1999 IEEE International Fuzzy Systems Conference Proceedings
August 22-25, 1999, Seoul, Korea

A Method for Describing the Rank of a Fuzzy Number and the *k*-th Largest Fuzzy Number with Fuzzy Sets

Jee-Hyong LEE[*], Hyung LEE-KWANG[*] and Keon-Myong LEE[**]

[*]Computer Science Deptartment, KAIST
Kusong-dong, Yousong-gu, Taejon 305-701, Korea
(E-mail : {leejh,khlee}@fuzzy.kaist.ac.kr)

[**]Computer Science Department, Chungbuk National University,
San48, Gaesin-dong, Cheongju, Chunbuk 361-763, Korea
(Email:kmlee@cbucc.chungbuk.ac.kr)

Abstract

Since fuzzy numbers represent vague numeric values, they may overlap with each other. So, the comparison between fuzzy numbers may be vague. For this reason, in a set of fuzzy numbers, the rank of a fuzzy number and the *k*-th largest fuzzy number are vague. In this paper, to determine the rank of a fuzzy number and the *k*-th largest fuzzy number in a set of fuzzy numbers, a fuzzy set based method is proposed. The proposed method uses a given greater-than relation defined on the set of fuzzy numbers. The rank of a fuzzy number is described with a fuzzy set of the ranks that the fuzzy number can take, and the *k*-th largest fuzzy number with a fuzzy set of fuzzy numbers which can be *k*-th ranked.

1. Introduction

Fuzzy numbers are one of important research areas of the fuzzy theory[1]. Since fuzzy numbers represent vague numeric values, there is an inherent vagueness in the comparison between fuzzy numbers; i.e., it is not easy to determine whether a fuzzy number is larger or smaller than others[2]. For this reason, in a set of fuzzy numbers, the rank of a fuzzy number and the *k*-th largest fuzzy number are also vague. That is, in a set of fuzzy numbers, a fuzzy number may take several ranks, and several fuzzy numbers may be the *k*-th largest.

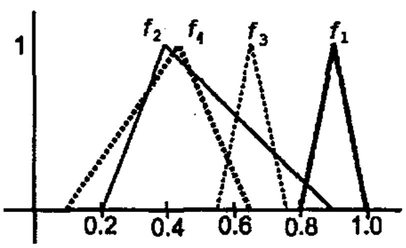

Figure 1. Four fuzzy numbers

For example, if there are four fuzzy numbers as shown in Figure 1, how does f_1 stand among them? If f_1 does not overlap with f_2, we can clearly state that f_1 is the largest. But f_1 does, so we cannot make such statement. That is, f_1 is strongly regarded as the largest, but f_2 also has the possibility to be the largest even if the possibility is small. Thus, the ranks which f_1 may take are the first and the second.

Among the four numbers in Figure 1, which number stands second? Is it proper to choose one fuzzy number? If we consider vagueness represented by fuzzy numbers, it is perplexing to choose the second largest. Considering all possible cases, we can say that all four fuzzy numbers may stand second. For example, as mentioned above, one of possible ranks of f_1 is 2, so f_1 may stand second. Since

0-7803-5406-0/99/$10.00 ©1999 IEEE

fuzzy number f_2 overlaps with f_1, f_3 and f_4, there is the possibility that f_2 is larger than f_3 and f_4, and smaller than f_1. So f_2 may stand second. Similarly, f_3 and f_4 may stand second.

A research area related to these is ranking fuzzy numbers. There are many proposed raking method[3-11]. Ranking fuzzy numbers is arranging fuzzy numbers in a sequence according to their magnitude. Thus, ranking is strongly related with determination of the rank of a fuzzy number, and selection of the *k*-th largest. However, the previously proposed ranking methods usually deal with only how to rank fuzzy numbers. So, by using those method, we cannot describe the vagueness of the rank of a fuzzy number and the *k*-th largest. For example, let us assume that we apply one of the previous ranking methods to the fuzzy numbers in Figure 1, so that we get sequence f_1, f_3, f_2, f_4 as the ranking result. Then, from this sequence, we can notify that f_1 is the largest, and the second largest is f_3. But these results are quite different from those of our previous example; f_1 may be the largest or the second largest, and the four fuzzy numbers may be the second largest. They give only one among several ranks which a fuzzy number may takes, and only one among several fuzzy numbers which may be the *k*-th largest. Thus, we can notify that other methods are needed to determine the rank of fuzzy number and selection of the *k*-th largest fuzzy number.

In this paper, we propose a new method for describing the rank of a fuzzy number and the *k*-th largest fuzzy number in a set of fuzzy numbers. Our method describes them with fuzzy sets by using a given greater-than relation defined on fuzzy numbers. In the next section, the position certainty degree is defined which is used in our method. In Section 3 a method for describing the rank of a fuzzy number and the *k*-th largest fuzzy numbers with fuzzy sets is proposed by using the position certainty degree. This paper will be concluded in the last section.

2. Position certainty degree

In this section, the position certainty degree is defined, which is used in our method. Before defining the position certainty degree, a greater-than relation assumed in this paper is mentioned.

A greater-than relation R is a fuzzy relation defined on $F \times F$, where $F = \{f_1, f_2, \cdots, f_n\}$, a set of fuzzy numbers. The membership degree of (f_i, f_j) to R, $\mu_R(f_i, f_j)$ or R_{ij}, is the degree to which f_i is thought to be greater than f_j. For example, if f_i is clearly greater than f_j, then $R_{ij} = 1$, and if f_i is clearly smaller than f_j, then $R_{ij} = 0$. If $0 < R_{ij} < 1$, it is the case where we cannot cleary determine whether f_i is greater than f_j or not, but in that case, the larger R_{ij} is, the more likely f_i is greater than f_j. A greater-than relation does not need to satisfy other conditions such as being reflexive, symmetric or transitive.

The position certainty degree to which a fuzzy number is the *k*-th largest in a set of fuzzy numbers is defined as follows:

Definition 1. For $F = \{f_1, f_2, \cdots, f_n\}$, a set of fuzzy numbers, and R, a given greater-than relation on $F \times F$, the *position certainty degree* to which f_i is the *k*-th largest in F, $C_F(f_i, k)$, is defined as follows:

$$C_F(f_i, k) = \max_{\forall S_i^k} \min\{R_{j_1 i}, R_{j_2 i}, \cdots, R_{j_{k-1} i}, R_{i j_{k+1}}, \cdots, R_{i j_n}\}$$

where $S_i^k = f_{j_1}, f_{j_2}, \cdots, f_{j_{k-1}}, f_i, f_{j_{k+1}}, \cdots, f_{j_n}$, $j_m = \{k | 1 \le k \le n, k \ne i\}$, is a sequence which contains all elements of F and where f_i is located at the *k*-th position. $R_{ij} \in R$, $1 \le i, j \le n$. For convenience we call $\min\{R_{j_1 i}, R_{j_2 i}, \cdots, R_{j_{k-1} i}, R_{i j_{k+1}}, \cdots, R_{i j_n}\}$ the *local position certainty degree* of f_i.

For example, let us assume that we have a greater-than relation on the fuzzy numbers in Figure 1, as shown in Table 1. Table 1 shows the degree to which a fuzzy number is considered as larger than the others. For and $I=\{R_{23}, R_{24}\}$. The minimum in $S \cup I = \{R_{12}, R_{32}, R_{42}, R_{23}, R_{24}\}$ is $R_{23}=0.30$, so it is removed in the second iteration. In this way, in the fourth iteration, $S=\{R_{12}\}$ and $I=\{R_{24}\}$. These satisfy one of the exit conditions: $|S| = 1 < 3\text{-}1$. So, the lastly deleted value, $R_{32}=0.65$, becomes $C_F(f_2, 3)$.

3. Fuzzy set of possible ranks and fuzzy set of the *k*-th largest

In this section, a *fuzzy set of possible ranks* and a *fuzzy set of the k-th largest* are defined based on the position certainty degree defined in Section 2. They describe the ranks which a fuzzy number may take, and the fuzzy numbers which may be the *k*-th largest with fuzzy sets.

Definition 2. For the given set of fuzzy numbers, $F=\{f_1, f_2, \cdots, f_n\}$, a fuzzy set of the possible ranks of f_i in F, $\tilde{R}_F(f_i)$, is defined as follows :

$$\tilde{R}_F(f_i) = \{(k, \mu_{\tilde{R}_F(f_i)}(k)) \mid \mu_{\tilde{R}_F(f_i)}(k) = C_F(f_i, k), \; k=1, \cdots, n\}.$$

The rank of a number represents how the number stands in the given set of numbers. In the case of crisp numbers, the rank of a number, usually, may be cleary determined, but in the case of fuzzy numbers, it may not because fuzzy numbers can overlap with each other. Thus, the rank of a fuzzy number is vague; it is not suitable to describe the rank with a number. The defined fuzzy set, $\tilde{R}_F(f_i)$, describes the rank of a fuzzy number with a fuzzy set. The membership degree of k to $\tilde{R}_F(f_i)$, $\mu_{\tilde{R}_F(f_i)}(k)$, is the possibility that fuzzy number f_i may stand k-th in F, and defined as $C_F(f_i, k)$.

For example, for $F=\{f_1, f_2, f_3, f_4\}$ whose greater-than relation is given as Table 1, the fuzzy sets of the possible ranks of all fuzzy numbers in F are as follows :

$\tilde{R}_F(f_1)$={ (1,0.98), (2,0.05), (3,0.00), (4,0.00) }

$\tilde{R}_F(f_2)$={ (1,0.05), (2,0.30), (3,0.65), (4,0.40) }

$\tilde{R}_F(f_3)$={ (1,0.00), (2,0.65), (3,0.30), (4,0.02) }

$\tilde{R}_F(f_4)$={ (1,0.00), (2,0.02), (3,0.40), (4,0.70) }.

The possibility that f_1 stands first in F is 0.98, and second in F is 0.05. The possibility for f_1 to take the other ranks is 0. In the case of f_4, the possibility to stand first is 0; second is 0.02; third is 0.40; and fourth is 0.70.

Definition 3. For $F=\{f_1, f_2, \cdots, f_n\}$, a set of fuzzy numbers, a fuzzy set of the *k*-th largest fuzzy numbers in F, $\tilde{F}_F(k)$, is defined as follows:

$$\tilde{F}_F(k) = \{(f_i, \mu_{\tilde{F}_F(k)}(f_i)) \mid \mu_{\tilde{F}_F(k)}(f_i) = C(f_i, k), \; f_i = f_1, \cdots, f_n\}.$$

In the case of crisp numbers, the *k*-th largest number can be cleary determined, but in the case of fuzzy numbers, it cannot be; since, as mentioned before, fuzzy numbers represent vague values, it is not easy to determine which fuzzy number is the *k*-th largest. The *k*-th largest fuzzy number is vague. Definition 3 describes the *k*-th largest fuzzy numbers with a fuzzy set. Fuzzy set $\tilde{F}_F(k)$ contains all fuzzy numbers which can be possibly the *k*-th largest. The membership degree of a fuzzy number to $\tilde{F}_F(k)$ is the possibility that the fuzzy number is the *k*-th largest.

For example, for $F=\{f_1, f_2, f_3, f_4\}$ whose greater-than relation is given as Table 1, fuzzy sets of the *k*-th largest fuzzy numbers are as follows :

$\tilde{F}_F(1)$={ (f_1,0.98), (f_2,0.05), (f_3,0.00), (f_4,0.00) }

$\tilde{F}_F(2)$={ (f_1,0.05), (f_2,0.30), (f_3,0.65), (f_4,0.02) }

$\tilde{F}_F(3)$={ (f_1,0.00), (f_2,0.65), (f_3,0.30), (f_4,0.40) }

$\tilde{F}_F(4)$={ (f_1,0.00), (f_2,0.40), (f_3,0.02), (f_4,0.70) }

The fuzzy numbers regarded as the largest in F are f_1 with possibility 0.98; f_2 with 0.05; and f_3 and f_4 with 0. All four fuzzy numbers are thought as the second

Table 1. A greater-than relation defined on the fuzzy numbers of Figure 1

	f_1	f_2	f_3	f_4
f_1	0.50	0.98	1.00	1.00
f_2	0.05	0.50	0.30	0.70
f_3	0.00	0.65	0.50	0.95
f_4	0.00	0.40	0.02	0.50

example, R_{12}=0.98 means that f_1 is considered as greater then f_2 with the degree of 0.98.

Let us evaluate $C_F(f_2, 3)$ with Table 1. There are 6 sequences where f_2 is located at the third position. The followings show them and the local position certainty degree of f_2 in each of them:

f_1, f_3, f_2, f_4 : $\min\{R_{12}, R_{32}, R_{24}\}$
= min{0.98, 0.65, 0.70} = 0.65

f_1, f_4, f_2, f_3 : $\min\{R_{12}, R_{42}, R_{23}\}$
= min{0.98, 0.40, 0.30} = 0.30

f_3, f_1, f_2, f_4 : $\min\{R_{32}, R_{12}, R_{24}\}$
= min{0.65, 0.98, 0.70} = 0.65

f_3, f_4, f_2, f_1 : $\min\{R_{32}, R_{42}, R_{21}\}$
= min{0.65, 0.40, 0.05} = 0.05

f_4, f_1, f_2, f_3 : $\min\{R_{42}, R_{12}, R_{23}\}$
= min{0.40, 0.98, 0.30} = 0.30

f_4, f_3, f_2, f_1 : $\min\{R_{42}, R_{32}, R_{21}\}$
= min{0.40, 0.65, 0.05} = 0.05

Therefore, the position certainty degree to which fuzzy number f_2 is the third largest, $C_F(f_2, 3)$, is

$$C_F(f_2, 3) = \max\{0.65, 0.30, 0.65, 0.05, 0.30. 0.05\} = 0.65$$

In order to evaluate the position certainty degree of fuzzy number f, we need to check $(n-1)!$ sequences which are composed of all elements of F except f. So the time complexity may be high. But, there is an algorithm with much lower time complexity. The algorithm is shown in Figure 2.

If we apply the algorithm in Figure 2 to Table 1 for evaluating $C_F(f_2, 3)$, the Do-Until loop makes four iterations as follows:

Initial : $S=\{R_{12}, R_{32}, R_{42}\}$ $I=\{R_{21}, R_{23}, R_{24}\}$
1st : $S=\{R_{12}, R_{32}, R_{42}\}$ $I=\{R_{23}, R_{24}\}$
2nd : $S=\{R_{12}, R_{32}, R_{42}\}$ $I=\{R_{24}\}$
3rd : $S=\{R_{12}, R_{32}\}$ $I=\{R_{24}\}$
4th : $S=\{R_{12}\}$ $I=\{R_{24}\}$.

Initially $S=\{R_{12}, R_{32}, R_{42}\}$ and $I=\{R_{21}, R_{23}, R_{24}\}$. In $S \cup I$, the minimum is $R_{21}=0.05$, so it is removed from I. Thus, after the first iteration $S=\{R_{12}, R_{32}, R_{42}\}$

```
S = {R_ki | k ≠ i}
I = {R_ik | k ≠ i}
U = sort S ∪ I
Do
    R_lm = select the minimum in U
    delete R_lm from U and from its source, S or I
Until ( |S| < k−1 or |I| < n−k or ( R_ml ∉ S and R_ml ∉ I) )
the lastly deleted R_lm is C_F(f_i, k)
```

Figure 2. The algorithm for the position certainty degree

largest with 0.05, 0.30, 0.65 and 0.02 respectively.

4. Concluding remarks

Fuzzy numbers represent uncertain and vague values, so there inherently exists vagueness in determining the rank of a fuzzy number and the k-th largest fuzzy number. In order to deal with such vagueness, a fuzzy set of possible ranks and a fuzzy set of the k-th largest have been defined based on a given greater-than relation. These can be applied to many areas such as decision-making.

References

[1] A. Kaufmann, M. M. Gupta, *Introduction to fuzzy arithmetic : theory and applications, Van Nostrand Reinhold,* New York, 1991.

[2] J. H. Lee, "Comparison, ranking and determination of the Ranks of fuzzy numbers based on a satisfaction function", *Ph.D. Dissertation, CS Dept., KAIST*, 1999.

[3] G. Bortolan, R. Degani, "A review of some methods for ranking fuzzy subsets", *Fuzzy Sets and Systems*, vol.15, pp.1-19, 1985.

[4] K. Kim, K. S. Park, "Ranking fuzzy numbers with index of optimism", *Fuzzy Sets and Systems*, vol.35, pp.143-150, 1990.

[5] V. Peneva, I. Popchev, "Comparison of clusters from fuzzy numbers", *Fuzzy Sets and Systems*, vol.97, pp.75-81, 1998.

[6] S.-H. Chen, "Ranking fuzzy numbers with maximizing set and minimizing set", *Fuzzy Sets and Systems*, vol.17, pp.113-129, 1985.

[7] L. M. Campos Ibanes, A. Gonzalez Munoz, "A subjective approach for ranking fuzzy numbers", *Fuzzy Sets and Systems*, vol.29, pp.145-153, 1989.

[8] K. M. Lee, C. H. Cho, H. Lee-Kwang, "Ranking fuzzy values with satisfaction function", *Fuzzy Sets and Systems*, vol.64, pp.295-309, 1994.

[9] T. S. Liou, M. J. Wang, "Ranking fuzzy numbers with integral value", *Fuzzy Sets and Systems*, vol.50, pp.247-255, 1992.

[10] P. Fortemps, M. Roubens, "Ranking and defuzzification methods based on area compensation", *Fuzzy Sets and Systems*, vol.82, pp.319-330, 1996.

[11] K. P. Yoon, "A probabilistic approach to rank complex fuzzy numbers", *Fuzzy Sets and Systems*, vol.80, pp.167-176, 1996.