Design of Non-Uniformly Spaced Linear Phase FIR Filters Using Mixed Integer Linear Programming

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ABSTRACT

An optimization problem for designing a non-uniformly spaced, linear phase FIR filter with minimal complexity is formulated, and solved by mixed integer linear programming (MILP). Design examples illustrate that the proposed method is useful for designing a wide range of filter types, and can outperform the subset selection-based design methods.

I. Introduction

One design approach to efficient FIR filters, which require fewer arithmetic operations than conventional ones, is to design non-uniformly spaced FIR filters [1]-[7] by using techniques such as the *subset selection* method [4]-[7]. This approach results in a filter requiring fewer multiplications and additions, at the expense of increased delays. In contrast to most of the other efficient FIR filter design techniques [8]-[11], which are mainly useful for narrowband filter design, the subset selection method is effective in designing broad range of filter types including wideband and nonlinear phase FIR filters. Filter design using the subset selection algorithm, however, is computationally inefficient and furthermore, cannot guarantee a desired filter with minimal complexity. This is because the "subset size" of the subset selection method is determined by trial and error. (To obtain a filter with minimal complexity, the subset size which is proportional to the number of nonzero terms in the impulse response should be minimized.)

In this paper, we formulate an optimization problem for designing a non-uniformly spaced, linear phase FIR filter with minimal complexity. Then this problem is solved by mixed integer linear programming (MILP)[12]. Through some design examples, we shall show that our technique is applicable over a wide range of filter types and can outperform the subset selection-based design methods.

II. The Filter Design Method

The technique proposed in this section can be applied to linear phase FIR filter design. In what follows, we shall illustrate our method for the case where the impulse response is symmetric and the filter length is even.

Let $h(n)$, n = 0, 1, ..., 2N-1, denote the impulse response of a nonuniformly spaced, linear phase FIR filter. It is assumed that $2N$ is greater than $2N_c$ which is the minimum length of a conventional filter needed to meet the filter specifications. If $h(n) = h(2N-1-n)$ for n = 0, 1, ..., N-1, its frequency response is given by (omitting the linear phase term $\exp(-j(N-1)\omega/2)$)

$$H(\omega) = \sum_{n=0}^{N-1} d(n)\cos\left(n - \frac{1}{2}\right)\omega \tag{1}$$

where $d(n) = 2h(N-n-1)$. We define sequences $I_d(n)$ and $P_d(n)$ indicating the existance and the location of the nonzero values of $d(n)$ as follows:

$$I_d(n) = \begin{cases} 0 & \text{when } d(n) = 0 \\ 1 & \text{when } d(n) \neq 0 \end{cases} \tag{2}$$

and

$$P_d(n) = \begin{cases} 0 & \text{when } d(n) = 0 \\ \text{n} & \text{when } d(n) \neq 0 \end{cases}. \tag{3}$$

Note that $\sum_{n=0}^{N-1} I_d(n)$ is the number of nonzero values of $d(n)$ and that $\max_n \left[P_{d_1}(n)\right]$ is proportional to the delay required in implementing $h(n)$. In order to design a filter with minimal complexity we formulate the following optimization problem.

$$\underset{d(n)}{Minimize}\ J(d(n)) = J_A(d(n)) + J_D(d(n)) \tag{4}$$

subject to

$$|H(\omega) - 1| \le k(\omega)\delta \quad \text{when } \omega \in \text{passband},$$
$$|H(\omega)| \le k(\omega)\delta \quad \text{when } \omega \in \text{stopband},$$

where $J_A(.)$ and $J_D(.)$ are the costs for arithmetic operations (multiplications and additions) and delay, respectively, $k(\omega)$ is the ripple weighting and δ is the ripple size given in the filter specifications. Our objective is to find $d(n)$ minimizing the cost $J(d(n))$ under the constraints of filter specifications. Before defining the costs $J_A(.)$ and $J_D(.)$, some properties that need to be satisfied by the costs are described below.

Desirable Properties of The Costs: Consider two sequences $d_1(n)$ and $d_2(n)$, $n = 0, 1, \ldots, N\text{-}1$, having M_1 and M_2 nonzero values, respectively.

(A) When $M_2 > M_1$, it is desired that

$$J(d_2(n)) > J(d_1(n)). \tag{5}$$

(B) When $M_1 = M_2$, it is desired that

$$J_A(d_1(n)) = J_A(d_2(n)) \tag{6}$$

and

$$J_D(d_1(n)) > J_D(d_2(n)) \text{ whenever}$$
$$\max_n \left[P_{d_1}(n)\right] > \max_n \left[P_{d_2}(n)\right]. \tag{7}$$

These properties indicate that between two sequences the one that needs more computations should have larger cost, and that when the required computational load is the same, the one with more delay have larger cost. We define

$$J_A(d(n)) = c_A \sum_{n=0}^{N-1} I_d(n) \tag{8}$$

with c_A a constant, since the number of arithmetic operations is proportional to $\sum_{n=0}^{N-1} I_d(n)$. Note that the desired property in (6) is satisfied by this $J_A(.)$. Through trial and error, we were able to find the following delay cost $J_D(.)$ that meets the property in (7).

$$J_D(d(n)) = \sum_{n=0}^{N-1} q(n) I_d(n) \tag{9}$$

where

$$q(n) = \begin{cases} 1 & when \;\; n = 0 \\ 2 & when \;\; n = 1 \\ \sum_{i=0}^{n-1} q(i) & when \;\; n \ge 2 \end{cases} \tag{10}$$

Note that $q(n) = \sum_{i=0}^{n-1} q(i) = 3 \cdot 2^{n-2} \;\; for \;\; n \ge 2$.

Observation 1: The delay cost in (9) satisfies the desired property in (7).

proof: Let $\max_n \left[P_{d_i}(n)\right] = m_i,\ i = 1, 2.$

$$\sum_{n=0}^{N-1} q(n) I_{d_1}(n) = \sum_{n=0}^{m_1} q(n) I_{d_1}(n) > \sum_{n=0}^{m_1-1} q(n)$$
$$\ge \sum_{n=0}^{m_1-1} q(n) I_{d_2}(n) \ge \sum_{n=0}^{m_2} q(n) I_{d_2}(n)$$
$$= \sum_{n=0}^{N-1} q(n) I_{d_2}(n)$$

where the first inequality comes from the fact that $q(n) \ge \sum_{i=0}^{n-1} q(i)$ for $n > 0$. ■

Now the overall cost

$$J(d(n)) = c_A \sum_{n=0}^{N-1} I_d(n) + \sum_{n=0}^{N-1} q(n) I_d(n) \tag{11}$$

that meets the desired property in (5) is found by adjusting c_A. The observation below addresses this issue.

Observation 2: The property in (5) is satisfied by $J(.)$ in (11) if

$$c_A \ge \sum_{n=0}^{N-1} q(n) = 3 \cdot 2^{N-2} \tag{12}$$

proof: Assume that $c_A \ge \sum_{n=0}^{N-1} q(n)$. Then

$$J_A(d_2(n)) = \sum_{n=0}^{N-1} c_A I_{d_2}(n) = M_2 c_A \ge (M_1 + 1) c_A$$
$$> M_1 c_A + \sum_{n=0}^{N-1} q(n)$$
$$\ge M_1 c_A + \sum_{n=0}^{N-1} q(n) I_{d_1}(n)$$
$$= J(d_1(n))$$ ■

The overall cost $J(.)$ in (11) is a linear function of $I_d(n)$. Next we shall show that $I_d(n)$ can be generated from $d(n)$ using linear inequalities.

Observation 3: Suppose that for a given $d(n)$, $n = 0, \ldots, N\text{-}1$, j and k are some positive constants satisfying $j < |d(n)| < k$ for all nonzero values of $d(n)$. If we define a sequence $I(n) \in \{0, 1\}$, $n = 0, \ldots, N\text{-}1$, satisfying

$$\frac{1}{k}|d(n)| \le I(n) \le \frac{1}{j}|d(n)|,\ n = 1, 2, \ldots, N, \tag{13}$$

then $I(n) = I_d(n)$.

The proof for this observation is simple, and omitted. Summarizing the above results, the optimization problem in (4) is rewritten as

$$\underset{d(n)}{Minimize}\ J(d(n)) = \sum_{n=0}^{N-1} [c_A + q(n)]\, I_d(n) \qquad (14)$$

subject to

$$|H(\omega) - 1| \le k(\omega)\delta \qquad \text{when } \omega \in \text{passband},$$

$$|H(\omega)| \le k(\omega)\delta \qquad \text{when } \omega \in \text{stopband},$$

$$|d(n)|/k \le I_d(n) \le |d(n)|/j\ ,\ n = 1, 2, ..., N,$$

$$0 \le I_d(n) \le 1,\ I_d(n) \text{ is an integer.}$$

Here $c_A = 3 \cdot 2^{N-2}$, and j and k, respectively, are set at sufficiently small and large values. (In our design examples, which are presented in the next section, j and k are set at 10^{-4} and 10^2, respectively.) This problem can be solved by MILP, treating $d(n)$, $I_d(n)$ as variables.

III. Examples

The MILP problems considered in this section were solved by using the commercial package in [13]. The required computation time for each problem was less than 5 minutes in Sparc 2.

Example 1 (Wideband Filter Design): This example illustrates that our method is useful for designing a wideband filter. The desired specifications in normalized frequency are:

passband : [0, 0.2], stopband : [0.25, 0.50],

dB_p = 0.2 dB maximum passband ripple,

dB_S = 60 dB minimum stopband attenuation.

Note that we have purposely chosen a "difficult" wideband problem where the passband and the stopband have somewhat different widths. The Parks-McClellan algorithm determined that a length $2N_c$=48 linear phase FIR filter met the specifications. Using our method with $2N$=50, we were able to find a non-uniformly spaced FIR filter satisfying the given specifications. The impulse response and the frequency response of this filter are shown in Fig. 1 and 2, respectively. The impulse response of this filter has 40 nonzero coefficients. Thus at least 4 multiplications and 8 additions are reduced at the expense of only 2 delays, as compared with the conventional filter.

Example 2 (Prefilter-Equalizer Design): This example compares our method with the subset selection-based method by considering the design example (Example 4) in [7]. The desired specifications in normalized frequency are as follows:

passband : [0, 0.021], stopband : [0.07, 0.50],

dB_p = 0.2 dB maximum passband ripple,

dB_S = 60 dB minimum stopband attenuation.

This filter is designed as a cascade of a multiplierless prefilter and an equalizer. Given the prefilter in [7], we designed an equalizer such that the cascade meets the above specifications, Table I and Fig. 3 show the impulse response and the frequency response of the equalizer, respectively. Table II compares the designed equalizer with the one in [7]. Our design reduced 2 multiplications, 4 additions and 23 delays.

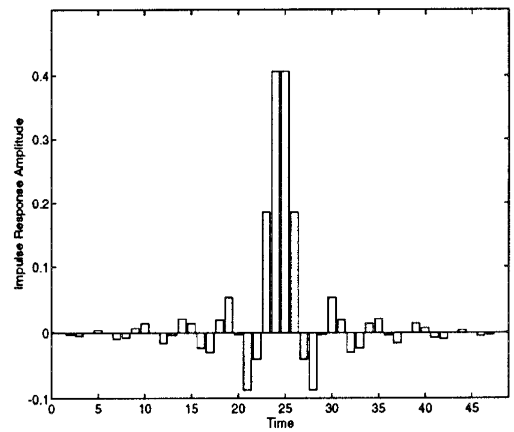

Fig. 1. Impulse response of the filter designed in Example 1.

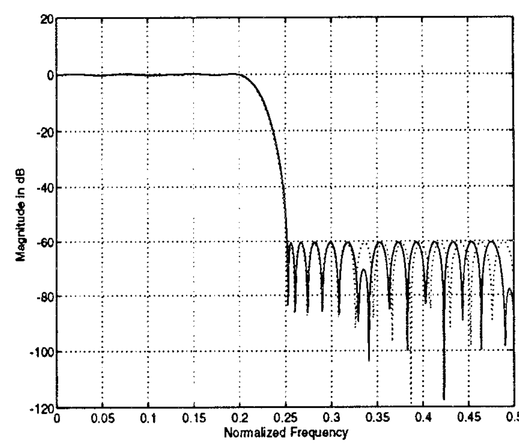

Fig. 2. Frequency response of the conventional filter (dotted line), and the filter designed in Example 1 (solid line).

Table I. Nonzero values of the impulse response of the equalizer designed in Example 2. Here $2N$=10 and $h(n)=h(2N-1-n)$, n = 0, 1, ..., 9.

n	0	4
$h(n)$	-1.802035	2.293609

Table II. Comparison between the subset selection-based method in [7] and our method.

	equalizer in [7]	designed equalizer
multi.	4	2
add.	7	3
delays	32	9

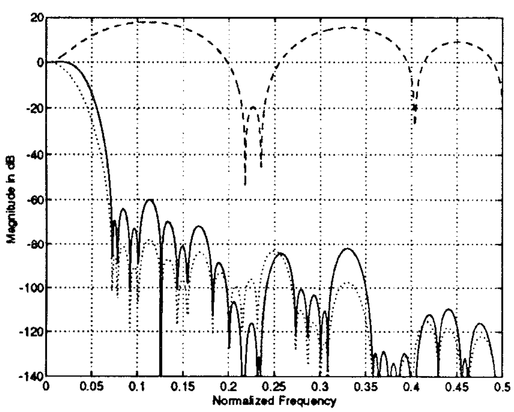

Fig. 3. Frequency response of prefilter in [7] (dotted line), the equalizer designed using proposed algorithm (dashed line), and the overall cascaded filter (solid line).

IV. Conclusion

An optimization problem for designing a non-uniformly spaced, linear phase FIR filter with minimal complexity has been formulated, and solved by MILP. Design examples demonstrated that our technique is useful for designing wide range of filter types and can outperform the subset selection-based design methods.

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