Rate of Convergence Towards Semi-Relativistic Hartree Dynamics

Ji Oon Lee

Abstract. We consider the semi-relativistic system of N gravitating Bosons with gravitation constant G. The time evolution of the system is described by the relativistic dispersion law, and we assume the mean-field scaling of the interaction where $N \to \infty$ and $G \to 0$ while $GN = \lambda$ fixed. In the super-critical regime of large λ, we introduce the regularized interaction where the cutoff vanishes as $N \to \infty$. We show that the difference between the many-body semi-relativistic Schrödinger dynamics and the corresponding semi-relativistic Hartree dynamics is at most of order N^{-1} for all λ, i.e., the result covers the sub-critical regime and the super-critical regime. The N dependence of the bound is optimal.

1. Introduction

We consider a system of N gravitating three-dimensional Bosons in $\mathbb{R}^3$. When the particles in the system have the relativistic dispersion with Newtonian gravity, the Hamiltonian of the system is

$$H_{\mathrm{grav}} = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} - G \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}. \tag{1.1}$$

The Hamiltonian H_{grav} acts on the Hilbert space $L^2(\mathbb{R}^{3N})_s$, the subspace of $L^2(\mathbb{R}^{3N})$ consisting of all symmetric functions with respect to the permutations of particles. Such a system is known as a Boson star.

We are interested in the mean-field limit, where we let $G \to 0$ and $N \to \infty$ with $\lambda := GN$ is fixed. The N particle Hamiltonian is thus defined by

$$H_N = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}. \tag{1.2}$$

In the Hamiltonian H_N, the kinetic energy and the interaction potential energy scale is of the same order (inverse length), hence the system is critical and its behavior hugely depends on the coupling constant λ. It was proved by Lieb and Yau in [24] that there exists a critical coupling constant $\lambda_{\mathrm{crit}}(N)$, depending on N, such that the minimum energy

$$E_N^\lambda = \inf_{\psi \in L^2(\mathbb{R}^{3N})} \frac{\langle \psi, H_N \psi \rangle}{\|\psi\|_{L^2}^2} \tag{1.3}$$

is bounded below if $\lambda < \lambda_{\mathrm{crit}}(N)$ and $E_N^\lambda = -\infty$ if $\lambda > \lambda_{\mathrm{crit}}(N)$. As $N \to \infty$, $\lambda_{\mathrm{crit}}(N)$ converges to a number $\lambda_{\mathrm{crit}}^H$, where

$$\frac{1}{\lambda_{\mathrm{crit}}^H} = \sup_{\|\varphi\|_{L^2(\mathbb{R}^3)}=1} \left(\frac{1}{2} \int \mathrm{d}x \mathrm{d}y \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} \right) \Big/ \left(\int \mathrm{d}x \left| |\nabla|^{1/2} \varphi(x) \right|^2 \right). \tag{1.4}$$

The exact value of $\lambda_{\mathrm{crit}}^H$ is not known, but it was shown in [23,24] that $4/\pi \leq \lambda_{\mathrm{crit}}^H \leq 2.7$.

In the subcritical case $\lambda < \lambda_{\mathrm{crit}}^H$, the Hamiltonian H_N defines a self-adjoint operator with domain $H^{1/2}(\mathbb{R}^{3N})$ when N is sufficiently large. (Technically, H_N is considered as the Friedrichs extension of (1.2).) Thus, it generates the one-parameter group of unitary operators e^{-itH_N} that describes the time evolution of the given system. We focus on the time evolution with respect to H_N of a factorized initial data $\psi_N := \varphi^{\otimes N}$ for some $\varphi \in H^1(\mathbb{R}^3)$. It is expected that $\psi_{N,t} := e^{-itH_N} \psi_N$ satisfies

$$\psi_{N,t} \simeq \varphi_t^{\otimes N}, \tag{1.5}$$

where φ_t is the solution of the semi-relativistic nonlinear Hartree equation

$$i\partial_t \varphi_t = (1 - \Delta)^{1/2} \varphi_t - \lambda \left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t \tag{1.6}$$

with initial data $\varphi_{t=0} = \varphi$.

The factorization (1.5) should be understood in terms of the marginal densities (reduced density matrices) associated with $\psi_{N,t}$. We define the k-particle marginal density through its kernel

$$\gamma_{N,t}^{(k)}(\mathbf{x}_k, \mathbf{x}_k')$$
$$:= \int \mathrm{d}x_{k+1} \cdots \mathrm{d}x_N \psi_{N,t}(\mathbf{x}_k, x_{k+1}, \ldots, x_N) \overline{\psi_{N,t}}(\mathbf{x}_k', x_{k+1}, \ldots, x_N), \tag{1.7}$$

where $\mathbf{x}_k = (x_1, x_2, \ldots, x_k)$ and $\mathbf{x}_k' = (x_1', x_2', \ldots, x_k')$. Since $\|\psi_{N,t}\|_{L^2} = 1$, we can see that $\mathrm{Tr}\, \gamma_{N,t}^{(k)} = 1$ for all $1 \leq k \leq N$. Thus, $\gamma_{N,t}^{(k)}$ is a trace class operator. Elgart and Schlein [6] proved that, in the large N limit, the

k-particle marginal density associated with $\psi_{N,t}$ converges to k-particle marginal density associated with the factorized wavefunction $\varphi_t^{\otimes N}$, under the condition that $\lambda < 4/\pi$ and $\varphi \in H^1(\mathbb{R})$. More precisely, for any fixed $t \in \mathbb{R}$,

$$\mathrm{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \to 0 \quad \text{as } N \to \infty, \tag{1.8}$$

where $|\varphi_t\rangle\langle\varphi_t|$ denotes the rank one projection onto φ_t. For $\lambda < \lambda_{\mathrm{crit}}^H$, it is proved by Lenzmann [22] that the semi-relativistic Hartree equation (1.6) is globally well-posed in $H^s(\mathbb{R}^3)$ for every $s \geq 1/2$. Therefore, (1.8) shows that the solution of the N-particle Schrödinger equation $\psi_{N,t}$ can be approximated by products of the solution of the semi-relativistic Hartree equation φ_t for all $t \in \mathbb{R}$.

The rate of convergence in (1.8) is attained by Knowles and Pickl [21] for the case $\lambda < 4/\pi$ with the initial condition $\varphi \in H^s$ for $s > 1$. More precisely,

$$\mathrm{Tr} \left| \gamma_{N,t}^{(k)} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \right| \leq \frac{C(k,t)}{\sqrt{N}} \tag{1.9}$$

for some constant $C(k,t)$ independent of N. Here, $C(k,t) = C_t\sqrt{k}$ where C_t grows at most exponentially in t.

In the supercritical regime $\lambda > \lambda_{\mathrm{crit}}^H$, on the other hand, solutions of (1.6) may blow up in finite time, which was proved by Fröhlich and Lenzmann [13]. Physically, the blowup of the solution of (1.6) describes the gravitational collapse of a Boson star whose mass is over a critical value, provided that the relativistic dynamics of the system can be approximated by the semi-relativistic Hartree dynamics as in the subcritical case. This assumption was proved by Michelangeli and Schlein [25] with the regularized Hamiltonian

$$H_N^\alpha = \sum_{j=1}^N (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j}^N \frac{1}{|x_i - x_j| + \alpha_N} \tag{1.10}$$

with $\alpha_N > 0$ and $\alpha_N \to 0$ as $N \to \infty$. The regularized Hamiltonian H_N^α defines a quadratic form, which is bounded below, hence we may consider its Friedrichs extension as a self-adjoint operator with domain $H^{1/2}(\mathbb{R}^{3N})$. If we let $\gamma_{N,t}^{\alpha,(1)}$ be the one-particle marginal density associated with $\psi_{N,t}^\alpha = e^{-itH_N^\alpha}\varphi^{\otimes N}$, then [25, Theorem 1.1] shows that with the initial condition $\varphi \in H^2(\mathbb{R}^3)$,

$$\mathrm{Tr} \left| \gamma_{N,t}^{\alpha,(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq \frac{C(t)}{\sqrt{N}} \tag{1.11}$$

for all $|t| \leq T$, where T is the maximal time of the existence of the solution of (1.6).

The corresponding results for non-relativistic dynamics is relatively well-established. Spohn [29] first proved that (1.8) holds when the interaction potential is bounded. This result was extended by Erdős and Yau [8] for the Coulomb type interaction. Rodnianski and Schlein [28] obtained an explicit bound on the rate of the convergence in (1.8) for the Coulomb type interaction. The result in [28], which showed that the rate of the convergence in (1.8) is $O(N^{-1/2})$, is extended further by Knowles and Pickl [21] for more singular potentials. On

the other hand, Erdős and Schlein [7] proved that the rate of convergence in (1.8) is $O(N^{-1})$ for bounded potentials, which is considered to be optimal. The same rate of convergence for more singular potentials including Coulomb type potential was obtained in [2,3]. Another important result in this direction is the derivation of the Gross–Pitaevskii equation for describing Bose–Einstein condensates by Erdős et al. [9–12]. (See also works by Pickl [26,27].) We also remark that other results concerning second-order correction to the mean-field limit are attained by Grillakis et al. [16,17]. (See also works by Chen [4].)

In this paper, we improve the bound (1.9) and (1.11) by applying the method developed in [28]. First introduced by Hepp [19] and extended by Ginibre and Velo [14,15], this method have been successful in proving various bounds on the rate of convergence as in [2,3,25,28]. We show that the left hand sides of (1.9) and (1.11), the differences between the one-particle marginal density associated with the solution of the time evolution of the factorized initial data and the orthogonal projection onto the solution of the semi-relativistic Hartree equation (1.6), are $O(N^{-1})$. The first main result of this paper, which considers the subcritical case, is the following theorem:

Theorem 1.1. *Suppose that* $\lambda < \lambda_{\text{crit}}^H$, $\varphi \in H^1(\mathbb{R}^3)$ *with* $\|\varphi\|_{L^2} = 1$, *and* $\psi_N = \varphi^{\otimes N}$. *Let* $\psi_{N,t} = e^{-itH_N}\psi_N$ *be the evolution of the initial wave function* ψ_N *with respect to the Hamiltonian* (1.2) *and let* $\gamma_{N,t}^{(1)}$ *be the one-particle marginal density associated with* $\psi_{N,t}$. *Let* φ_t *be the solution of the* (1.6) *with initial data* $\varphi_{t=0} = \varphi$. *Let*

$$\nu(t) := \sup_{|s| \leq t} \|\varphi_s\|_{H^1}. \tag{1.12}$$

Then, there exists a constant C, *depending only on* λ *and* $\nu(t)$, *such that*

$$Tr \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq CN^{-1}. \tag{1.13}$$

Remark 1.2. Since the semi-relativistic Hartree equation (1.6) is globally well-posed in H^1 for the subcritical case, $\nu(t) < \infty$ for all $t \in \mathbb{R}$. See [5,22] for more detail.

In the supercritical case, while we should introduce the regularized Hamiltonian (1.10) to define a self-adjoint operator, the approximating semi-relativistic Hartree equation does not need to contain the regularized non-linear term, i.e., it suffices to consider the equation (1.6) for approximating the evolution of the N-particle factorized initial state. The second main result of this paper is the following theorem:

Theorem 1.3. *Suppose that* $\lambda \geq \lambda_{\text{crit}}^H$, $\varphi \in H^1(\mathbb{R}^3)$ *with* $\|\varphi\|_{L^2} = 1$, *and* $\psi_N = \varphi^{\otimes N}$. *Let* $\psi_{N,t}^\alpha = e^{-itH_N^\alpha}\psi_N$ *be the evolution of the initial wave function* ψ_N *with respect to the Hamiltonian* (1.10) *with* $\alpha_N \leq N^{-4}$ *and let* $\gamma_{N,t}^{\alpha,(1)}$ *be the one-particle marginal density associated with* $\psi_{N,t}^\alpha$. *Let* φ_t *be the solution of the* (1.6) *with initial data* $\varphi_{t=0} = \varphi$. *Fix* T *such that*

$$\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \tag{1.14}$$

Then, there exists a constant C, depending only on λ, $\|\varphi\|_{H^1}$, T, and κ, such that

$$Tr \left| \gamma_{N,t}^{\alpha,(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq CN^{-1} \tag{1.15}$$

for all $|t| \leq T$.

Remark 1.4. The existence of such T follows from the local well-posedness of the semi-relativistic Hartree equation (1.6). See [5,22] for more detail.

As in [2,3,25,28], we first consider the case where the initial state is the coherent state in the Fock space. (See (3.18) and (3.20).) For the evolution of the coherent state, we need to control the fluctuation $\mathcal{U}_N(t;s)$, which is defined in (3.32), around the semi-relativistic Hartree dynamics. It was proved in [25, Theorem 4.1] that for the evolution of the coherent state we can achieve the optimal rate of convergence $O(N^{-1})$ towards the semi-relativistic Hartree dynamics. We then use the information on the evolution of the coherent state to estimate the fluctuations for the dynamics of the factorized state (1.5).

The main technical difficulty here is that the conversion procedure from the coherent state to the factorized state generates a factor of order $N^{3/4}$, which makes the rate of convergence to be of order $N^{-1/4}$ if no further treatment is applied. (See the term E_t^2 in (4.15) for the detail.) To compensate the loss in the rate of convergence, Rodnianski and Schlein [28] used an estimate on this term, which is equivalent to Lemma 7.1 in this paper. This improves the rate by $N^{1/4}$, which gives the $O(N^{-1/2})$ rate of convergence in [28] and [25].

In the non-relativistic case, as in [2,3], it was possible to overcome the difficulty by controlling the fluctuation $\mathcal{U}(t;s)$ first by comparing it with an approximate dynamics $\mathcal{U}_2(t;s)$, whose generator is $\mathcal{L}_2(t)$ (see (3.29)), which was introduced by Ginibre and Velo [14] as a limiting dynamics. While this technique circumvents the problem simply by not generating the term with a factor of order $N^{3/4}$, it requires to estimate the square of the interaction potential energy by kinetic energy, which does not work for the semi-relativistic case.

In this paper, we use an approximate evolution $\widetilde{\mathcal{U}}(t;s)$ as in [28] and attain an additional factor of order $N^{-1/2}$ by improving the estimate Lemma 7.1 as in Lemma 7.2. While this improved bound holds only for Fock states with odd number of particles, it turns out by counting the parity that this bound is enough to achieve the optimal rate of convergence for the factorized initial data. This estimate shows we can convert the results for the coherent states to the factorized states without any loss in terms of N dependence, and in particular, it can also be applied for other problems including the rate of convergence problem for the non-relativistic case.

Another technical difficulty in the semi-relativistic case appears in the case $4/\pi \leq \lambda < \lambda_{\mathrm{crit}}$. In this case, while the system is still subcritical, it is harder to control the interaction potential energy by the kinetic energy unlike the case $\lambda \leq 4/\pi$ where we may use Kato's inequality. See Lemmas 2.1 and 6.1 for the technical details.

The paper is organized as follows. In Sect. 2, we show that the time evolution with original Hamiltonian can be well approximated by the time evolution

with regularized Hamiltonian, provided that the cutoff approaches zero sufficiently fast. In Sect. 3, we define the Fock space and reformulate the problem using the operators defined on the Fock space. In Sect. 4, we prove Proposition 2.4, which implies the main results of the paper. A series of estimates will be proved in Sects. 5–8.

Remark 1.5. Throughout the paper, C and K will denote various constants independent of N. The L^p-space norm for $1 \leq p \leq \infty$ will be denoted by $\|\cdot\|_p$. The sequence α_N is positive and satisfies $\alpha_N \leq N^{-4}$. The norm $\|\cdot\|$ will denote the Fock space norm, which will be defined later via the scalar product (3.2), except for the case we denote by $\|J\|$ the operator norm of an operator J as in Lemma 3.2.

2. Regularization of the Interaction

Recall that the regularized Hamiltonian is defined by

$$H_N^\alpha = \sum_{j=1}^N (1 - \Delta_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j}^N \frac{1}{|x_i - x_j| + \alpha_N}. \tag{2.1}$$

As in [3], we first prove an estimate for the difference between the evolution of the initial N-particle wavefunction with respect to the original Hamiltonian H_N and with respect to the regularized Hamiltonian H_N^α.

Lemma 2.1. *Let $\psi_N = \varphi^{\otimes N}$ for some $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ and $\psi_{N,t}^\alpha = e^{-iH_N^\alpha t}\psi_N$. If $\lambda < \lambda_{\text{crit}}^H$, then there exist constants $C > 0$ and N_0 such that, for all $t \in \mathbb{R}$ and positive integer $N > N_0$,*

$$\left\| \psi_{N,t} - \psi_{N,t}^\alpha \right\|_2^2 \leq CN^2 \alpha_N |t|. \tag{2.2}$$

Proof. We first consider the derivative

$$\frac{d}{dt} \left\| \psi_{N,t} - \psi_{N,t}^\alpha \right\|_2^2 = -2 \operatorname{Re} \frac{d}{dt} \langle \psi_{N,t}, \psi_{N,t}^\alpha \rangle$$
$$= 2 \operatorname{Im} \langle \psi_{N,t}, (H_N - H_N^\alpha)\psi_{N,t}^\alpha \rangle. \tag{2.3}$$

Next, we note that

$$\left| \langle \psi_{N,t}, (H_N - H_N^\alpha)\psi_{N,t}^\alpha \rangle \right|$$

$$= \frac{\lambda}{N} \left| \left\langle \psi_{N,t}, \sum_{i<j}^N \left(\frac{1}{|x_i - x_j|} - \frac{1}{|x_i - x_j| + \alpha_N} \right) \psi_{N,t}^\alpha \right\rangle \right|$$

$$\leq \frac{\lambda \alpha_N}{N} \sum_{i<j}^N \left| \left\langle \psi_{N,t}, \left(\frac{1}{|x_i - x_j|(|x_i - x_j| + \alpha_N)} \right) \psi_{N,t}^\alpha \right\rangle \right|$$

$$\leq \frac{\lambda \alpha_N}{N} \sum_{i<j}^N \langle \psi_{N,t}, (1 - \Delta_i)^{1/2}(1 - \Delta_j)^{1/2} \psi_{N,t} \rangle^{1/2}$$

$$\times \langle \psi_{N,t}^\alpha, (1 - \Delta_i)^{1/2}(1 - \Delta_j)^{1/2} \psi_{N,t}^\alpha \rangle^{1/2}$$

$$\leq \frac{\lambda \alpha_N}{N} \sum_{i<j}^{N} \left(\left\langle \psi_{N,t}, (1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}\psi_{N,t} \right\rangle \right.$$

$$\left. \times (1-\Delta_j)^{1/2}\psi_{N,t}^\alpha \right), \tag{2.4}$$

where we used the operator inequality

$$\frac{1}{|x_i - x_j|^2} \leq C(1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}. \tag{2.5}$$

(See [6, Lemma 9.1] for the proof.)

In Lemma 6.1, we show that

$$\sum_{i<j}^{N} \langle \psi_{N,t}, (1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}\psi_{N,t} \rangle \leq CN^3 \tag{2.6}$$

and

$$\sum_{i<j}^{N} \langle \psi_{N,t}^\alpha, (1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}\psi_{N,t}^\alpha \rangle \leq CN^3. \tag{2.7}$$

Thus, from (2.3), (2.4), and Lemma 6.1, we find that

$$\frac{d}{dt} \left\| \psi_{N,t} - \psi_{N,t}^\alpha \right\|_2^2 \leq CN^2 \alpha_N. \tag{2.8}$$

The desired lemma follows after integrating over t. □

From Lemma 2.1, we obtain a bound on the difference between the marginal densities associated with the $\psi_{N,t}$ and $\psi_{N,t}^\alpha$.

Corollary 2.2. *Let* $\psi_N = \varphi^{\otimes N}$ *for some* $\varphi \in H^1(\mathbb{R}^3)$ *with* $\|\varphi\|_2 = 1$. *Let* $\psi_{N,t} = e^{-iH_N t}\psi_N$ *and* $\psi_{N,t}^\alpha = e^{-iH_N^\alpha t}\psi_N$. *For any* $k \in \mathbb{N}$, *let* $\gamma_{N,t}^{(k)}$ *and* $\gamma_{N,t}^{\alpha,(k)}$ *be the k-particle reduced densities associated with* $\psi_{N,t}$ *and* $\psi_{N,t}^\alpha$, *respectively. Suppose* $\alpha_N \leq N^{-4}$ *in* (2.1). *If* $\lambda < \lambda_{\mathrm{crit}}^H$, *then there exist constants* $C > 0$ *and* N_0 *such that, for all* $t \in \mathbb{R}$ *and positive integer* $N > N_0$,

$$Tr \left| \gamma_{N,t}^{(k)} - \gamma_{N,t}^{\alpha,(k)} \right| \leq C|t|^{1/2}N^{-1}. \tag{2.9}$$

Proof. See [3, Corollary 2.1]. □

We next estimate the difference between the solutions of the semi-relativistic Hartree equations with the Coulomb potential and with the regularized potential. The proof of the following proposition will be given in Sect. 6.

Proposition 2.3. *Let* $\varphi \in H^1(\mathbb{R}^3)$ *with* $\|\varphi\|_2 = 1$. *Let* φ_t *denote the solution of the nonlinear Hartree equation* (1.6) *with initial condition* $\varphi_{t=0} = \varphi$ *and* φ_t^α *the solution of the regularized semi-relativistic Hartree equation*

$$i\partial_t \varphi_t^\alpha = (1-\Delta)^{1/2}\varphi_t^\alpha - \lambda \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2 \right) \varphi_t^\alpha, \tag{2.10}$$

with the same initial condition $\varphi_{t=0}^\alpha = \varphi$. *Fix* T *such that*

$$\kappa = \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \tag{2.11}$$

Then, there exist constants C and K, depending only on λ, κ, T, and $\|\varphi\|_{H^1}$, such that

$$\|\varphi_t - \varphi_t^\alpha\|_{H^{1/2}} \leq C\alpha_N^{1/2} \tag{2.12}$$

for all $|t| < T$. Therefore,

$$Tr \; ||\varphi_t^\alpha\rangle\langle\varphi_t^\alpha| - |\varphi_t\rangle\langle\varphi_t|| \leq \|\varphi_t - \varphi_t^\alpha\|_2 \leq C\alpha_N^{1/2}. \tag{2.13}$$

As a consequence of Corollary 2.2 and Proposition 2.3, Theorems 1.1 and 1.3 follow from the next proposition.

Proposition 2.4. *Let $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let $\gamma_{N,t}^{\alpha,(1)}$ be the one-particle marginal density associated with $e^{-itH_N^\alpha}\varphi^{\otimes N}$ and φ_t^α the solution of the regularized semi-relativistic Hartree equation* (2.10) *with initial data $\varphi_{t=0} = \varphi$. Suppose $\alpha_N \leq N^{-4}$ in* (2.1). *Then, there exists a constant C, depending only on λ, $\|\varphi\|_{H^1}$, T, and κ, such that*

$$Tr \left|\gamma_{N,t}^{\alpha,(1)} - |\varphi_t^\alpha\rangle\langle\varphi_t^\alpha|\right| \leq CN^{-1} \tag{2.14}$$

for all $|t| \leq T$.

The proof of Proposition 2.4 will be given in Sect. 4, where we will use the Fock space representation of the problem.

3. Fock Space Representation

Let $\mathcal{F}$ be the Fock space of symmetric functions, i.e.

$$\mathcal{F} := \bigoplus_{n \geq 0} (L^2(\mathbb{R}^{3n}))_s, \tag{3.1}$$

where we let $L^2(\mathbb{R}^{3n})_s = \mathbb{C}$ when $n = 0$ and s denotes the subspace of symmetric functions with respect to the permutation of particles $x_1, x_2, \ldots, x_n$. A vector ψ in $\mathcal{F}$ is a sequence $\psi = \{\psi^{(n)}\}_{n \geq 0}$ of n-particle wavefunctions $\psi^{(n)} \in (L^2(\mathbb{R}^{3n}))_s$. The scalar product between $\psi_1, \psi_2 \in \mathcal{F}$ is defined by

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{F}} = \sum_{n \geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^{3n})} \tag{3.2}$$

and we will omit the subscript $\mathcal{F}$ from now on. We let

$$\Omega := \{1, 0, 0, \ldots\} \in \mathcal{F}, \tag{3.3}$$

which is called the vacuum. We will also make use of an operator P_n, the projection onto the n-particle sector of the Fock space, which is defined by $P_n\psi = \{0, 0, \ldots, \psi^{(n)}, 0, \ldots\}$ for a vector ψ in $\mathcal{F}$.

On $\mathcal{F}$, the creation operator a_x^* and the annihilation operator a_x for $x \in \mathbb{R}^3$ are defined by

$$(a_x^*\psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{N}} \sum_{j=1}^n \delta(x - x_j)\psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \tag{3.4}$$

$$(a_x\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1}\, \psi^{(n+1)}(x, x_1, \ldots, x_n).$$

For $f \in L^2(\mathbb{R}^3)$, $a^*(f)$ and $a(f)$ are given by

$$a^*(f) = \int \mathrm{d}x f(x) a_x^*$$
$$a(f) = \int \mathrm{d}x \overline{f(x)} a_x, \tag{3.5}$$

or equivalently,

$$(a_x^* \psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{N}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$
$$(a_x \psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int \mathrm{d}x \, \overline{f(x)} \psi^{(n+1)}(x, x_1, \ldots, x_n). \tag{3.6}$$

We also use the self-adjoint operator

$$\phi(f) = a^*(f) + a(f) \tag{3.7}$$

for $f \in L^2(\mathbb{R}^3)$. We have the following lemma that will be used to bound the creation operator and the annihilation operator:

Lemma 3.1. *For any $f \in L^2(\mathbb{R}^3)$ and $\psi \in \mathcal{D}(\mathcal{N}^{1/2})$, we have*

$$\|a(f)\psi\| \le \|f\|_2 \|\mathcal{N}^{1/2}\psi\|, \tag{3.8}$$
$$\|a^*(f)\psi\| \le \|f\|_2 \|(\mathcal{N}+1)^{1/2}\psi\|, \tag{3.9}$$
$$\|a(f)\psi\| \le 2\|f\|_2 \|(\mathcal{N}+1)^{1/2}\psi\|. \tag{3.10}$$

Proof. See [28, Lemma 2.1]. □

For an operator J acting on $L^2(\mathbb{R}^3)$, we define the second quantization of J, $d\Gamma(J)$, as the operator on $\mathcal{F}$ whose action on the n-particle sector is given by

$$(d\Gamma(J)\psi)^{(n)} = \sum_{j=1}^{n} J_j \psi^{(n)}, \tag{3.11}$$

where $J_j = 1 \otimes 1 \otimes \cdots 1 \otimes J \otimes 1 \otimes \cdots 1$ is the operator acting only on the j-th particle. If J has a kernel $J(x;y)$, then $d\Gamma(J)$ can be written as

$$d\Gamma(J) = \int \mathrm{d}x \mathrm{d}y \, J(x;y) a_x^* a_y. \tag{3.12}$$

The number operator $\mathcal{N}$ is defined by

$$\mathcal{N} := d\Gamma(1) = \int \mathrm{d}x \, a_x^* a_x \tag{3.13}$$

and it also satisfies

$$(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}. \tag{3.14}$$

We will use the following lemma to estimate $d\Gamma(J)$:

Lemma 3.2. *For any bounded one-particle operator J on $L^2(\mathbb{R}^3)$ and for every $\psi \in \mathcal{D}(\psi)$, we have*

$$\|d\Gamma(J)\psi\| \leq \|J\|\|\mathcal{N}\psi\|. \tag{3.15}$$

Here, $\|J\|$ denotes the operator norm of J.

Proof. See [3, Lemma 3.1]. □

We define the Hamiltonian $\mathcal{H}_N$ on $\mathcal{F}$ by

$$\mathcal{H}_N := \int dx\, a_x^*(1 - \Delta_x)^{1/2}a_x - \frac{\lambda}{2N}\iint dxdy \frac{1}{|x-y|}a_x^*a_y^*a_ya_x. \tag{3.16}$$

Note that for any function $\psi^{(N)} \in L^2(\mathbb{R}^{3N})_s$, $\mathcal{H}_N\psi^{(N)} = H_N\psi^{(N)}$. Similarly, we define the regularized Hamiltonian $\mathcal{H}_N^\alpha$ on $\mathcal{F}$ by

$$\mathcal{H}_N^\alpha := \int dx\, a_x^*(1 - \Delta_x)^{1/2}a_x - \frac{\lambda}{2N}\iint dxdy \frac{1}{|x-y|+\alpha_N}a_x^*a_y^*a_ya_x, \tag{3.17}$$

which also satisfies $\mathcal{H}_N^\alpha\psi^{(N)} = H_N^\alpha\psi^{(N)}$ for any function $\psi^{(N)} \in L^2(\mathbb{R}^{3N})_s$.

For $f \in L^2(\mathbb{R}^3)$, the Weyl operator $W(f)$ is defined by

$$W(f) := \exp(a^*(f) - a(f)), \tag{3.18}$$

and it satisfies

$$W(f) = e^{-\|f\|_2^2/2}\exp(a^*(f))\exp(-a(f)). \tag{3.19}$$

The coherent state with a one-particle wave function f is $W(f)\Omega$, which satisfies

$$W(f)\Omega = e^{-\|f\|_2^2/2}\exp(a^*(f))\Omega = e^{-\|f\|_2^2/2}\sum_{n\geq 0}\frac{(a^*(f))^n}{\sqrt{n!}}\Omega. \tag{3.20}$$

Let $\Gamma_{N,t}^{\alpha,(1)}(x;y)$ be the kernel of the one-particle marginal density associated with the time evolution of the coherent state $W(\sqrt{N}\varphi)\Omega$ with respect to the regularized Hamiltonian $\mathcal{H}^\alpha$. By definition,

$$\Gamma_{N,t}^{\alpha,(1)}(x;y) = \frac{1}{N}\langle e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega, a_y^*a_x e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega\rangle, \tag{3.21}$$

We expect that the limit of the kernel of one particle marginal density is $\overline{\varphi_t^\alpha}(y)\varphi_t^\alpha(x)$, thus we expand $\Gamma_{N,t}^{\alpha,(1)}(x;y)$ in terms of $(a_x - \sqrt{N}\varphi_t^\alpha(x))$ and $(a_y^* - \sqrt{N}\overline{\varphi_t^\alpha}(x))$. Then, we get

$$\begin{aligned}
\Gamma_{N,t}^{\alpha,(1)}(x;y) = {} & \varphi_t^\alpha(x)\overline{\varphi_t^\alpha}(y) \\
& + \frac{1}{N}\langle\Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_y^* - \sqrt{N}\overline{\varphi_t^\alpha}(y))(a_x - \sqrt{N}\varphi_t^\alpha(x)) \\
& \times e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega\rangle \\
& + \frac{\varphi_t^\alpha(x)}{\sqrt{N}}\langle\Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_y^* - \sqrt{N}\overline{\varphi_t^\alpha}(y))e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega\rangle \\
& + \frac{\overline{\varphi_t^\alpha}(y)}{\sqrt{N}}\langle\Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_x - \sqrt{N}\varphi_t^\alpha(x))e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega\rangle.
\end{aligned} \tag{3.22}$$

It is well known that the Weyl operator satisfies for any $f \in L^2(\mathbb{R}^3)$ that

$$W^*(f)a_x W(f) = a_x + f(x). \tag{3.23}$$

(See [28, Lemma 2.2].) This shows that

$$a_x - \sqrt{N}\varphi_t^\alpha(x) = W(\sqrt{N}\varphi_t^\alpha)a_x W^*(\sqrt{N}\varphi_t^\alpha), \tag{3.24}$$

which allows us to simplify the terms in the right hand side of (3.22), for example,

$$W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_x - \sqrt{N}\varphi_t^\alpha(x))e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)$$
$$= W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi_t^\alpha)a_x W^*(\sqrt{N}\varphi_t^\alpha)e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi). \tag{3.25}$$

To further understand the operator $W^*(\sqrt{N}\varphi_t^\alpha)e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)$, we consider the time derivative of it. As in [19,14], it turns out that

$$i\partial_t W^*(\sqrt{N}\varphi_t^\alpha)e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)$$
$$=: \left(\sum_{k=0}^4 \mathcal{L}_k(t)\right) W^*(\sqrt{N}\varphi_t^\alpha)e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi), \tag{3.26}$$

where the operators $\mathcal{L}_k(t)$ contains k creation/annihilation operators in it. More precisely, we have

$$\mathcal{L}_0(t) = \frac{N}{2}\int_0^t d\tau \int dx \left(\frac{\lambda}{|\cdot| + \alpha_N} * |\varphi_\tau^\alpha|^2\right)(x)|\varphi_\tau^\alpha(x)|^2, \tag{3.27}$$

$$\mathcal{L}_1(t) = 0, \tag{3.28}$$

$$\mathcal{L}_2(t) = \int dx\, a_x^*(1-\Delta_x)^{1/2}a_x + \lambda \int dx \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2\right)(x)a_x^* a_x$$
$$+ \lambda \iint dxdy \frac{1}{|x-y| + \alpha_N}\overline{\varphi_t^\alpha}(x)\varphi_t^\alpha(y)a_y^* a_x \tag{3.29}$$
$$+ \frac{\lambda}{2} \iint dxdy \frac{1}{|x-y| + \alpha_N}(\varphi_t^\alpha(x)\varphi_t^\alpha(y)a_x^* a_y^* + \overline{\varphi_t^\alpha}(x)\overline{\varphi_t^\alpha}(y)a_x a_y),$$

$$\mathcal{L}_3(t) = \frac{\lambda}{\sqrt{N}} \iint dxdy \frac{1}{|x-y| + \alpha_N}a_x^* \left(\varphi_t^\alpha(y)a_y^* + \overline{\varphi_t^\alpha}(y)a_y\right)a_x, \tag{3.30}$$

$$\mathcal{L}_4 = \frac{\lambda}{2N} \iint dxdy \frac{1}{|x-y| + \alpha_N}a_x^* a_y^* a_x a_y. \tag{3.31}$$

Note that $\mathcal{L}_0(t)$ is not an operator but a complex-valued function on t, which we call the phase factor. Although this term contains the factor N, we may ignore this term by using a function $e^{-i\mathcal{L}_0(t)}$ whose derivative can offset the term $\mathcal{L}_0(t)$ in the right hand side of (3.26).

Generalizing the idea explained above, we define the unitary evolution

$$\mathcal{U}(t;s) := e^{-i\omega(t;s)}W^*(\sqrt{N}\varphi_t^\alpha)e^{-i(t-s)\mathcal{H}_N^\alpha}W(\sqrt{N}\varphi_s^\alpha) \tag{3.32}$$

with the phase factor

$$\omega(t;s) := \frac{N}{2} \int\limits_{s}^{t} d\tau \int dx \big(\frac{\lambda}{|\cdot| + \alpha_N} * |\varphi_\tau^\alpha|^2\big)(x)|\varphi_\tau^\alpha(x)|^2. \qquad (3.33)$$

We can find from the above construction that $\mathcal{U}(t;s)$ is a unitary operator satisfying

$$i\partial_t \mathcal{U}(t;s) = (\mathcal{L}_2(t) + \mathcal{L}_3(t) + \mathcal{L}_4)\mathcal{U}(t;s) \quad \text{and} \quad \mathcal{U}(s;s) = I. \qquad (3.34)$$

Furthermore, since $e^{-i\omega(t;s)}$ commutes with the operators a_x and a_x^*, we find from (3.25) that

$$W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_x - \sqrt{N}\varphi_t^\alpha(x))e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi) = \mathcal{U}^*(t;0)a_x\mathcal{U}(t;0). \qquad (3.35)$$

Let

$$\mathcal{K} := \int dx\, a_x^*(1 - \Delta_x)^{1/2}a_x. \qquad (3.36)$$

We consider a modified evolution $\widetilde{\mathcal{U}}(t;s)$, which is a unitary operator satisfying

$$i\partial_t \widetilde{\mathcal{U}}(t;s) = (\mathcal{L}_2(t) + \mathcal{L}_4)\widetilde{\mathcal{U}}(t;s) \quad \text{and} \quad \widetilde{\mathcal{U}}(s;s) = I \qquad (3.37)$$

We remark that $\widetilde{\mathcal{U}}(t;s)$ is bounded in $\mathcal{Q}(\mathcal{K}+\mathcal{N}^2)$, the form domain of the operator $(\mathcal{K}+\mathcal{N}^2)$, and is strongly differentiable from $\mathcal{Q}(\mathcal{K}+\mathcal{N}^2)$ to $\mathcal{Q}^*(\mathcal{K}+\mathcal{N}^2)$. See Sect. 8 for more detail.

For simplicity, we will use notations

$$\mathcal{U}(t) := \mathcal{U}(t;0), \quad \widetilde{\mathcal{U}}(t) := \widetilde{\mathcal{U}}(t;0). \qquad (3.38)$$

When written through the kernel form, the one-particle marginal density associated with the time evolution of the factorized states with respect to the regularized Hamiltonian, $\gamma_{N,t}^{\alpha,(1)}(x;y)$, satisfies

$$\gamma_{N,t}^{\alpha,(1)}(x;y) = \frac{1}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, e^{it\mathcal{H}_N^\alpha}a_y^*a_x e^{-it\mathcal{H}_N^\alpha}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\rangle. \qquad (3.39)$$

4. Proof of Main Results

In this section, we prove Proposition 2.4, which implies Theorems 1.1 and 1.3. We will use the following lemmas, which will be proved in Sect. 5.

Lemma 4.1. *For a Hermitian Operator J on $L^2(\mathbb{R}^3)$, let*

$$E_t^1(J) := \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)d\Gamma(J)\mathcal{U}(t)\Omega \right\rangle. \qquad (4.1)$$

Then, there exist constants C and K, depending only on λ and $\sup_{|s|\leq t}\|\varphi_s\|_{H^1}$, such that

$$|E_t^1(J)| \leq \frac{C\|J\|e^{Kt}}{N}. \qquad (4.2)$$

Lemma 4.2. *For a Hermitian Operator J on $L^2(\mathbb{R}^3)$, let*

$$E_t^2(J) := \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)\phi(J\varphi_t)\mathcal{U}(t)\Omega \right\rangle. \qquad (4.3)$$

Then, there exist constants C and K, depending only on λ and $\sup_{|s|\le t}\|\varphi_s\|_{H^1}$, such that

$$|E_t^2(J)| \le \frac{C\|J\|e^{Kt}}{N}. \qquad (4.4)$$

We are now ready to prove Proposition 2.4.

Proof of Proposition 2.4. We have seen in Sect. 3 that

$$W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N^\alpha t}(a_x - \sqrt{N}\varphi_t^\alpha(x))e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi) = \mathcal{U}^*(t)a_x\mathcal{U}(t). \quad (4.5)$$

By definition, we have that

$$P_N W(\sqrt{N}\varphi)\Omega = e^{-N/2}\frac{\left(a^*(\sqrt{N}\varphi)\right)^N}{N!}\Omega = \frac{1}{d_N}\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega, \qquad (4.6)$$

where P_N is the projection onto the N-particle sector of the Fock space. Here, d_N denotes the constant

$$d_N := \frac{\sqrt{N!}}{N^{N/2}e^{-N/2}} \simeq N^{1/4}. \qquad (4.7)$$

For the factorized initial data, we first rewrite (3.39) using the coherent state and the projection operator P_N. From (4.6) we get

$$\begin{aligned}\gamma_{N,t}^{\alpha,(1)}(x;y) &= \frac{1}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, e^{i\mathcal{H}_N^\alpha t}a_y^*a_x e^{-i\mathcal{H}_N^\alpha t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\rangle \\ &= \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, e^{i\mathcal{H}_N^\alpha t}a_y^*e^{-i\mathcal{H}_N^\alpha t}e^{i\mathcal{H}_N^\alpha t}a_x e^{-i\mathcal{H}_N^\alpha t}P_N W(\sqrt{N}\varphi)\Omega \right\rangle \end{aligned}$$
$$(4.8)$$

By counting the number of particles, we find that

$$e^{i\mathcal{H}_N^\alpha t}a_y^*e^{-i\mathcal{H}_N^\alpha t}e^{i\mathcal{H}_N^\alpha t}a_x e^{-i\mathcal{H}_N^\alpha t}P_N = P_N e^{i\mathcal{H}_N^\alpha t}a_y^*e^{-i\mathcal{H}_N^\alpha t}e^{i\mathcal{H}_N^\alpha t}a_x e^{-i\mathcal{H}_N^\alpha t}, \quad (4.9)$$

where we used the fact that the evolution operator $e^{-i\mathcal{H}_N^\alpha t}$ conserves the number of particles in a Fock state. Moreover, it is obvious that

$$\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, P_N\psi \right\rangle = \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, \psi \right\rangle \qquad (4.10)$$

for any $\psi \in \mathcal{F}$. Thus, we obtain from (4.8) that

$$\gamma_{N,t}^{\alpha,(1)}(x;y) = \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, e^{i\mathcal{H}_N^\alpha t}a_y^*e^{-i\mathcal{H}_N^\alpha t}e^{i\mathcal{H}_N^\alpha t}a_x e^{-i\mathcal{H}_N^\alpha t}W(\sqrt{N}\varphi)\Omega \right\rangle$$
$$(4.11)$$

To simplify it further, we apply (4.5) to find that

$$e^{i\mathcal{H}_N^\alpha t}a_x e^{-i\mathcal{H}_N^\alpha t} = W(\sqrt{N}\varphi)\mathcal{U}^*(t)(a_x + \sqrt{N}\varphi_t^\alpha(x))\mathcal{U}(t)W^*(\sqrt{N}\varphi) \quad (4.12)$$

and an analogous result for the creation operator. Hence, we get

$$
\begin{aligned}
\gamma_{N,t}^{\alpha,(1)}&(x;y)\\
&= \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)\big(a_y^* + \sqrt{N}\overline{\varphi_t^\alpha}(y)\big)\mathcal{U}(t)W^*(\sqrt{N}\varphi)\right.\\
&\qquad \times \left. W(\sqrt{N}\varphi)\mathcal{U}^*(t)\big(a_x + \sqrt{N}\varphi_t^\alpha(x)\big)\mathcal{U}(t)W^*(\sqrt{N}\varphi)W(\sqrt{N}\varphi)\Omega\right\rangle\\
&= \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)\big(a_y^* + \sqrt{N}\overline{\varphi_t^\alpha}(y)\big)\right.\\
&\qquad \times \left. \big(a_x + \sqrt{N}\varphi_t^\alpha(x)\big)\mathcal{U}(t)\Omega\right\rangle.
\end{aligned}
\tag{4.13}
$$

Expanding the term $\big(a_y^* + \sqrt{N}\overline{\varphi_t^\alpha}(y)\big)\big(a_x + \sqrt{N}\varphi_t^\alpha(x)\big)$, we obtain the following equation for the one-particle marginal.

$$
\begin{aligned}
\gamma_{N,t}^{\alpha,(1)}&(x;y) - \overline{\varphi_t^\alpha}(y)\varphi_t^\alpha(x)\\
&= \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)a_y^* a_x\mathcal{U}(t)\Omega\right\rangle\\
&\quad + \overline{\varphi_t^\alpha}(y)\frac{d_N}{\sqrt{N}}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)a_x\mathcal{U}(t)\Omega\right\rangle\\
&\quad + \varphi_t^\alpha(x)\frac{d_N}{\sqrt{N}}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)a_y^*\mathcal{U}(t)\Omega\right\rangle.
\end{aligned}
\tag{4.14}
$$

For any compact one-particle Hermitian operator J on $L^2(\mathbb{R}^3)$, we find

$$
\begin{aligned}
\mathrm{Tr}\, J&\left(\gamma_{N,t}^{\alpha,(1)} - |\varphi_t^\alpha\rangle\langle\varphi_t^\alpha|\right)\\
&= \frac{d_N}{N}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)d\Gamma(J)\mathcal{U}(t)\Omega\right\rangle\\
&\quad + \frac{d_N}{\sqrt{N}}\left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{U}^*(t)\phi(J\varphi_t^\alpha)\mathcal{U}(t)\Omega\right\rangle\\
&= E_t^1(J) + E_t^2(J).
\end{aligned}
\tag{4.15}
$$

Lemma 4.1 and Lemma 4.2 show that

$$
\left|\mathrm{Tr}\, J\left(\gamma_{N,t}^{\alpha,(1)} - |\varphi_t^\alpha\rangle\langle\varphi_t^\alpha|\right)\right| \le |E_t^1(J)| + |E_t^2(J)| \le \frac{C}{N}\|J\|e^{Kt}
\tag{4.16}
$$

for all compact Hermitian operators J on $L^2(\mathbb{R}^3)$. Since the space of compact operators is the dual to the space of trace class operators, and since $\gamma_{N,t}^{\alpha,(1)}$ and $|\varphi_t^\alpha\rangle\langle\varphi_t^\alpha|$ are Hermitian, we obtain that

$$
\mathrm{Tr}\,\left|\gamma_{N,t}^{\alpha,(1)} - |\varphi_t^\alpha\rangle\langle\varphi_t^\alpha|\right| \le \frac{Ce^{Kt}}{N},
\tag{4.17}
$$

which was to be proved. $\square$

5. Comparison of Dynamics

In this section, we prove important lemmas that were used in the proof of Theorem 1.1 by estimating the difference between $\mathcal{U}(t;s)$ and $\widetilde{\mathcal{U}}(t;s)$.

Lemma 5.1. *For any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist constants C and K, depending on λ, j, and $\sup_{\tau \leq |t|,|s|} \|\varphi_\tau\|_{H^{1/2}}$, such that*

$$\langle \widetilde{\mathcal{U}}(t;s)\psi, \mathcal{N}^j \widetilde{\mathcal{U}}(t;s)\psi \rangle \leq C e^{K|t-s|} \langle \psi, (\mathcal{N}+1)^j \psi \rangle. \tag{5.1}$$

Proof. Let $\widetilde{\psi} = \widetilde{\mathcal{U}}(t;s)\psi$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \widetilde{\psi}, (\mathcal{N}+1)^j \widetilde{\psi} \rangle = \langle \widetilde{\psi}, [i\mathcal{L}_2, (\mathcal{N}+1)^j] \widetilde{\psi} \rangle$$

$$= \mathrm{Im} \iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|} \varphi_t(x)\varphi_t(y) \langle \widetilde{\psi}, [a_x^* a_y^*, (\mathcal{N}+1)^j] \widetilde{\psi} \rangle$$

$$= \mathrm{Im} \iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|} \varphi_t(x)\varphi_t(y) \langle \widetilde{\psi}, a_x^* a_y^* ((\mathcal{N}+1)^j - (\mathcal{N}+3)^j) \widetilde{\psi} \rangle$$

$$= \mathrm{Im} \iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|} \varphi_t(x)\varphi_t(y)$$

$$\times \langle (\mathcal{N}+3)^{\frac{j}{2}-1} a_x a_y \widetilde{\psi}, (\mathcal{N}+3)^{1-\frac{j}{2}} ((\mathcal{N}+1)^j - (\mathcal{N}+3)^j) \widetilde{\psi} \rangle. \tag{5.2}$$

Thus, from Schwarz inequality, we obtain that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \widetilde{\psi}, (\mathcal{N}+1)^j \widetilde{\psi} \rangle \right|$$

$$\leq \iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|} |\varphi_t(x)||\varphi_t(y)| \|(\mathcal{N}+3)^{\frac{j}{2}-1} a_x a_y \widetilde{\psi}\|$$

$$\times \|(\mathcal{N}+3)^{1-\frac{j}{2}} ((\mathcal{N}+1)^j - (\mathcal{N}+3)^j) \widetilde{\psi}\|$$

$$\leq \lambda \|(\mathcal{N}+3)^{1-\frac{j}{2}} ((\mathcal{N}+1)^j - (\mathcal{N}+3)^j) \widetilde{\psi}\| \left(\iint \mathrm{d}x\mathrm{d}y \frac{|\varphi_t(x)|^2 |\varphi_t(y)|^2}{|x-y|^2} \right)^{1/2}$$

$$\times \left(\iint \mathrm{d}x\mathrm{d}y \|(\mathcal{N}+3)^{\frac{j}{2}-1} a_x a_y \widetilde{\psi}\|^2 \right)^{1/2} \tag{5.3}$$

Easy algebra shows that $(\mathcal{N}+3)^{1-(j/2)} |(\mathcal{N}+1)^j - (\mathcal{N}+3)^j| \leq C(\mathcal{N}+1)^{j/2}$. From Hardy-Littlewood-Sobolev inequality we have that

$$\iint \mathrm{d}x\mathrm{d}y \frac{|\varphi_t(x)|^2 |\varphi_t(y)|^2}{|x-y|^2} \leq C\|\varphi_t\|_3^4 \leq \|\varphi_t\|_{H^{1/2}}^4. \tag{5.4}$$

We also have that

$$\iint \mathrm{d}x\mathrm{d}y \|(\mathcal{N}+3)^{(j/2)-1} a_x a_y \psi^\alpha\|^2 = \iint \mathrm{d}x\mathrm{d}y \|a_x a_y (\mathcal{N}+1)^{(j/2)-1} \psi^\alpha\|^2$$

$$\leq \|(\mathcal{N}+1)^{j/2} \psi^\alpha\|^2. \tag{5.5}$$

Altogether, we have shown that

$$\left|\frac{d}{dt}\langle\widetilde{\mathcal{U}}(t;s)\psi,(\mathcal{N}+1)^j\widetilde{\mathcal{U}}(t;s)\psi\rangle\right| \le C\|(\mathcal{N}+1)^{j/2}\widetilde{\mathcal{U}}(t;s)\psi\|^2$$
$$= C\langle\widetilde{\mathcal{U}}(t;s)\psi,(\mathcal{N}+1)^j\widetilde{\mathcal{U}}(t;s)\psi\rangle. \quad (5.6)$$

Since $\widetilde{\mathcal{U}}(s;s) = I$, we also have

$$\langle\widetilde{\mathcal{U}}(s;s)\psi,(\mathcal{N}+1)^j\widetilde{\mathcal{U}}(s;s)\psi\rangle = \langle\psi,(\mathcal{N}+1)^j\psi\rangle. \quad (5.7)$$

Using (5.6) and (5.7) the conclusion follows directly from the Gronwall's lemma. □

Lemma 5.2. *For any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist constants C and K, depending on λ, j, and $\sup_{|\tau|\le|t|,|s|}\|\varphi_\tau\|_{H^1}$ such that*

$$\langle\mathcal{U}(t;s)\psi,\mathcal{N}^j\mathcal{U}(t;s)\psi\rangle \le Ce^{K|t-s|}\langle\psi,(\mathcal{N}+1)^{2j+2}\psi\rangle. \quad (5.8)$$

Proof. See [28, Proposition 3.3]. □

Lemma 5.3. *For any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist a constant C, depending on λ, j, and $\|\varphi_t\|_{H^1}$ such that*

$$\|(\mathcal{N}+1)^{j/2}\mathcal{L}_3(t)\psi\| \le \frac{C}{\sqrt{N}}\|(\mathcal{N}+1)^{(j+3)/2}\psi\|. \quad (5.9)$$

Proof. While this lemma can be proved as in [3, Lemma 6.3], we give a shorter proof here. Let

$$A_3(t) = \iint dxdy\,\frac{1}{|x-y|+\alpha_N}\overline{\varphi}_t(y)a_x^*a_ya_x. \quad (5.10)$$

Then,

$$(\mathcal{N}+1)^{j/2}\mathcal{L}_3(t) = \frac{\lambda}{\sqrt{N}}\left((\mathcal{N}+1)^{j/2}A_3(t)+(\mathcal{N}+1)^{j/2}A_3^*(t)\right). \quad (5.11)$$

We estimate $(\mathcal{N}+1)^j A_3(t)$ and $(\mathcal{N}+1)^j A_3^*(t)$ separately. The first term $(\mathcal{N}+1)^j A_3(t)$ satisfies for any $\xi \in \mathcal{F}$ that

$$|\langle\xi,(\mathcal{N}+1)^{j/2}A_3(t)\psi\rangle| = \left|\iint dxdy\,\frac{\overline{\varphi}_t(y)}{|x-y|+\alpha_N}\langle\xi,(\mathcal{N}+1)^{j/2}a_x^*a_ya_x\psi\rangle\right|$$
$$= \left|\iint dxdy\,\frac{\overline{\varphi}_t(y)}{|x-y|+\alpha_N}\langle(\mathcal{N}+1)^{-1/2}\xi,(\mathcal{N}+1)^{(j+1)/2}a_x^*a_ya_x\psi\rangle\right|$$
$$\le \left(\iint dxdy\,\frac{|\varphi_t(y)|^2}{|x-y|^2}\|a_x(\mathcal{N}+1)^{-1/2}\xi\|^2\right)^{1/2}$$
$$\times \left(\iint dxdy\|a_ya_x\mathcal{N}^{(j+1)/2}\psi\|^2\right)^{1/2}$$
$$\le C\|\varphi_t\|_{H^1}\|\xi\|\|\mathcal{N}^{(j+3)/2}\psi\|, \quad (5.12)$$

where we used Hardy inequality in the last inequality. Since ξ was arbitrary, we obtain that

$$\|(\mathcal{N}+1)^{j/2}A_3(t)\psi\| \le C\|\varphi_t\|_{H^1}\|\mathcal{N}^{(j+3)/2}\psi\|. \quad (5.13)$$

Similarly, we can find that

$$\|(\mathcal{N}+1)^{j/2}A_3^*(t)\psi\| \le C\|\varphi_t\|_{H^1}\|(\mathcal{N}+2)^{(j+3)/2}\psi\|. \tag{5.14}$$

Hence, from (5.11), (5.13), and (5.14) we get

$$\|(\mathcal{N}+1)^{j/2}\mathcal{L}_3(t)\psi\| \le \frac{C}{\sqrt{N}}\|(\mathcal{N}+1)^{(j+3)/2}\psi\|, \tag{5.15}$$

which was to be proved. $\qquad\square$

Lemma 5.4. *For all $j \in \mathbb{N}$, there exist constants C and K depending only on λ, j, and $\sup_{|s|\le t}\|\varphi_s\|_{H^1}$ such that, for any $f \in L^2(\mathbb{R}^3)$,*

$$\left\|(\mathcal{N}+1)^{j/2}\left(\mathcal{U}^*(t)\phi(f)\mathcal{U}(t) - \widetilde{\mathcal{U}}^*(t)\phi(f)\widetilde{\mathcal{U}}(t)\right)\Omega\right\| \le \frac{C\|f\|_2 e^{Kt}}{N}. \tag{5.16}$$

Proof. Let

$$\mathcal{R}_1(f) := \left(\mathcal{U}^*(t) - \widetilde{\mathcal{U}}^*(t)\right)\phi(f)\widetilde{\mathcal{U}}(t) \tag{5.17}$$

and

$$\mathcal{R}_2(f) := \mathcal{U}^*(t)\phi(f)\left(\mathcal{U}(t) - \widetilde{\mathcal{U}}(t)\right) \tag{5.18}$$

so that

$$\mathcal{U}^*(t)\phi(f)\mathcal{U}(t) - \widetilde{\mathcal{U}}^*(t)\phi(f)\widetilde{\mathcal{U}}(t) = \mathcal{R}_1(f) + \mathcal{R}_2(f). \tag{5.19}$$

Then, from Lemmas 5.1, 5.2, and 5.3, we find that

$$\left\|(\mathcal{N}+1)^{j/2}\mathcal{R}_1(f)\Omega\right\|$$

$$= \left\|\int_0^t ds\,(\mathcal{N}+1)^{j/2}\mathcal{U}^*(s;0)\mathcal{L}_3(s)\widetilde{\mathcal{U}}^*(t;s)\phi(f)\widetilde{\mathcal{U}}(t)\Omega\right\|$$

$$\le \int_0^t ds\left\|(\mathcal{N}+1)^{j/2}\mathcal{U}^*(s;0)\mathcal{L}_3(s)\widetilde{\mathcal{U}}^*(t;s)\phi(f)\widetilde{\mathcal{U}}(t)\Omega\right\|$$

$$\le Ce^{Kt}\int_0^t ds\left\|(\mathcal{N}+1)^{j+1}\mathcal{L}_3(s)\widetilde{\mathcal{U}}^*(t;s)\phi(f)\widetilde{\mathcal{U}}(t)\Omega\right\|$$

$$\le \frac{Ce^{Kt}}{\sqrt{N}}\int_0^t ds\left\|(\mathcal{N}+1)^{j+(5/2)}\phi(f)\widetilde{\mathcal{U}}(t)\Omega\right\|$$

$$\le \frac{Ce^{Kt}}{\sqrt{N}}\left\|(\mathcal{N}+1)^{j+(5/2)}\phi(f)\widetilde{\mathcal{U}}(t)\Omega\right\| \tag{5.20}$$

Thus, we can get the following bound for $\mathcal{R}_1(f)$.

$$\|(\mathcal{N}+1)^{j/2}\mathcal{R}_1(f)\Omega\|$$

$$\le \frac{Ce^{Kt}}{\sqrt{N}}\left(\|a(f)(\mathcal{N}+1)^{j+(5/2)}\widetilde{\mathcal{U}}(t)\Omega\| + \|a^*(f)(\mathcal{N}+1)^{j+(5/2)}\widetilde{\mathcal{U}}(t)\Omega\|\right)$$

$$\leq \frac{C\|f\|_2 e^{Kt}}{\sqrt{N}}\|(\mathcal{N}+1)^{j+3}\widetilde{\mathcal{U}}(t)\Omega\| \leq \frac{C\|f\|_2 e^{Kt}}{\sqrt{N}}\|(\mathcal{N}+1)^{j+(5/2)}\Omega\|$$

$$\leq \frac{C\|f\|_2 e^{Kt}}{\sqrt{N}}. \tag{5.21}$$

The study of $\mathcal{R}_2(f)$ is similar and gives

$$\|(\mathcal{N}+1)^{j/2}\mathcal{R}_2(f)\Omega\| \leq \frac{C\|f\|_2 e^{Kt}}{\sqrt{N}}. \tag{5.22}$$

Therefore,

$$\left\|(\mathcal{N}+1)^{j/2}\left(\mathcal{U}^*(t)\phi(f)\mathcal{U}(t) - \widetilde{\mathcal{U}}^*(t)\phi(f)\widetilde{\mathcal{U}}(t)\right)\Omega\right\|$$

$$\leq \left\|(\mathcal{N}+1)^{j/2}\mathcal{R}_1(f)\Omega\right\| + \left\|(\mathcal{N}+1)^{j/2}\mathcal{R}_2(f)\Omega\right\|$$

$$\leq \frac{C\|f\|_2 e^{Kt}}{\sqrt{N}}, \tag{5.23}$$

which was to be proved. $\qquad\square$

In Sect. 7, we will prove the following estimates:

There exists a constant $C > 0$ such that, for any $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$, we have

$$\left\|(\mathcal{N}+1)^{-1/2}W^*(\sqrt{N}\varphi)\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega\right\| \leq \frac{C}{d_N}. \tag{5.24}$$

Moreover, for all non-negative integers $k \leq (1/2)N^{1/3}$,

$$\left\|P_{2k+1}W^*(\sqrt{N}\varphi)\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega\right\| \leq \frac{2(k+1)^{3/2}}{d_N\sqrt{N}}. \tag{5.25}$$

We are now ready to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. We first observe that

$$|E_t^1(J)| = \left|\frac{d_N}{N}\left\langle W^*(\sqrt{N}\varphi)\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega, \mathcal{U}^*(t)d\Gamma(J)\mathcal{U}(t)\Omega\right\rangle\right|$$

$$\leq \frac{d_N}{N}\left\|(\mathcal{N}+1)^{-1/2}W^*(\sqrt{N}\varphi)\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega\right\|\left\|(\mathcal{N}+1)^{1/2}\mathcal{U}^*(t)d\Gamma(J)\mathcal{U}(t)\Omega\right\|. \tag{5.26}$$

From the estimate (5.24), we have

$$\left\|(\mathcal{N}+1)^{-1/2}W^*(\sqrt{N}\varphi)\frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}}\Omega\right\| \leq \frac{C}{d_N}. \tag{5.27}$$

Lemma 5.2 shows that

$$\|(\mathcal{N}+1)^{1/2}\mathcal{U}^*(t)d\Gamma(J)\mathcal{U}(t)\Omega\| \leq Ce^{Kt}\|(\mathcal{N}+1)^2 d\Gamma(J)\mathcal{U}(t)\Omega\|$$

$$\leq C\|J\|e^{Kt}\|(\mathcal{N}+1)^3\mathcal{U}(t)\Omega\| \leq C\|J\|e^{Kt}\|(\mathcal{N}+1)^7\Omega\| = C\|J\|e^{Kt} \tag{5.28}$$

Thus, we obtain

$$|E_t^1(J)| \leq \frac{C\|J\|e^{Kt}}{N}, \tag{5.29}$$

which proves the desired lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 4.2. Let

$$\mathcal{R}(J\varphi_t) := \mathcal{U}^*(t)\phi(J\varphi_t)\mathcal{U}(t) - \widetilde{\mathcal{U}}^*(t)\phi(J\varphi_t)\widetilde{\mathcal{U}}(t). \tag{5.30}$$

From the parity, we have that $P_{2k}\widetilde{\mathcal{U}}^*(t)\phi(J\varphi_t)\widetilde{\mathcal{U}}(t)\Omega = 0$ for any $k = 0, 1, \ldots$. (See Lemma 8.2.) Thus, we have

$$
\begin{aligned}
|E_t^2(J)| &= \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\widetilde{\mathcal{U}}^*(t)\phi(J\varphi_t)\widetilde{\mathcal{U}}(t)\Omega \right\rangle \\
&\quad + \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, W(\sqrt{N}\varphi)\mathcal{R}(J\varphi_t)\Omega \right\rangle \\
&\leq \frac{d_N}{\sqrt{N}} \left\| \sum_{k=1}^{\infty}(\mathcal{N}+1)^{-5/2}P_{2k-1}W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\| \\
&\quad \times \left\| (\mathcal{N}+1)^{5/2}\widetilde{\mathcal{U}}^*(t)\phi(J\varphi_t)\widetilde{\mathcal{U}}(t)\Omega \right\| \\
&\quad + \frac{d_N}{\sqrt{N}} \left\| (\mathcal{N}+1)^{-1/2}W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\| \\
&\quad \times \left\| (\mathcal{N}+1)^{1/2}\mathcal{R}(J\varphi_t)\Omega \right\|
\end{aligned} \tag{5.31}
$$

Let $M := (1/2)N^{1/3}$. We have from the estimate (5.25) that

$$
\begin{aligned}
&\left\| \sum_{k=1}^{\infty}(\mathcal{N}+1)^{-5/2}P_{2k-1}W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\|^2 \\
&\leq \sum_{k=1}^{M} \left\| (\mathcal{N}+1)^{-5/2}P_{2k-1}W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\|^2 \\
&\quad + \frac{1}{M^5}\sum_{k=M}^{\infty} \left\| P_{2k-1}W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\|^2 \\
&\leq \left(\sum_{k=1}^{M}\frac{C}{k^2 d_N^2 N} \right) + \frac{C}{N^{5/3}} \left\| W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \right\|^2 \leq \frac{C}{d_N^2 N}. \tag{5.32}
\end{aligned}
$$

Lemma 5.1 shows that

$$
\begin{aligned}
\|(\mathcal{N}+1)^{5/2}\widetilde{\mathcal{U}}^*(t)\phi(J\varphi_t)\widetilde{\mathcal{U}}(t)\Omega\| &\leq Ce^{Kt}\|(\mathcal{N}+1)^{5/2}\phi(J\varphi_t)\widetilde{\mathcal{U}}(t)\Omega\| \\
&\leq C\|J\varphi_t\|_2 e^{Kt}\|(\mathcal{N}+1)^3\widetilde{\mathcal{U}}(t)\Omega\|^2 \leq C\|J\|e^{Kt}\|(\mathcal{N}+1)^3\Omega\|^2 = C\|J\|e^{Kt}. \tag{5.33}
\end{aligned}
$$

We find from (5.24) that

$$\left\| (\mathcal{N}+1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C}{d_N}. \tag{5.34}$$

Finally, Lemma 5.4 shows that

$$\|(\mathcal{N}+1)^{1/2} \mathcal{R}(J\varphi_t)\Omega\| \leq \frac{C\|J\varphi_t\|_2 e^{Kt}}{N} \leq \frac{C\|J\| e^{Kt}}{N}. \tag{5.35}$$

Therefore,

$$|E_t^2(J)| \leq \frac{C\|J\| e^{Kt}}{N}, \tag{5.36}$$

which was to be proved.   □

6. Properties of Regularized Dynamics

In this section, we prove various lemmas, which allows us to use the regularized dynamics instead of the full dynamics.

Lemma 6.1. *Let* $\psi_N = \varphi^{\otimes N}$ *for some* $\varphi \in H^1(\mathbb{R}^3)$ *with* $\|\varphi\| = 1$. *Let* $\psi_{N,t} = e^{-iH_N t}\psi_N$ *and* $\psi_{N,t}^\alpha = e^{-i\mathcal{H}_N^\alpha t}\psi_N$. *If* $\lambda < \lambda_{\mathrm{crit}}^H$, *then there exists a constant* $C > 0$ *and* N_0 *such that, for all* $t \in \mathbb{R}$ *and for any positive integer* $N > N_0$,

$$\sum_{\substack{i<j}}^{N} \langle \psi_{N,t}, (1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}\psi_{N,t} \rangle \leq CN^3 \tag{6.1}$$

and

$$\sum_{\substack{i<j}}^{N} \langle \psi_{N,t}^\alpha, (1-\Delta_i)^{1/2}(1-\Delta_j)^{1/2}\psi_{N,t}^\alpha \rangle \leq CN^3. \tag{6.2}$$

Proof. Let

$$S_j = (1-\Delta_j)^{1/2}, \qquad V_{ij} = \frac{\lambda}{|x_i - x_j|}, \tag{6.3}$$

so that

$$H_N = \sum_{j=1}^{N} S_j - \frac{1}{N} \sum_{\substack{i<j}}^{N} V_{ij}. \tag{6.4}$$

We first consider the operator

$$H_{N-1} = \sum_{j=1}^{N-1} S_j - \frac{1}{N-1} \sum_{\substack{i<j}}^{N-1} V_{ij}. \tag{6.5}$$

Let $\eta = (\lambda_{\mathrm{crit}}^H/\lambda)^{1/2}$ so that $\eta > 1$ and $\lambda\eta < \lambda_{\mathrm{crit}}^H$. Then,

$$H_{N-1} = \eta^{-1}\left((\eta-1)\sum_{j=1}^{N-1} S_j + \sum_{j=1}^{N-1} S_j - \frac{\lambda\eta}{N-1} \sum_{\substack{i<j}}^{N-1} V_{ij} \right). \tag{6.6}$$

When N is sufficiently large, we have the following operator inequality

$$\sum_{j=1}^{N-1} S_j - \frac{\lambda\eta}{N-1} \sum_{i<j}^{N-1} V_{ij} \geq -M(N-1) \tag{6.7}$$

for some $M \geq 0$. (See [24, Theorem 1].) Thus,

$$H_{N-1} \geq -\eta^{-1} MN + (1-\eta^{-1}) \sum_{j=1}^{N-1} S_j. \tag{6.8}$$

Let

$$H_N^{(N-1)} = \sum_{j=1}^{N-1} S_j - \frac{1}{N} \sum_{i<j}^{N-1} V_{ij}. \tag{6.9}$$

We consider the operator

$$\begin{aligned}
H_N^2 &= \left(H_N^{(N-1)} + S_N - \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2 \\
&= \left(H_N^{(N-1)} - \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2 + S_N^2 + 2H_N^{(N-1)} S_N \\
&\quad - S_N \left(\frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) - \left(\frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) S_N,
\end{aligned} \tag{6.10}$$

where we used that $[H_N^{(N-1)}, S_N] = 0$. Now, we find that

$$H_N^2 \geq S_N^2 + 2H_N^{(N-1)} S_N - S_N \left(\frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) - \left(\frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right) S_N. \tag{6.11}$$

Since

$$H_N^{(N-1)} \geq H_{N-1} \geq -\eta^{-1} MN + (1-\eta^{-1}) \sum_{j=1}^{N-1} S_j, \tag{6.12}$$

we have that

$$\begin{aligned}
H_N^{(N-1)} S_N = S_N^{1/2} H_N^{(N-1)} S_N^{1/2} &\geq S_N^{1/2} \left(-\eta^{-1} MN + (1-\eta^{-1}) \sum_{j=1}^{N-1} S_j \right) S_N^{1/2} \\
&\geq -\eta^{-1} MN S_N + (1-\eta^{-1}) \sum_{j=1}^{N-1} S_j S_N.
\end{aligned} \tag{6.13}$$

Let C_0 be a constant satisfying the operator inequality

$$C_0 S_j S_N \geq V_{jN}^2, \tag{6.14}$$

and choose N_1 large so that $(1 - \eta^{-1}) \geq C_0 N_1^{-1}$. Then, for all $N > N_1$,

$$2H_N^{(N-1)} S_N + 2MNS_N$$

$$\geq 2(1 - \eta^{-1}) \sum_j = 1^{N-1} S_j S_N \geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N + \frac{1}{N} \sum_{j=1}^{N-1} V_{jN}^2$$

$$\geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N + \left(\frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2, \tag{6.15}$$

where the last inequality comes from the Schwarz inequality. Hence, we obtain from (6.11) and (6.15) that

$$H_N^2 + 2MNS_N \geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N + \left(S_N - \frac{1}{N} \sum_{j=1}^{N-1} V_{jN} \right)^2$$

$$\geq (1 - \eta^{-1}) \sum_{j=1}^{N-1} S_j S_N. \tag{6.16}$$

Similarly, for any $1 \leq j \leq N$,

$$H_N^2 + 2MNS_j \geq (1 - \eta^{-1}) \sum_{i:i \neq j}^{N} S_i S_j. \tag{6.17}$$

Thus, summing (6.17) over j, we get

$$NH_N^2 + 2MN \sum_{j=1}^{N} S_j \geq (1 - \eta^{-1}) \sum_{i \neq j}^{N} S_i S_j. \tag{6.18}$$

For the operator H_N, similarly to (6.8), we have

$$H_N \geq -\eta^{-1} MN + (1 - \eta^{-1}) \sum_{j=1}^{N} S_j, \tag{6.19}$$

thus,

$$(1 - \eta^{-1})^{-1} H_N + (\eta - 1)^{-1} MN \geq \sum_{j=1}^{N} S_j. \tag{6.20}$$

Together with (6.18), we have shown that

$$\frac{\eta N}{\eta - 1} H_N^2 + 2 \left(\frac{\eta}{\eta - 1} \right)^2 MNH_N + \frac{2\eta M^2 N^2}{(\eta - 1)^2} \geq \sum_{i \neq j}^{N} S_i S_j. \tag{6.21}$$

Since H_N and H_N^2 have the upper bounds

$$H_N \leq \sum_{j=1}^{N} S_j \tag{6.22}$$

and

$$H_N^2 = \left(\sum_{j=1}^{N} S_j - \frac{1}{N} \sum_{i<j}^{N} V_{ij} \right)^2 \le 2 \left(\sum_{j=1}^{N} S_j \right)^2 + \frac{2}{N^2} \left(\sum_{i<j}^{N} V_{ij} \right)^2$$

$$\le 2N \sum_{j=1}^{N} S_j^2 + \frac{N-1}{N} \sum_{i<j}^{N} V_{ij}^2 \le CN \sum_{j=1}^{N} S_j^2, \tag{6.23}$$

respectively, we have that

$$\langle \psi_{N,t}, H_N \psi_{N,t} \rangle = \langle \varphi^{\otimes N}, H_N \varphi^{\otimes N} \rangle$$

$$\le \langle \varphi^{\otimes N}, \sum_{j=1}^{N} S_j \varphi^{\otimes N} \rangle \le CN \|\varphi\|_{H^{1/2}}^2 \tag{6.24}$$

and

$$\langle \psi_{N,t}, H_N^2 \psi_{N,t} \rangle = \langle \varphi^{\otimes N}, H_N^2 \varphi^{\otimes N} \rangle \le CN \langle \varphi^{\otimes N}, \sum_{j=1}^{N} S_j^2 \varphi^{\otimes N} \rangle \le CN^2 \|\varphi\|_{H^1}^2. \tag{6.25}$$

Therefore, from (6.21), (6.24), and (6.25), we find

$$\sum_{i<j}^{N} \langle \psi_{N,t}, S_i S_j \psi_{N,t} \rangle \le CN^3, \tag{6.26}$$

which proves the first part of the lemma. The second part of the lemma can be proved analogously. $\square$

We consider the regularized semi-relativistic Hartree equation (2.10) given by

$$i\partial_t \varphi_t^\alpha = (1 - \Delta)^{1/2} \varphi_t^\alpha - \lambda \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2 \right) \varphi_t^\alpha, \tag{6.27}$$

and study properties of the solution of (2.10).

The following results will be used in the proof of Proposition 2.3:

Lemma 6.2 (Generalized Leibniz Rule). *Suppose that* $1 < p < \infty$, $s \ge 0$, $\alpha \ge 0$, $\beta \ge 0$, *and* $1/p_i + 1/q_i = 1/p$ *with* $i = 1, 2$, $1 < q_1 \le \infty$, $1 < p_2 \le \infty$. *Then*

$$\|(-\Delta)^{s/2}(fg)\|_p$$
$$\le C \left(\|(-\Delta)^{(s+\alpha)/2} f\|_{p_1} \|(-\Delta)^{\alpha/2} g\|_{q_1} + \|(-\Delta)^{\beta/2} f\|_{p_2} \|(-\Delta)^{(s+\beta)/2} g\|_{q_2} \right), \tag{6.28}$$

where the positive constant C *depends on all of the parameters above but not on* f *and* g.

Proof. See [18, Theorem 1.4]. $\square$

Lemma 6.3 (Propagation of regularity). *Fix $s > 1/2$. Let $\varphi \in H^s(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let φ_t and φ_t^α denote the solutions of the semi-relativistic Hartree equations (1.6) and Hartree equation with cutoff, respectively, with the initial condition $\varphi_{t=0} = \varphi$. Fix $T > 0$ such that*

$$\kappa = \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \tag{6.29}$$

Then, there exists a constant $\nu = \nu(\kappa, T, s, \|\varphi\|_{H^s}) < \infty$ (but independent of α_N) such that

$$\sup_{|t| \leq T} \|\varphi_t\|_{H^s}, \sup_{|t| \leq T} \|\varphi_t^\alpha\|_{H^s} \leq \nu. \tag{6.30}$$

Proof. See [25, Proposition 2.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To prove Proposition 2.3, we first consider the following a priori bound on the difference in L^2-norm:

Lemma 6.4. *Suppose that the assumptions of Proposition 2.3 are satisfied. Then, there exist constants C and K, depending only on λ, κ, T, and $\|\varphi\|_{H^1}$, such that*

$$\|\varphi_t - \varphi_t^\alpha\|_2 \leq C\alpha_N \tag{6.31}$$

for all $|t| < T$.

Proof. See [25, Proposition 2.2] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using Lemma 6.4, we prove Proposition 2.3. In the following proof, we generally follow the proof of [25, Proposition 2.2] except in a few estimates.

Proof of Proposition 2.3. First, note that, for any $|t| \leq T$, $\|\varphi_t\|_{H^1} \leq \nu$ for some constant ν depending only on T, κ, and $\|\varphi\|_{H^1}$, which follows from Lemma 6.2. To prove the proposition, it suffices to show that

$$\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2 \leq C\alpha_N^{1/2}. \tag{6.32}$$

From Schwarz inequality, we obtain that

$$\left| \frac{d}{dt} \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2^2 \right|$$

$$= \left| -2\lambda \operatorname{Im}\left\langle (-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha), \right. \right.$$

$$\left. \left. (-\Delta)^{1/4}\left[\left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2 \right) \varphi_t^\alpha \right] \right\rangle \right|$$

$$\leq 2\lambda \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2$$

$$\times \left\| (-\Delta)^{1/4}\left[\left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2 \right) \varphi_t^\alpha \right] \right\|_2.$$

$$\tag{6.33}$$

To estimate the right hand side, we use the following decomposition:

$$\left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \varphi_t - \left(\frac{1}{|\cdot| + \alpha_N} * |\varphi_t^\alpha|^2 \right) \varphi_t^\alpha \right] \right\|_2$$

$$\leq \left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) (\varphi_t - \varphi_t^\alpha) \right] \right\|_2$$

$$+ \left\| (-\Delta)^{1/4} \left[\left(\left(\frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right) (\varphi_t - \varphi_t^\alpha) \right] \right\|_2$$

$$+ \left\| (-\Delta)^{1/4} \left[\left(\left(\frac{1}{|\cdot|} - \frac{1}{|\cdot| + \alpha_N} \right) * |\varphi_t|^2 \right) \varphi_t \right] \right\|_2$$

$$+ \left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) (\varphi_t - \varphi_t^\alpha) \right] \right\|_2$$

$$+ \left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \varphi_t \right] \right\|_2. \tag{6.34}$$

The first term in the right hand side of (6.34) is bounded by

$$\left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) (\varphi_t - \varphi_t^\alpha) \right] \right\|_2$$

$$\leq C \left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \right\|_6 \| \varphi_t - \varphi_t^\alpha \|_3$$

$$+ C \left\| \frac{1}{|\cdot|} * |\varphi_t|^2 \right\|_\infty \| (-\Delta)^{1/4} (\varphi_t - \varphi_t^\alpha) \|_2 \tag{6.35}$$

where we used the generalized Leibniz rule, Lemma 6.2. By Sobolev inequality,

$$\| \varphi_t - \varphi_t^\alpha \|_3 \leq C \| (-\Delta)^{1/4} (\varphi_t - \varphi_t^\alpha) \|_2, \tag{6.36}$$

and by Kato's inequality,

$$\left\| \frac{1}{|\cdot|} * |\varphi_t|^2 \right\|_\infty \leq C \| \varphi_t \|_{H^{1/2}}. \tag{6.37}$$

Since

$$\frac{1}{|\cdot|} * |\varphi_t|^2 = -4\pi(-\Delta)^{-1}|\varphi_t|^2, \tag{6.38}$$

we find that

$$(-\Delta)^{1/4} \left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) = -4\pi(-\Delta)^{-3/4}|\varphi_t|^2 = -4\pi G_{3/2} * |\varphi_t|^2, \tag{6.39}$$

where $G_{3/2}$ is the kernel of the operator $(-\Delta)^{-3/4}$ that is given by

$$G_{3/2}(x) = \frac{\pi^2 \sqrt{2}}{\Gamma(3/4)} |x|^{-3/2}. \tag{6.40}$$

Thus, from Hardy–Littlewood–Sobolev inequality and Sobolev inequality,

$$\left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot|} * |\varphi_t|^2 \right) \right\|_6 = C \left\| |\cdot|^{-3/2} * |\varphi_t|^2 \right\|_6 \leq C \| \varphi_t \|_3^2 \leq C \| \varphi_t \|_{H^{1/2}}^2. \tag{6.41}$$

From (6.35), (6.36), (6.37), and (6.41), we get

$$\left\|(-\Delta)^{1/4}\left[\left(\frac{1}{|\cdot|}*|\varphi_t|^2\right)(\varphi_t-\varphi_t^\alpha)\right]\right\|_2 \le C\|(-\Delta)^{1/4}(\varphi_t-\varphi_t^\alpha)\|_2. \quad (6.42)$$

The second term in the right hand side of (6.34) can be bounded analogously, hence it satisfies

$$\left\|(-\Delta)^{1/4}\left[\left(\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right)(\varphi_t-\varphi_t^\alpha)\right]\right\|_2$$
$$\le C\|(-\Delta)^{1/4}(\varphi_t-\varphi_t^\alpha)\|_2. \quad (6.43)$$

The third term in the right hand side of (6.34) is again bounded using Lemma 6.2 by

$$\left\|(-\Delta)^{1/4}\left[\left(\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right)\varphi_t\right]\right\|_2$$
$$\le C\left\|(-\Delta)^{1/4}\left[\left(\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right)\right]\right\|_3\|\varphi_t\|_6$$
$$+C\left\|\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right\|_\infty\|(-\Delta)^{1/4}\varphi_t\|_2. \quad (6.44)$$

We have from Hardy–Littlewood–Sobolev inequality, generalized Leibniz rule, and Sobolev inequality that

$$\left\|(-\Delta)^{1/4}\left[\left(\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right)\right]\right\|_3 \le \alpha_N\left\|\frac{1}{|\cdot|^2}*(-\Delta)^{1/4}|\varphi_t|^2\right\|_3$$
$$\le C\alpha_N\|(-\Delta)^{1/4}(\overline{\varphi_t}\varphi_t)\|_{3/2} \le C\alpha_N\|(-\Delta)^{1/4}\varphi_t\|_2\|\varphi_t\|_6 \le C\alpha_N\kappa\nu. \quad (6.45)$$

From Hardy inequality, we get that

$$\left\|\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right\|_\infty \le \alpha_N\left\|\frac{1}{|\cdot|^2}*|\varphi_t|^2\right\|_\infty \le C\alpha_N\nu^2. \quad (6.46)$$

Thus, from (6.44), (6.45), and (6.44), we obtain that

$$\left\|(-\Delta)^{1/4}\left[\left(\left(\frac{1}{|\cdot|}-\frac{1}{|\cdot|+\alpha_N}\right)*|\varphi_t|^2\right)\varphi_t\right]\right\|_2 \le C\alpha_N. \quad (6.47)$$

The fourth term in the right hand side of (6.34) is bounded by

$$\left\|(-\Delta)^{1/4}\left[\left(\frac{1}{|\cdot|+\alpha_N}*(|\varphi_t|^2-|\varphi_t^\alpha|^2)\right)(\varphi_t-\varphi_t^\alpha)\right]\right\|_2$$
$$\le C\left\|(-\Delta)^{1/4}\left(\frac{1}{|\cdot|+\alpha_N}*(|\varphi_t|^2-|\varphi_t^\alpha|^2)\right)\right\|_\infty\|\varphi_t-\varphi_t^\alpha\|_2$$
$$+C\left\|\frac{1}{|\cdot|+\alpha_N}*(|\varphi_t|^2-|\varphi_t^\alpha|^2)\right\|_\infty\|(-\Delta)^{1/4}(\varphi_t-\varphi_t^\alpha)\|_2 \quad (6.48)$$

We notice that

$$\left((-\Delta)^{1/4}\frac{1}{|\cdot|+\alpha_N}\right)(x) \le \frac{C}{(|x|+\alpha_N)^{3/2}}, \quad (6.49)$$

which is proved in [25, Proposition 2.2]. Thus,

$$\left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \right\|_\infty \|\varphi_t - \varphi_t^\alpha\|_2$$

$$\leq C\alpha_N^{-3/2} \left\| |\varphi_t|^2 - |\varphi_t^\alpha|^2 \right\|_1 \|\varphi_t - \varphi_t^\alpha\|_2$$

$$\leq C\alpha_N^{-3/2} \left\| |\varphi_t| + |\varphi_t^\alpha| \right\|_2 \|\varphi_t - \varphi_t^\alpha\|_2^2 \leq C\alpha_N^{1/2}, \qquad (6.50)$$

where we used Lemma 6.4 in the last inequality. We also have that

$$\left\| \frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right\|_\infty \leq \alpha_N^{-1} \left\| |\varphi_t|^2 - |\varphi_t^\alpha|^2 \right\|_1 \leq C, \qquad (6.51)$$

where we used the same argument as in (6.50). Thus, from (6.48), (6.50), and (6.51), we obtain that

$$\left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) (\varphi_t - \varphi_t^\alpha) \right] \right\|_2$$

$$\leq C\alpha_N^{1/2} + C\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2. \qquad (6.52)$$

The last term of the right hand side (6.34) is bounded by

$$\left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \varphi_t \right] \right\|_2$$

$$\leq C \left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \right\|_3 \|\varphi_t\|_6$$

$$+ C \left\| \frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right\|_6 \|(-\Delta)^{1/4}\varphi_t\|_3. \qquad (6.53)$$

The first term in the right hand side of (6.53) is bounded by

$$\left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \right\|_3$$

$$\leq \left\| (-\Delta)^{1/4} \frac{1}{|\cdot| + \alpha_N} \right\|_3 \left\| |\varphi_t|^2 - |\varphi_t^\alpha|^2 \right\|_1$$

$$\leq C\alpha_N \left\| \frac{1}{(|\cdot| + \alpha_N)^{3/2}} \right\|_3, \qquad (6.54)$$

where we used the bound (6.49). An explicit computation shows that

$$\left\| \frac{1}{(|\cdot| + \alpha_N)^{3/2}} \right\|_3^3 = 4\pi \int_0^\infty \frac{r^2}{(r + \alpha_N)^{9/2}} \mathrm{d}r = \frac{64\pi}{105} \alpha_N^{-3/2}. \qquad (6.55)$$

Hence,

$$\left\| (-\Delta)^{1/4} \left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \right\|_3 \leq C\alpha_N^{1/2}. \qquad (6.56)$$

The second term in the right hand side of (6.53) is estimated as

$$\left\| \frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right\|_6 \|(-\Delta)^{1/4}\varphi_t\|_3$$

$$\leq \left\| \frac{1}{|\cdot|} * ||\varphi_t|^2 - |\varphi_t^\alpha|^2| \right\|_6 \|\varphi_t\|_{H^1}$$

$$\leq C \left\| |\varphi_t|^2 - |\varphi_t^\alpha|^2 \right\|_{6/5} \|\varphi_t\|_{H^1} \leq C \left\| |\varphi_t| + |\varphi_t^\alpha| \right\|_2 \|\varphi_t - \varphi_t^\alpha\|_3 \|\varphi_t\|_{H^1}$$

$$\leq C\nu\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2, \tag{6.57}$$

where we used Sobolev inequality and Hardy–Littlewood–Sobolev inequality. Thus, from (6.53), (6.56), and (6.57), we obtain that

$$\left\| (-\Delta)^{1/4} \left[\left(\frac{1}{|\cdot| + \alpha_N} * (|\varphi_t|^2 - |\varphi_t^\alpha|^2) \right) \varphi_t \right] \right\|_2$$

$$\leq C\alpha_N^{1/2} + C\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2. \tag{6.58}$$

Therefore, from (6.33), (6.34), (6.42), (6.43), (6.47), (6.52), and (6.58), we find that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2^2 \right|$$

$$\leq C\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2 \left(\alpha_N^{1/2} + \|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2 \right)$$

$$\leq C\alpha_N + C\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^\alpha)\|_2^2. \tag{6.59}$$

Now, (6.32) follows from Gronwall's lemma. This concludes the proof of the Proposition 2.3. $\qquad\square$

7. Properties of Weyl Operator

In this section, we prove various estimates on the following state:

$$W^*(\sqrt{N}\varphi) \frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}} \Omega. \tag{7.1}$$

Lemma 7.1. *There exists a constant $C > 0$ such that, for any $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$, we have*

$$\left\| (\mathcal{N} + 1)^{-1/2} W^*(\sqrt{N}\varphi) \frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C}{d_N}. \tag{7.2}$$

Proof. See [2, Lemma 6.3]. $\qquad\square$

In the next lemma, we prove an estimate on the state (7.1), which primarily shows that the state has a very small probability of having an odd number of particles.

Lemma 7.2. *For all non-negative integers* $k \leq (1/2)N^{1/3}$,

$$\left\| P_{2k} W^*(\sqrt{N}\varphi) \frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2}{d_N} \tag{7.3}$$

and

$$\left\| P_{2k+1} W^*(\sqrt{N}\varphi) \frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2(k+1)^{3/2}}{d_N \sqrt{N}}. \tag{7.4}$$

Proof. Since

$$W^*(\sqrt{N}\varphi) = W(-\sqrt{N}\varphi) = e^{-N/2} \exp\left(a^*(-\sqrt{N}\varphi)\right) \exp\left(a(\sqrt{N}\varphi)\right), \tag{7.5}$$

we find for any $\ell \leq N$ that

$$P_\ell W^*(\sqrt{N}\varphi) \frac{\left(a^*(\varphi)\right)^N}{\sqrt{N!}} \Omega$$

$$= \frac{e^{-N/2}}{\sqrt{N!}} \sum_{m=0}^{\ell} \frac{\left(a^*(-\sqrt{N}\varphi)\right)^m}{m!} \frac{\left(a(\sqrt{N}\varphi)\right)^{N-\ell+m}}{(N-\ell+m)!} \left(a^*(\varphi)\right)^N \Omega$$

$$= \frac{e^{-N/2}}{\sqrt{N!}} \sqrt{N}^{N-\ell} \sum_{m=0}^{\ell} \frac{(-1)^m N^m}{m!(N-\ell+m)!} \left(a^*(\varphi)\right)^m \left(a(\varphi)\right)^{N-\ell+m} \left(a^*(\varphi)\right)^N \Omega$$

$$= \frac{e^{-N/2}}{\sqrt{N!}} \sqrt{N}^{N-\ell} \sum_{m=0}^{\ell} \binom{N}{\ell-m} \frac{(-1)^m N^m}{m!} \left(a^*(\varphi)\right)^\ell \Omega$$

$$= \frac{1}{d_N} N^{-\ell/2} L_\ell^{(N-\ell)}(N) \left(a^*(\varphi)\right)^\ell \Omega, \tag{7.6}$$

where $L_n^{(\alpha)}(x)$ denotes the generalized Laguerre polynomial.

Generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the following recurrence relations:

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x), \tag{7.7}$$

$$x L_n^{(\alpha+1)}(x) = (n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x). \tag{7.8}$$

(See [1] for more detail.) From the recurrence relations, we find that

$$x L_{n-2}^{(\alpha+2)}(x) = x L_{n-1}^{(\alpha+2)}(x) - x L_{n-1}^{(\alpha+1)}(x)$$

$$= \left[(n+\alpha+1)L_{n-1}^{(\alpha+1)}(x) - n L_n^{(\alpha+1)}(x) \right] - x L_{n-1}^{(\alpha+1)}(x)$$

$$= (\alpha+1-x)L_{n-1}^{(\alpha+1)}(x) - n L_n^{(\alpha+1)}(x) + n L_{n-1}^{(\alpha+1)}(x)$$

$$= (\alpha+1-x)L_{n-1}^{(\alpha+1)}(x) - n L_n^{(\alpha)}(x). \tag{7.9}$$

Hence we get,

$$L_n^{(\alpha)}(x) = \frac{\alpha+1-x}{n} L_{n-1}^{(\alpha+1)}(x) - \frac{x}{n} L_{n-2}^{(\alpha+2)}(x). \tag{7.10}$$

Define

$$A_\ell := \begin{cases} N^{(1-\ell)/2} L_\ell^{(N-\ell)}(N) & \text{if } \ell \text{ odd} \\ N^{-\ell/2} L_\ell^{(N-\ell)}(N) & \text{if } \ell \text{ even} \end{cases}. \tag{7.11}$$

Then, from (7.10), we can find the following recurrence relations for A_ℓ:

$$A_{2k+1} = -\frac{2k}{2k+1} A_{2k} - \frac{A_{2k-1}}{2k+1}, \qquad A_{2k} = -\frac{2k-1}{2k} \cdot \frac{A_{2k-1}}{N} - \frac{A_{2k-2}}{2k}, \tag{7.12}$$

where k is a non-negative integer. It can be easily computed that $A_0 = 1$ and $A_1 = 0$. Now, we consider the following claim:

Claim. For any $1 \le k \le (1/2)N^{1/3}$,

$$|A_{2k-2}| \le \frac{1}{\sqrt{(2k-2)!}}, \qquad |A_{2k-1}| \le \frac{k\sqrt{k}}{\sqrt{(2k-1)!}}. \tag{7.13}$$

We prove the claim inductively. It is trivial that A_0 and A_1 satisfy the claim. If $A_0, A_1, \ldots, A_{2k-1}$ satisfies (7.13), then from (7.12), we obtain that

$$|A_{2k}| \le \frac{k\sqrt{k}}{N\sqrt{(2k-1)!}} + \frac{1}{k\sqrt{(2k-2)!}} = \frac{1}{\sqrt{(2k)!}} \left(\frac{\sqrt{2}k^2}{N} + \sqrt{\frac{2k-1}{2k}} \right)$$

$$\le \frac{1}{\sqrt{(2k)!}} \left(\frac{\sqrt{2}k^2}{N} + 1 - \frac{1}{4k} \right) \le \frac{1}{\sqrt{(2k)!}}, \tag{7.14}$$

since $k \le (1/2)N^{1/3}$. We also have that

$$|A_{2k+1}|$$

$$\le \frac{1}{\sqrt{(2k)!}} + \frac{k\sqrt{k}}{(2k+1)\sqrt{(2k-1)!}} = \frac{1}{\sqrt{(2k+1)!}} \left(\sqrt{2k+1} + \frac{\sqrt{2}k^2}{\sqrt{2k+1}} \right)$$

$$= \frac{1}{\sqrt{(2k+1)!}} \left(2k+1 + \frac{2k^4}{2k+1} + 2\sqrt{2}k^2 \right)^{1/2}$$

$$\le \frac{1}{\sqrt{(2k+1)!}} (k^3 + 3k^2 + 2k + 1)^{1/2} \le \frac{(k+1)\sqrt{k+1}}{\sqrt{(2k+1)!}}. \tag{7.15}$$

Thus, the claim (7.13) is proved.

Now, we observe that

$$\left\| P_{2k} W^*(\sqrt{N}\varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| = \frac{A_{2k}}{d_N} \left\| (a^*(\varphi))^{2k} \Omega \right\|$$

$$\le \frac{1}{d_N} \left\| \frac{(a^*(\varphi))^{2k}}{\sqrt{(2k)!}} \Omega \right\| \le \frac{1}{d_N} \tag{7.16}$$

and

$$\left\| P_{2k+1} W^*(\sqrt{N}\varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{A_{2k+1}}{d_N \sqrt{N}} \left\| (a^*(\varphi))^{2k+1} \Omega \right\|$$

$$\leq \frac{(k+1)^{3/2}}{d_N \sqrt{N}} \left\| \frac{(a^*(\varphi))^{2k+1}}{\sqrt{(2k+1)!}} \Omega \right\| \leq \frac{(k+1)^{3/2}}{d_N \sqrt{N}}. \tag{7.17}$$

This proves the desired lemma. $\square$

8. Properties of the Evolution Operator $\widetilde{\mathcal{U}}(t;s)$

In this section, we prove some basic properties of the operator $\widetilde{\mathcal{U}}(t;s)$.

Following [15, Proposition 2.2], we can prove that $\widetilde{\mathcal{U}}(t;s)$ is bounded in $\mathcal{Q}(\mathcal{K}+\mathcal{N}^2)$, provided that $(\mathcal{L}_2(t)+\mathcal{L}_4)$ is stable. (See [20, Proposition 3.4] and [2, Lemma 7.1] for more detail.) The following lemma shows that $(\mathcal{L}_2(t)+\mathcal{L}_4)$ is stable:

Lemma 8.1. *Assume that $\nu(t) = \sup_{|s| \leq t} \|\varphi_s^\alpha\|_{H^1} < \infty$. Then, there exist constants $C', K' > 0$, depending on N, α_N, λ, t, and $\nu(t)$, such that, for the operator $A_2(t) = \mathcal{L}_2(t) + \mathcal{L}_4 + C(\mathcal{N}^2 + 1)$, we have the operator inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} A_2(t) \leq K' A_2(t). \tag{8.1}$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} A_2(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_2(t)$$

$$= -\iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|+\alpha_N} \overline{\varphi_t^\alpha(y)} \dot{\varphi}_t^\alpha(y) a_x^* a_x$$

$$-\iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|+\alpha_N} \overline{\varphi_t^\alpha(x)} \dot{\varphi}_t^\alpha(y) a_y^* a_x$$

$$-\iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|+\alpha_N} \varphi_t^\alpha(x) \dot{\varphi}_t^\alpha(y) a_x^* a_y^* + \text{h.c.} \tag{8.2}$$

where h.c. denotes the Hermitian conjugate and $\dot{\varphi}_t^\alpha = \partial_t \varphi_t^\alpha$. Recall that φ_t^α is the solution of (2.10). Since

$$\|(1-\Delta)^{1/2} \phi_t\|_2 \leq \|\phi_t\|_{H^1} \tag{8.3}$$

and

$$\left\| \frac{1}{|\cdot|+\alpha_N} * |\phi_t|^2 \right\|_\infty \leq \|\phi_t\|_{H^{1/2}} < \infty, \tag{8.4}$$

we find that $\dot{\varphi}_t^\alpha \in L^2(\mathbb{R}^3)$. Thus, for any $\psi \in \mathcal{F}$,

$$\left| \left\langle \psi, \iint \mathrm{d}x\mathrm{d}y \frac{\lambda}{|x-y|+\alpha_N} \varphi_t^\alpha(x) \dot{\varphi}_t^\alpha(y) a_x^* a_y^* \psi \right\rangle \right|$$

$$= \left| \int \mathrm{d}x\mathrm{d}y \left\langle a_x a_y \psi, \frac{\lambda}{|x-y|+\alpha_N} \varphi_t^\alpha(x) \dot{\varphi}_t^\alpha(y) \psi \right\rangle \right|$$

$$\leq \int \mathrm{d}x\mathrm{d}y\|a_xa_y\psi\|^2 + \int \mathrm{d}x\mathrm{d}y|\frac{\lambda}{|x-y|+\alpha_N}|^2|\varphi_t^\alpha(x)|^2|\dot\varphi_t^\alpha(y)|^2\|\psi\|^2$$

$$\leq \langle\psi,\mathcal{N}^2\psi\rangle + C\|\varphi_t^\alpha\|_{H^1}\|\dot\varphi_t^\alpha\|_2\langle\psi,\psi\rangle. \tag{8.5}$$

Other terms in the right hand side of (8.2) can be bounded similarly. Thus, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}A_2(t) \leq C(\mathcal{N}^2 + 1). \tag{8.6}$$

Lemma 6.1 of [3] shows that $-C(\mathcal{N}+1) \leq \mathcal{L}_2(t) - \mathcal{K} \leq C(\mathcal{N}+1)$ for some constant $C > 0$. Moreover, for any $\psi \in \mathcal{F}$,

$$|\langle\psi,\mathcal{L}_4\psi\rangle| = \left|\left\langle\psi,\frac{\lambda}{2N}\iint \mathrm{d}x\mathrm{d}y\frac{1}{|x-y|+\alpha_N}a_x^*a_y^*a_xa_y\psi\right\rangle\right|$$

$$\leq CN^{-1}\alpha_N^{-1}\langle\psi,\mathcal{N}^2\psi\rangle, \tag{8.7}$$

hence $\mathcal{L}^4 \leq CN^{-1}\alpha_N^{-1}\mathcal{N}^2$. In summary, we showed that that there exist constants $C',K' \geq 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}A_2(t) \leq K'(\mathcal{N}^2 + 1) \leq K'(\mathcal{L}_2(t) + \mathcal{L}_4 + C'(\mathcal{N}^2 + 1)) = K'A_2(t), \tag{8.8}$$

which was to be proved. $\qquad\square$

The following lemma that shows the number of the particles in the state $\widetilde{\mathcal{U}}^*(t)\phi(f)\widetilde{\mathcal{U}}(t)\Omega$ cannot be even.

Lemma 8.2. *Let $f \in L^2(\mathbb{R}^3)$. Then, for any $k = 0, 1, 2, \ldots,$*

$$P_{2k}\widetilde{\mathcal{U}}^*(t)\phi(f)\widetilde{\mathcal{U}}(t)\Omega = 0. \tag{8.9}$$

Proof. We first show that the parity $(-1)^\mathcal{N}$ and the operator $\widetilde{\mathcal{U}}(t)$ commute. We note that

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\mathcal{U}}^*(t)(-1)^\mathcal{N}\widetilde{\mathcal{U}}(t)\right) = \widetilde{\mathcal{U}}^*(t)[(-1)^\mathcal{N},(\mathcal{L}_2(t) + \mathcal{L}_4)]\widetilde{\mathcal{U}}(t). \tag{8.10}$$

Since $(\mathcal{L}_2(t) + \mathcal{L}_4)$ and $(-1)^\mathcal{N}$ commute, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\mathcal{U}}^*(t)(-1)^\mathcal{N}\widetilde{\mathcal{U}}(t)\right) = 0. \tag{8.11}$$

We also know that $\widetilde{\mathcal{U}}(0) = I$, hence,

$$\widetilde{\mathcal{U}}^*(t)(-1)^\mathcal{N}\widetilde{\mathcal{U}}(t) = \widetilde{\mathcal{U}}^*(0)(-1)^\mathcal{N}\widetilde{\mathcal{U}}(0) = (-1)^\mathcal{N}. \tag{8.12}$$

Thus, $(-1)^\mathcal{N}\widetilde{\mathcal{U}}(t) = \widetilde{\mathcal{U}}(t)(-1)^\mathcal{N}$ for all t. Similarly, $\widetilde{\mathcal{U}}^*(t)$ and $(-1)^\mathcal{N}$ also commute.

Since $\widetilde{\mathcal{U}}(t)$ and $\widetilde{\mathcal{U}}^*(t)$ commute with the parity $(-1)^\mathcal{N}$, we have that for any non-negative integer k and any $\eta \in \mathcal{F}$,

$$\langle\eta, P_{2k}\widetilde{\mathcal{U}}^*(t)a(f)\widetilde{\mathcal{U}}(t)\Omega\rangle$$

$$= \langle\eta, P_{2k}(-1)^\mathcal{N}\widetilde{\mathcal{U}}^*(t)a(f)\widetilde{\mathcal{U}}(t)\Omega\rangle = \langle\eta, P_{2k}\widetilde{\mathcal{U}}^*(t)(-1)^\mathcal{N}a(f)\widetilde{\mathcal{U}}(t)\Omega\rangle$$

$$= \langle\eta, P_{2k}\widetilde{\mathcal{U}}^*(t)a(f)(-1)^{\mathcal{N}-1}\widetilde{\mathcal{U}}(t)\Omega\rangle = \langle\eta, P_{2k}\widetilde{\mathcal{U}}^*(t)a(f)\widetilde{\mathcal{U}}(t)(-1)^{\mathcal{N}-1}\Omega\rangle$$

$$= -\langle\eta, P_{2k}\widetilde{\mathcal{U}}^*(t)a(f)\widetilde{\mathcal{U}}(t)\Omega\rangle, \tag{8.13}$$

which shows that $P_{2k}\widetilde{\mathcal{U}}^*(t)a(f)\widetilde{\mathcal{U}}(t)\Omega = 0$. The proof for that

$$P_{2k}\widetilde{\mathcal{U}}^*(t)a^*(f)\widetilde{\mathcal{U}}(t)\Omega = 0 \tag{8.14}$$

is similar. Therefore, we get the desired lemma. $\qquad\square$

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References

[1] Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York (1972)

[2] Chen, L., Lee, J.O.: Rate of convergence in nonlinear Hartree dynamics with factorized initial data. J. Math. Phys. **52**, 052108 (2011)

[3] Chen, L., Lee, J.O., Schlein, B.: Rate of convergence towards Hartree dynamics. J. Stat. Phys. **144**, 872–903 (2011)

[4] Chen, X.: Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions. Arch. Rational Mech. Anal. **203**, 455–497 (2012)

[5] Cho, Y., Ozawa, T.: On the semirelativistic Hartree-type equation. Siam J. Math. Anal. **38**, 1060–1074 (2006)

[6] Elgart, A., Schlein, B.: Mean field dynamics of Boson stars. Commun. Pure Appl. Math. **60**, 500–545 (2007)

[7] Erdős, L., Schlein, B.: Quantum dynamics with mean field interactions: a new approach. J. Stat. Phys. **134**, 859–870 (2009)

[8] Erdős, L., Yau, H.-T.: Derivation of the nonlinear Schrödinger equation from a many body Coulom system. Adv. Theor. Math. Phys. **5**, 1169–1205 (2001)

[9] Erdős, L., Schlein, B., Yau, H.-T.: Derivation of the cubic nonlinear Schrödinger equation from quantum dynamics of many-body systems. Invent. Math. **167**, 515–614 (2007)

[10] Erdős, L., Schlein, B., Yau, H.-T.: Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate. Ann. Math. (2) **172**, 291–370 (2010)

[11] Erdős, L., Schlein, B., Yau, H.-T.: Rigorous derivation of the Gross–Pitaevskii equation. Phys. Rev. Lett. **98**, 040404 (2007)

[12] Erdős, L., Schlein, B., Yau, H.-T.: Rigorous derivation of the Gross–Pitaevskii equation with a large interaction potential. J. Am. Math. Soc. **22**, 1099–1156 (2009)

[13] Fröhlich, J., Lenzmann, E.: Blowup for nonlinear wave equations describing Boson stars. Commun. Pure Appl. Math. **60**, 1691–1705 (2007)

[14] Ginibre, J., Velo, G.: The classical field limit of scattering theory for non-relativistic many-Boson systems. I. Commun. Math. Phys. **66**, 37–76 (1979)

[15] Ginibre, J., Velo, G.: The classical field limit of scattering theory for non-relativistic many-Boson systems. II. Commun. Math. Phys. **68**, 45–68 (1979)

[16] Grillakis, M.G., Machedon, M., Margetis, D.: Second order corrections to mean field evolution of weakly interacting Bosons. I. Commun. Math. Phys. **294**, 273–301 (2010)

[17] Grillakis, M.G., Machedon, M., Margetis, D.: Second order corrections to mean field evolution of weakly interacting Bosons. II. Adv. Math. **228**, 1788–1815 (2011)

[18] Gulisashvili, A., Kon, M.E.: Exact smoothing properties of Schrödinger semigroups. Am. J. Math. **118**, 1215–1248 (1987)

[19] Hepp, K.: The classical limit for quantum mechanical correlation functions. Commun. Math. Phys. **35**, 265–277 (1974)

[20] Kato, T.: Linear evolution equations of "Hyperbolic" type. J. Fac. Sci. Univ. Tokyo, Sec. I **17**, 241–258 (1970)

[21] Knowles, A., Pickl, P.: Mean-field dynamics: singular potentials and rate of convergence. Commun. Math. Phys. **298**, 101–138 (2010)

[22] Lenzmann, E.: Well-posedness for semi-relativistic Hartree equations of critical type. Math. Phys. Anal. Geom. **10**, 43–64 (2007)

[23] Lieb, E., Thirring, W.: Gravitational collapse in quantum mechanics with relativistic kinetic energy. Ann. Phys. **155**, 494–512 (1984)

[24] Lieb, E., Yau, H.-T.: The chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Commun. Math. Phys. **112**, 147–174 (1987)

[25] Michelangeli, A., Schlein, B.: Dynamical collapse of Boson stars. Commun. Math. Phys. **311**, 645–687 (2012)

[26] Pickl, P.: Derivation of the time dependent Gross–Pitaevskii equation without positivity condition on the interaction. J. Stat. Phys. **140**, 76–89 (2010)

[27] Pickl, P.: Derivation of the time dependent gross Pitaevskii equation with external fields. arXiv:1001.4894v2 (2010)

[28] Rodnianski, I., Schlein, B.: Quantum fluctuations and rate of convergence towards mean field dynamics. Commun. Math. Phys. **291**, 31–61 (2009)

[29] Spohn, H.: Kinetic equations from hamiltonian dynamics. Rev. Mod. Phys. **52**, 569–615 (1980)

Ji Oon Lee
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejeon, 305701, Republic of Korea
e-mail: `jioon.lee@kaist.edu`