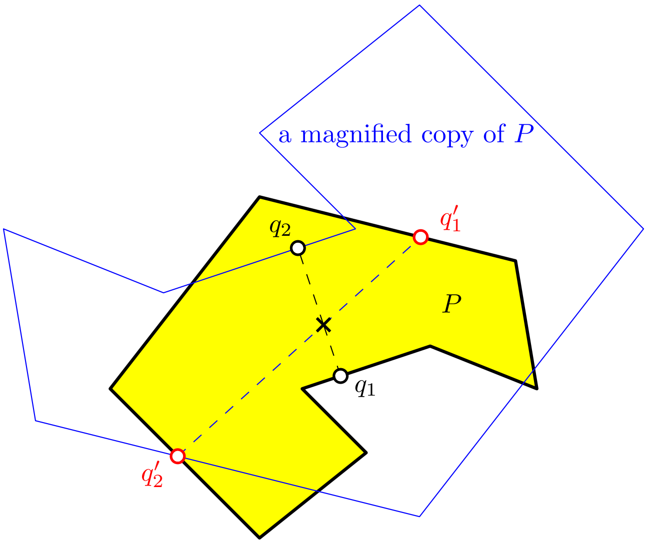

Figure 1: Second paragraph of the Proof of Theorem 1. If the weights w_1 and w_2 are initially at $q_1 \in \partial P$ and $q_2 \in P$, and w_1 moves along ∂P, then to keep the barycenter fixed, w_2 must move along a magnified (and reflected) copy of ∂P, so that it hits ∂P at some point q_2'.

giving a configuration where w_1 and w_2 are both on ∂P. See Figure 1 for illustation.

Now fix w_1 at this point and let w_2 run along ∂P, while moving w_3 accordingly so that the barycenter of all points remains at the origin. This means moving w_3 along a copy of ∂P magnified by $-w_2/w_3$. Since this is at least as big as ∂P, and w_3 initially lies inside P, it must come out at some point, giving a configuration where w_1, w_2, w_3 are on ∂P.

Repeating this, we can bring all weights to ∂P. This proves Theorem 1.

In this proof, we moved points "along" the boundary ∂P (or its copies) and argued that they must "come out" of P. This may sound as if we used the fact that ∂P is a curve, but a closer look at the proof shows that the theorem holds as long as (P is a closed bounded set and) ∂P is connected.

Algorithmic aspects

We consider the computational problem that corresponds to Theorem 1: Given a region P and a set of k weights, we want to determine whether we can balance the weights by putting them on ∂P, and if so, find such a location. We restrict ourselves to the case where P is a simple polygon with n vertices, and design algorithms in terms of n and k.

If none of the weights exceeds the sum of the others, the proof of Theorem 1 implies a polynomial-time algorithm. In order to replace q_s and q_{s+1} with a pair of boundary points q_s' and q_{s+1}' (Figure 1), we need to find an intersection point of ∂P and its reflected and scaled copy as shown in the proof. This can be done in $O(n \log n)$ time. The initial location of the largest weight can be found in $O(n)$ time. Thus, we have an $O(kn \log n)$ time algorithm.

We can design a faster algorithm as follows. We greedily divide the weights into three groups so that no group weighs more than the sum of the rest. This is always possible in $O(k)$ time (as long as no single weight exceeds the sum of the others). Thus, we have an instance for $k = 3$, which we solve in $O(n \log n)$ time. This gives an $O(k + n \log n)$-time algorithm, although the output may look a little artificial since all weights will be located at (at most) three points.

If we are given a set of unknown number of weights that may contain a weight exceeding the sum of the rest, the problem is NP-complete.

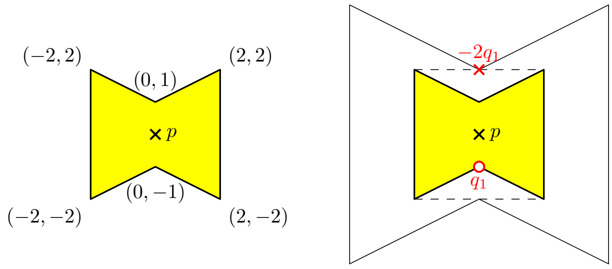

Figure 2: A hard instance of the balancing location problem.

Proposition 1. *There exists a polygon $P \subseteq \mathbb{R}^2$ containing the origin such that it is NP-complete to determine if a given set of weights can be placed on the boundary ∂P so that their barycenter is at the origin.*

Proof. The problem is clearly in NP, and we prove its NP-hardness by reducing the PARTITION problem to it. The input of PARTITION problem is a set of N nonnegative integers $a_1, a_2, \ldots a_N$ and ask whether there is a subset $X \subset \{1, \ldots, N\}$ such that

$$\sum_{i \in X} a_i = \sum_{i \in \{1,\ldots,N\} \setminus X} a_i.$$

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_N$ (this can be done by reordering the weights if necessary). We transform the problem into a weight balancing problem as follows: we set $k = N + 1$, $w_i = a_{i-1}$ for $i = 2, 3, \ldots, k$, and $w_1 = 2\sum_{i=2}^{k} w_i$.

Let P be the non-convex polygon with vertices $(0, 1)$, $(2, 2)$, $(2, -2)$, $(0, -1)$, $(-2, -2)$, and $(-2, 2)$ (see the left picture of Figure 2). Note that $P = -P$ and $-2P$ contains the convex hull $\mathrm{conv}(P)$ of P. Moreover, the two reflex vertices of $-2P$ are the only points of $2P \cap \mathrm{conv}(P)$, and each of the points is the midpoint of an edge of $\mathrm{conv}(P)$ as shown in the right picture.

Observe that the only possible location q_1 of weight w_1 should be one of the two reflex vertices since its reflection $-2q_1$ should be written as a convex combination of other points on ∂P, and hence contained in $\mathrm{conv}(P)$. That is, $-2q_1$ lies on the midpoint of an edge of $\mathrm{conv}(P)$ (without loss of generality, we may assume it is the edge e from $(-2, 2)$ to $(2, 2)$). In particular, there is a solution if and only if we can place the remaining points in a way that their barycenter lies on the midpoint of e.

Since the new target point lies on the edge e of $\mathrm{conv}(P)$, the only possible location for the remaining points is $(-2, 2)$ or $(2, 2)$. Moreover, the barycenter becomes the midpoint if and only if the weights are equally divided. Thus, the PARTITION problem is reduced to the balancing location problem. Since PARTITION is NP-complete, we conclude that detecting the existence of a balancing location is also NP-complete. □

3. TRIPODAL POINTS

In this section, we consider a 3-dimensional (bounded, closed, not necessarily convex) polyhedron P, and prove Theorem 2, which states that there are tripodal points on the boundary ∂P. Note that tripodal points are a natural analogue of antipodal points: saying that three points are tripodal is equivalent to requiring that they are at the same distance from the origin and their barycenter is the origin.

Let p_0 and p_1 be a nearest and a farthest point on ∂P, respectively, from the origin o. They exist because ∂P is compact. Consider a simple piecewise-linear path L from p_0 to p_1 on ∂P, parametrized by a one-to-one continuous function $\gamma\colon [0,1] \to L$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.

We claim that there exist three points $a \in L$, $b \in \partial P$, and $c \in \partial P$ that are tripodal. For each $q \in L$, let $H(q)$ be the set of vectors perpendicular to the line through o to q. We use the following fact.

LEMMA 1. *There exists a continuous piecewise algebraic function $\mathbf{v}\colon L \to \mathbb{S}^2$ such that $\mathbf{v}(q) \in H(q)$ for all $q \in L$.*

PROOF. Let S be a set of points on L that contains both endpoints and all the joints of L (recall that L is piecewise linear). We first set $\mathbf{v}(u)$ arbitrarily on each point u of S. Thus, it suffices to define $\mathbf{v}$ on the line segment $e = uv$ spanned by a consecutive pair of points u, v in S along L. By the choice of S, e is contained in L. We parametrize e by $x\colon [0,1] \to e$ such that $x(0) = u$ and $x(1) = v$. For every $t \in [0,1]$, let $h(x(t))$ be the projection of $t\mathbf{v}(u) + (1-t)\mathbf{v}(v)$ to $H(x(t))$. If $h(x(t))$ is not a zero vector, we scale $h(x(t))$ to have a unit vector and define it as $\mathbf{v}(x(t))$. Since $h(x(t))$ becomes zero only if the vector $t\mathbf{v}(u) + (1-t)\mathbf{v}(v)$ is perpendicular to $H(x(t))$, a suitably large choice of S can ensure that $h(x(t))$ is not zero for any $t \in [0,1]$. We can easily check that $\mathbf{v}$ is continuous and piecewise algebraic. $\square$

We fix such a function $\mathbf{v}\colon L \to \mathbb{S}^2$. For each $t \in [0,1]$ and each angle $\theta \in [0, 2\pi)$, let $b(t,\theta)$ and $c(t,\theta)$ be the unique pair of points such that $\gamma(t)$, $b(t,\theta)$, $c(t,\theta)$ are tripodal points and the vector $b(t,\theta) - c(t,\theta) \in H(\gamma(t))$ makes an angle of $+\theta$ with $\mathbf{v}(\gamma(t))$. Define $f_1(t,\theta) \in \{+,-,0\}$ by whether the point $b(t,\theta)$ lies inside (the interior of) P, outside P, or on ∂P. Define $f_2(t,\theta)$ analogously using the point $c(t,\theta)$.

If there is (t,θ) such that $f_1(t,\theta) = f_2(t,\theta) = 0$, then $\gamma(t)$, $b(t,\theta)$, $c(t,\theta)$ are tripodal points and we are done. Suppose otherwise. Define the *signature* of (t,θ), denoted $F(t,\theta)$, as

$$\begin{cases} ++ & \text{if } (f_1(t,\theta), f_2(t,\theta)) \in \{(+,+),(+,0),(0,+)\}, \\ -- & \text{if } (f_1(t,\theta), f_2(t,\theta)) \in \{(-,-),(-,0),(0,-)\}, \\ +- & \text{if } (f_1(t,\theta), f_2(t,\theta)) = (+,-), \\ -+ & \text{if } (f_1(t,\theta), f_2(t,\theta)) = (-,+) \end{cases}$$

(Figure 3). Since p_0 and p_1 are the nearest and the farthest points, it holds that $F(0,\theta) = ++$ and $F(1,\theta) = --$.

For each θ, consider the transition of $F(t,\theta)$ as t changes from 0 to 1. Since $\mathbf{v}$ is piecewise algebraic, the number of transitions is finite, and we obtain a finite walk $\mathcal{W}(\theta)$ from $++$ to $--$ in the graph C shown in Figure 4.

Consider the edge e between $++$ and $+-$. A walk from $++$ to $--$ is called *even* if it uses e an even number (possibly zero) of times, and it is called *odd* otherwise. For example, the path $++, +-, --$ is odd and $++, -+, --$ is even.

As we increase θ continuously from one angle to another, the walk $\mathcal{W}(\theta)$ may change, but the parity remains invariant, because all that can happen are a finite number of

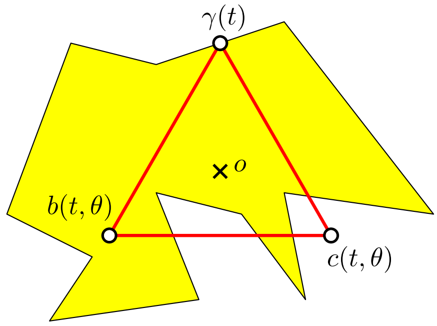

Figure 3: The signature is $+-$ for this (t,θ).

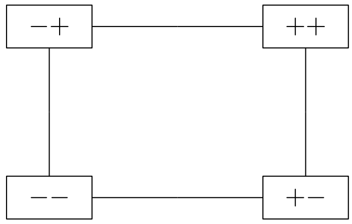

Figure 4: The cycle C.

- insertions, where an entry a in $\mathcal{W}(\theta)$ is replaced by a sequence a, b, a, where b is a neighbor of a in C, and
- deletions, where a sequence a, b, a is replaced by a,

and these events do not change the parity of the walk.

On the other hand, the walks $\mathcal{W}(0)$ and $\mathcal{W}(\pi)$ have different parities. To see this, let e' be the edge between $++$ and $-+$. Since e and e' form a cut separating $++$ and $--$ in C, each walk must use them an odd number of times in total. Since $b(t,0) = c(t,\pi)$ and $c(t,0) = b(t,\pi)$, the walk $\mathcal{W}(\pi)$ is obtained from $\mathcal{W}(0)$ by exchanging $-+$ and $+-$, and hence has opposite parity.

This is a contradiction, and Theorem 2 follows.

We here note that Theorem 2 extends to the case when ∂P is a smooth manifold or a PL manifold. A proof is given in the appendix.

Theorem 2 was about tripodal points in 3-dimensional polyhedra. We may ask similar questions for three points forming other shapes, or for higher dimensions.

CONJECTURE 1. *For a d-dimensional polyhedron P containing the origin, there exist d points on ∂P forming a regular $(d-1)$-dimensional simplex centered at the origin.*

Algorithmic aspects need further investigation. It is easy to devise an $O(n^3)$-time algorithm to find a tripodal location guaranteed by Theorem 2 for a polyhedron with n vertices, just by going through all the triples of faces. It is not clear if this can be improved.

4. LOCATING WEIGHTS ON EDGES

A d-dimensional (closed bounded) polyhedron P decomposes into faces of dimensions $i = 0, 1, \ldots d$. Let F_i be the set of i-dimensional faces. The union $S_k(P) = \bigcup_{i=0}^{k} \bigcup_{f \in F_i} f$ of faces of at most k dimensions is called the k-*skeleton* of P. In particular, the 1-skeleton $S_1(P)$ is the union of edges (including vertices), and the $(d-1)$-skeleton is ∂P. Thus, another natural higher-dimensional analogue of the 2-dimensional Theorem 0 (where we put weights on $S_1(P) = \partial P$) is to try to place weights on the 1-skeleton $S_1(P)$:

CONJECTURE 2. *On the* 1*-skeleton of any* d*-dimensional (bounded) polyhedron containing the origin, there exist* d *points whose barycenter is at the origin.*

In other words,

$$dP \subseteq \underbrace{S_1(P) \oplus \cdots \oplus S_1(P)}_{d \text{ times}}$$

for any d-dimensional polyhedron $P \subseteq \mathbb{R}^d$.

We have been unable to prove the conjecture, even for $d = 3$. However, we observe that having an additional point makes the problem much easier.

PROPOSITION 2. *On the* 1*-skeleton of any* 3*-dimensional (bounded convex) polyhedron containing the origin, there exist four points whose barycenter is at the origin.*

PROOF. We consider a plane H through the origin, and find an antipodal pair (q, q') on $\partial P \cap H$ by Theorem 0. Let F and F' be the faces containing q and q'. Again by Theorem 0, we can find pairs (q_1, q_2) and (q_3, q_4) on edges of F and F' with barycenter $q \in F$ and $q' \in F'$, respectively. These four points q_1, q_2, q_3, q_4 satisfy our criteria. $\square$

In the remainder of this section, we consider the case in which P is convex. Using an elementary argument, we can show that Conjecture 2 is true for convex polyhedra when d is a power of 2. A key tool is the following lemma.

LEMMA 2. *For any convex polyhedron* $P \subset \mathbb{R}^d$*, we have*

$$2P \subseteq S_{\lfloor d/2 \rfloor}(P) \oplus S_{\lceil d/2 \rceil}(P).$$

PROOF. Choose any point of the left-hand side, $2P$. We will show that this point is in the right-hand side. We may assume that this point is in the interior of $2P$, since the right-hand side is a closed set. Also, without loss of generality, we may assume that this point is the origin. Thus, assuming that P contains the origin in its interior, we need to show that the origin belongs to the right-hand side, or equivalently, that $S_{\lfloor d/2 \rfloor}(P) \cap S_{\lceil d/2 \rceil}(-P)$ is nonempty.

For simplicity of notation, we assume that d is even. The odd case is shown identically by replacing $d/2$ by $\lfloor d/2 \rfloor$ and $\lceil d/2 \rceil$ accordingly.

Since P contains the origin in its interior, the intersection $P \cap (-P)$ is a d-dimensional convex polyhedron. Moreover, its boundary C is centrally symmetric (i.e., $C = -C$). It suffices to show that C has a vertex in $S_{d/2}(P) \cap S_{d/2}(-P)$.

A facet ($(d-1)$-dimensional face) of C is a subset of a facet of either P or $-P$. We start with the special case in which C is simple. That is, every vertex of C is contained in exactly d facets of P or $-P$. A vertex of C is of *type* $(j, d-j)$ if it is contained in j facets of P and $d-j$ facets of $-P$. Let v be any vertex of C, and let $(k, d-k)$ be the type of v. If $k = d/2$, we are done. Thus, we assume without loss of generality that $k < d/2$. Since C is centrally symmetric, $-v \in C$, and $-v$ is of type $(d-k, k)$. Since the 1-skeleton of C is connected, there exists a path P in the skeleton from v to $-v$. Let (x, y) be an edge of P with x and y of type $(i, d-i)$ and $(j, d-j)$, respectively. Then $j \in \{i-1, i, i+1\}$. Thus, there exists a vertex w on P of type $(d/2, d/2)$.

Now, we consider the general case where C might have a vertex that is an intersection of more than d facets. We consider an infinitesimal perturbation of hyperplanes defining facets of P to make C simple. Then, the perturbed version $\tilde{C}$ of C has a vertex $\tilde{v}$ of type $(d/2, d/2)$, which corresponds to a vertex v of C. Thus, v must lie at an intersection of $S_{d/2}(P)$ and $S_{d/2}(-P)$. $\square$

Thus we can always find an antipodal pair of points from $\lfloor d/2 \rfloor$- and $\lceil d/2 \rceil$-dimensional faces. However, this does not extend to other pairs of dimensions k and $d-k$.

PROPOSITION 3. *There exists a convex polyhedron* $P \subseteq \mathbb{R}^d$ *containing the origin such that for any* $k < \lfloor d/2 \rfloor$*, it holds that* $S_k(P) \cap S_{d-k}(-P) = \emptyset$.

PROOF. First, we consider the case where $d = 2m$ is even (thus, $k < m$). Consider an equilateral triangle T centered at the origin. Then, we observe that all three vertices of T lie outside $-T$. Let $T^m = T \times \cdots \times T$ be the Cartesian product of T in $\mathbb{R}^{2m}$. Then, a k-dimensional face of $P = T^m$ is the Cartesian product of k edges and $m-k$ vertices of T. Since $m - k > 0$ and a vertex of T lies outside $-T$, the face cannot intersect $-P = (-T)^m$. If $d = 2m+1 \geq 3$ is odd, we consider $P = I \times T^m$, where $I = [-1, 2]$ is an interval. The remaining argument is analogous. $\square$

We then prove the following proposition, implying that Conjecture 2 is true for convex polyhedra if $d = 2^i$ for any $i \geq 0$.

PROPOSITION 4. *Let* k *be a positive integer and let* $d \leq 2^k$*. Then, on the* 1*-skeleton of any* d*-dimensional convex polyhedron, there are* 2^k *points whose barycenter is at the origin.*

PROOF. We use induction on k. The statement is true for $k = 1$ (Theorem 0). It follows from Lemma 2 that there are antipodal points $x \in F$ and $-x \in F'$, where F and F' are faces from $S_{\lfloor d/2 \rfloor}(P)$ and $S_{\lceil d/2 \rceil}(P)$, respectively. By the induction hypothesis applied to F and F' (translated by $-x$ and x), we have 2^{k-1} points on the skeleton of F with barycenter x, and 2^{k-1} points on the skeleton of F' with barycenter $-x$. These 2^k points together satisfy our requirement. $\square$

We note that our method is constructive, and such a location of points can be computed in polynomial time for any fixed dimension.

Using a generalization of the Borsuk-Ulam theorem in terms of $\mathbb{Z}_p$-valued Euler class of a vector bundle, we can extend the proof to other values of d.

THEOREM 4. *Conjecture 2 is true for convex polyhedra if* $d = 2^i 3^j$ *for any* $i, j \geq 0$.

Note that Theorem 3 in the introduction is a special case of this. Proving Theorem 4 requires some mathematical tools, and we give an outline of the proof in the appendix.

Recently, Conjecture 2 has been affirmatively settled for convex polyhedra, for all d, by Dobbins [2].

Algorithmic aspects of Proposition 4 and Theorem 4 are also unexplored.

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APPENDIX

A. POSTPONED PROOFS

A.1 Generalized tripodal points

An approach similar to Theorem 2 can be used to obtain a more general statement.

PROPOSITION 5. *Let P be a compact region in $\mathbb{R}^3$ containing the origin. Assume that there exists a number N such that for any $n > N$ there exists a polyhedron P_n containing the origin such that ∂P is located in the $\frac{1}{n}$-neighborhood of ∂P_n (with respect to the Hausdorff metric). Then, there exist three points on ∂P that are tripodal.*

The requirement for P is satisfied when ∂P is a smooth manifold or a PL manifold [10]. We conjecture that this result extends to higher dimensions.

PROOF OF PROPOSITION 5. By Theorem 2, there exist tripodal points $(q_1(n), q_2(n), q_3(n))$ on each ∂P_n. Since $\{(q_1(n), q_2(n), q_3(n))\}_{n \geq N}$ is an infinite sequence in a compact subset of the 9-dimensional space, it has a subsequence that converges to a point (q_1, q_2, q_3) in $\partial P \times \partial P \times \partial P$. This gives tripodal points on ∂P. $\square$

A.2 Proof of Theorem 4

We give an outline of the proof of Theorem 4.

PROPOSITION 6. *Let $d = 3k$ be a multiple of 3. For a d-dimensional convex compact polyhedron P containing the origin o in its interior, we can find three points q_1, q_2 and q_3 in the k-skeleton of P such that $q_1 + q_2 + q_3 = 0$.*

Theorem 4 easily follows from Proposition 6: If $d = 2^i 3^j$, we can reduce the dimension to 3^j by using Lemma 2, and then reduce the dimension similarly to 1 by using Proposition 6 analogously to the argument in Proposition 4. The proof of Proposition 6 uses results from topology that are not widely known and require considerable machinery to develop. Thus, we first give an informal introduction, then provide further details assuming familiarity with algebraic topology and the Euler cohomology class. A proof of the proposition in a more general form with further consequences for Conjecture 2 will be published in a companion paper.

We start with an informal introduction to the Euler class. Recall that a vector bundle is a generalization of the product space of a vector space (the fiber space) with some manifold (the base space). Unlike a product space, which comes with a pair of coordinate projections, a vector bundle only has a projection to the base space. When a vector bundle is "twisted," it is not possible to also define a projection to the fiber space such that the product of these projections is a homeomorphism. The Euler class indicates that a vector bundle is twisted by detecting the intersection of a pair of generic sections. For example, both the (unbounded) cylinder and Möbius strip are rank-1 vector bundles over a circle. The $\mathbb{Z}_2$-Euler class of the cylinder is trivial, but that of the Möbius strip is not. Thus, the cylinder has disjoint sections, but the Möbius strip does not. Moreover, the cylinder is a product space, but the Möbius strip is not.

A vector bundle can sometimes be formed by starting with a product space and taking the quotient by a group that acts on both of these spaces. The resulting vector bundle might then be twisted. If we are given some function $\phi\colon X \to Y$ respecting a group action G that is free on X where Y is a vector space, we may show that ϕ vanishes by forming the vector bundle $\pi\colon (X \times Y)/G \to X/G$. Then, ϕ defines a section of this vector bundle, and if the Euler class of the bundle is non-trivial, ϕ must intersect the zero section. For example, let $G \simeq \mathbb{Z}_2$ with generator acting on the circle $\mathbb{S}$ by rotating by the angle π and acting on the line $\mathbb{R}$ by scaling by -1. The cylinder is the product space $\mathbb{S}\times\mathbb{R}$, and the Möbius strip is the cylinder twisted by the $\mathbb{Z}_2$-action of G. That is, the Möbius strip is the quotient space $(\mathbb{S} \times \mathbb{R})/G$, which becomes a vector bundle with projection $\pi\colon (\mathbb{S} \times \mathbb{R})/G \to \mathbb{S}/G$. The fact that the $\mathbb{Z}_2$-Euler class of the Möbius strip is non-trivial implies that any function $\phi\colon \mathbb{S} \to \mathbb{R}$ respecting the action of G must vanish somewhere.

We may think of a homology class as an equivalence class of weighted subspaces of a certain dimension. When the coefficients are in a field, homology groups become vector spaces, and the corresponding cohomology space is the dual space of linear functionals. Here we will consider coefficients in the field $\mathbb{Z}_3$ of integers modulo 3. The Euler class, as a cohomology class, is then a certain linear functional on homology classes that detects the kernel of a section in the following way.

Proposition 7. *For a vector bundle over a closed manifold with $\mathbb{Z}_3$-Euler class e, a section σ, and a homology class a of the base space, if $e(a) \neq 0$, then the support of any representative of a intersects the kernel of σ.*

Proof of Proposition 6. Let

$$V = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ & \vdots & \\ x_{d,1} & x_{d,2} & x_{d,3} \end{bmatrix} \in \mathbb{R}^{d\times 3} \;\middle|\; \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{d,1} \end{bmatrix} + \begin{bmatrix} x_{1,2} \\ \vdots \\ x_{d,2} \end{bmatrix} + \begin{bmatrix} x_{1,3} \\ \vdots \\ x_{d,3} \end{bmatrix} = 0 \right\}$$

and let $Q = P^3 \cap V$ where $P^3 = P \times P \times P$. That is, Q is the convex compact polyhedron consisting of triples of points in P with barycenter at the origin. Note Q has dimension $3d - d = 2d$. For now, we assume that V intersects P^3 generically. That is, every face of Q is of the form $F = (F_1 \times F_2 \times F_3) \cap V$ where each F_i is a face of P and $\dim F = \dim F_1 + \dim F_2 + \dim F_3 - d$. Later we will use a perturbation argument to deal with the non-generic case. We define a of map $\phi = (\phi_1, \phi_2, \phi_3)\colon \partial Q \to \mathbb{R}^3$ as follows. We first define ϕ_i on a vertex v of ∂Q where $v = (v_1, v_2, v_3)$ and each component $v_i = (v_{1,i}, \ldots, v_{d,i})$ is a point in P. Each point v_i is in some face of dimension $d_i = d - c_i$ and $\sum_{i=1}^{3} d_i = d = 3k$. Let $\phi_i(v) = d_i - k$. Now extend ϕ_i to the vertices of a barycentric subdivision of ∂Q as follows. For each face F extend ϕ_i to the barycenter of F by the mean of ϕ_i on the vertices of F, so $\phi_i(v_F) = \frac{1}{h}\sum_{j=1}^{h} \phi_i(v_{F,j})$ where $v_{F,1}, \ldots, v_{F,h}$ are the vertices of F, $v_F = \frac{1}{h}\sum_{j=1}^{h} v_{F,j}$ is the barycenter of F. Finally, extend ϕ_i to all of ∂Q by linear interpolation of vertices on the simplices of the barycentric subdivision. Observe that ϕ is a continuous function of ∂Q.

Our goal is to find a vertex v of Q such that $\phi(v) = 0$. If we have such v, then $d_i = k$ for all $i = 1, 2, 3$, which implies each v_i is in the k-skeleton of P. Thus we have Proposition 6. We will find such a vertex in two parts. First, we will see that ϕ must vanish somewhere on a 2-dimensional face of Q. Second, a minimal face where ϕ vanishes cannot have dimension 2 or 1, so ϕ must vanish on a vertex. For the first part we make use of a topological result given by Munkholm [9] to prove a Borsuk-Ulam type theorem for $\mathbb{Z}_p$-actions, which we now state for $p = 3$.

Theorem 5 ([9]). *For a group $G \simeq \mathbb{Z}_3$ acting freely on the m-sphere $\mathbb{S}^m$ and acting on V by cyclically permuting coordinates in $\mathbb{R}^3$, the vector bundle*

$$\pi\colon E = (\mathbb{S}^m \times V)/G \;\to\; B = \mathbb{S}^m/G, \quad \pi([\theta, x]_G) = [\theta]_G$$

has non-trivial $\mathbb{Z}_3$-Euler class.

To see that ϕ vanishes on some 2-face of Q, form such a vector bundle,

$$\pi\colon E = (\partial Q \times V)/G \;\to\; B = \partial Q/G, \quad \pi([\theta, x]_G) = [\theta]_G.$$

By Theorem 5, the $\mathbb{Z}_3$-Euler class $e \in H^2(B; \mathbb{Z}_3)$ is non-trivial. Thus there is some $a \in H_2(B; \mathbb{Z}_3)$ such that $e(a) \neq 0$. By the canonical isomorphism of singular and cellular homology, a has some representative with support A in the 2-skeleton of B, which lifts to a subset of the 2-skeleton of Q. Let $Z = \ker\phi$. By Proposition 7, A intersects $\ker\phi/G$, and this lifts to the intersection of Z with certain 2-faces of Q.

Now that we know Z intersects a 2-face of Q, we will see that it must contain a vertex. Let F be a minimal face intersecting Z. F is the intersection of V with some face $F_1 \times F_2 \times F_3$ of P^3, and since Q has dimension $2d$, F has codimension at least $2d - 2$, which means the total codimension of the faces F_1, F_2, F_3 is at least $2d - 2$, which makes the sum of their dimensions is at most $d + 2$. By the pigeonhole principle, at most $(3k + 2)(k + 1)^{-1} < 3$ faces can have dimension at least $k + 1$. We may assume by symmetry that F_3 is a face of dimension k or smaller. That is, $\phi_3(p) \leq 0$ for all $p \in F$. We can conclude from this that $\phi_3(p) = 0$ for all $p \in F$. To see this, suppose there is some $p \in F$ such that $\phi_3(p) < 0$. This would give $\phi_3(v_F) < 0$, which implies $\phi_3(p) < 0$ for all p in the interior of F, but then Z must intersect F on its boundary, contradicting the minimality of F.

Suppose F has dimension 2. Then, ϕ is a continuous map from F to the line $\{(t, -t, 0) \mid t \in \mathbb{R}\}$ that attains 0, so ϕ must attain 0 on the boundary of F, contradicting the minimality of F.

Suppose F has dimension 1. Let v, w be their vertices of F. Since V intersects P^3 generically exactly one face of P defining v differs from the corresponding faces defining F and this face is one dimension lower; likewise for w. That is, up to symmetry, we have two cases:

$$v = G_1 \times F_2 \times F_3, \quad w = H_1 \times F_2 \times F_3 \quad \text{or} \quad w = F_1 \times H_2 \times F_3,$$

where $\dim G_1 = \dim F_1 - 1$ and $\dim H_i = \dim F_i - 1$. This gives $\phi(v) - \phi(w) =$ either $(0, 0, 0)$ or $(-1, 1, 0)$. Since ϕ is integral on v, w and attains 0 on F, we must have either $\phi(v) = 0$ or $\phi(w) = 0$, contradicting the minimality of F.

As long as V intersects P^3 genericaly, the dimension of a minimal face F intersecting the kernel of ϕ is at most 2, but cannot be 2 or 1, so $F = v = (v_1, v_2, v_3)$ must be a vertex where v_i is in the k-skeleton of P.

Now consider the case where V does not intersect P^3 generically. The polyhedron can be defined by a linear vector inequality $P = \{x \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{n\times d}$, $b \in \mathbb{R}^n$ where P has n facets. For a linear operator given by $\varepsilon \in \mathbb{R}^{n\times d}$, let $P_\varepsilon = \{x \mid (A+\varepsilon)x \leq b\}$. For almost every ε sufficiently small, V does intersect P_ε^3 generically, and by the argument just given, there is some $v(\varepsilon) \in S_1(P_\varepsilon)^3 \cap V$. Since $S_1(P_\varepsilon)$ is bounded for ε small, there is some limit point $\lim_{\varepsilon\to 0} v(\varepsilon) \in S_1(P)^3 \cap V$, where $\lim_{\varepsilon\to 0} v(\varepsilon)$ is the limits of all convergent sequences. $\square$