STACK FILTERS AND SELECTION PROBABILITIES

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ABSTRACT

Based on the facts that the output of a given stack filter can be determined if the ranks of the input samples is known and that this output always equals one of the samples in the input window, rank and sample selection probabilities are defined. The output distribution function of a stack filter of size N with continuous, independent identically distributed (i.i.d.) inputs can be expressed as a weighted sum of the distribution functions of the $i^{th}, i = 1, 2, \ldots, N$, order statistics, where the rank selection probabilities are the weights. The sample selection probabilities equal the impulse response coefficients of the FIR filter whose output spectrum is closest, of all linear filters, to that of the stack filter for i.i.d. Gaussian inputs. Some statistical properties of stack filters are then derived. A method to compute the selection probabilities from the positive Booloean function of the stack filter is also given.

1. INTRODUCTION

Recently, a class of filters called stack filters, which have threshold decomposition and stacking properties but are otherwise unconstrained, were introduced [1]. The stack filter is specified by a positive Boolean function [2] which represents the binary output at each threshold level of the multi-valued signal. The binary outputs are combined using threshold decomposition [3] to give the multi-valued signal. Stack filters can also be treated as a special case of morpological filters [4]. Their definition can then be extended to the continuous case in a very natural manner. Stack filters include all rank order filters, standard median and weighted median filters, and a number of other non-linear filters [5], which points to the importance of stack filters in non-linear filtering.

In [1] Wendt et al. derived some deterministic and statistical properties of stack filters. Subsequently, it was indicated that stack filters are effective in minimizing the mean absolute deviation of filtered signals [5]. Based on this error criterion a design method for stack filters has also been suggested [6].

In this paper we will define selection probabilities and relate them to the output distribution and spectral characteristics of the stack filters. A method to evaluate these probabilities from the Boolean function of the stack filter will be presented. Some statistical properties of stack filters will also be derived.

The rest of this paper is organized as follows. In Section 2 selection probabilities are introduced in the context of stack filters, followed by a discussion of the properties of stack filters with i.i.d. inputs. In Section 3 a method to evaluate the selection probabilities from a Boolean function is given. Conclusions are presented in Section 4

2. SELECTION PROBABILITIES

Consider a stack filter of size N, with continuous valued inputs $\mathrm{X} = (X_1, X_2, \ldots, X_N)$ specified by the Boolean function $g(\mathrm{x})$ where $\mathrm{x} = (x_1, x_2, \ldots, x_N)$ are the corresponding threshold level binary signals. Borrowing notations from [1] we will write $g(\mathrm{x})$ as

$$g(\mathrm{x}) = \pi_1 + \pi_2 + \cdots + \pi_m \tag{1}$$

where $\pi_p, 1 \leq p \leq m$ is a product of the variables $x_1, x_2, \ldots, x_N$ and is given by

$$\pi_p = x_{j(p,1)} x_{j(p,2)} \cdots x_{j(p,n_p)} \tag{2}$$

where $j(p,q)$ are the indices of the variables in π_p in increasing order and n_p is the number of variables in π_p.

The output Y can also be written in terms of the input variables $X_1, X_2, \ldots, X_N$ as follows [3]

$$Y = MAX(MIN(\Pi_1), MIN(\Pi_2), \ldots, MIN(\Pi_m)) \tag{3}$$

where

$$MIN(\Pi_p) = MIN(X_{j(p,1)}, X_{j(p,2)}, \ldots, X_{j(p,n_p)}) \tag{4}$$

and $X_{j(p,q)}$ are the multi-level signals corresponding to the threshold level binary signal $x_{j(p,q)}$.

From (3) and (4) it is clear that the output of the stack filter is always one of the samples in the input window. Further, the MAX and the MIN operations indicate that if the rank of the samples is known then it is possible to determine which ranked sample is the output. It is natural to ask what the probability will be of the i^{th} smallest sample $X_{(i)}$, or the j^{th} sample X_j being the output, $1 \leq i, j \leq N$. Taking our cue from this we will define rank and sample selection probabilities and investigate their importance in characterizing stack filters.

A. Notations and Definitions

The inputs $X_1, X_2, \ldots, X_N$ can be permuted in $N!$ possible ways where each permutation of the samples $z_k, k = 1, 2, \ldots, N!$ is called an ordering. Any ordering $z_k, k = 1, 2, \ldots N!$ is an arrangement of time-indexed samples $X_{k(1)}, X_{k(2)}, \ldots, X_{k(N)}$ such that $X_{k(1)} \leq X_{k(2)} \leq \ldots \leq X_{k(N)}$. The i^{th} ranked sample is denoted by $X_{(i)}, i = 1, 2, \ldots, N$ and is the i^{th} smallest sample in any of the $N!$ orderings. For any given ordering let the output $Y = X_{(i)} = X_j, 1 \leq i, j, \leq N$. The sets $\mathcal{G}_{ij}^k$ and $\mathcal{H}_{ij}^k$ are defined as follows: $\mathcal{G}_{ij}^k = \{X_{(n)} \mid n = 1, 2, \ldots, i-1$ for the ordering $z_k, X_{(i)} = X_j\}$ and $\mathcal{H}_{ij}^k = \{X_{(n)} \mid n = i+1, i+2, \ldots, N$ for the ordering $z_k, X_{(i)} = X_j\}$. The superscripts in $\mathcal{G}_{ij}^k$ and $\mathcal{H}_{ij}^k$ will be omitted in most cases. Different orderings may have identical outputs $Y = X_{(i)} = X_j$ as well as identical sets $\mathcal{G}_{ij}$ and

$\mathcal{H}_{ij}$. However, having identical outputs does not guarantee identical sets $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$. The number of distinct sets $\mathcal{G}_{ij}$ (or $\mathcal{H}_{ij}$) for which $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$ is denoted by C_{ij}. The *rank* and the *sample selection probabilities* are now defined.

Definition 1 *Rank Selection Probability (RSP):* The i^{th} rank selection probability is denoted by $P(Y = X_{(i)}), i = 1, 2, \ldots, N$ and is the probability that the output Y equals the i^{th} smallest sample $X_{(i)}$.

Definition 2 *Sample Selection Probability (SSP):* The j^{th} sample selection probability is denoted by $P(Y = X_j), j = 1, 2, \ldots, N$ and is the probability that the output Y equals the j^{th} sample X_j.

The distribution function $F_i(.)$ of the i^{th} ranked sample with continuous i.i.d. inputs is given by

$$F_i(y) = \sum_{r=i}^{N} \binom{N}{r} F_X^i(y)(1 - F_X(y))^{N-i} \tag{5}$$

where $F_X(.)$ is the distribution function of the i.i.d. input.

B. Rank Selection Probability and Stack Filters

Consider a stack filter defined in the binary domain by the Boolean function $f(x_1, x_2, x_3, x_4) = x_1 x_2 + x_3 x_4$ and in the continuous domain by $Y = MAX(MIN(X_1, X_2), MIN(X_3, X_4))$. For the input ordering $z_k : X_1 \leq X_2 \leq X_3 \leq X_4$, the output is found to be $Y = X_{(3)} = X_3$. Here $\mathcal{G}_{33} = \{X_1, X_2\}$ and $\mathcal{H}_{33} = \{X_4\}$. It is interesting to note that if the order of the samples X_1 and X_2 was reversed the output would still be the same. In the general case with $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$, if the rank of the samples in $\mathcal{G}_{ij}$ and/or $\mathcal{H}_{ij}$, are permuted the output does not change. This allows us to lump all such orderings as one event and leads to a representation of the output distribution in terms of the rank selection probabilities.

Lemma 1: Let $z_k, k = 1, 2, \ldots, N!$ represent the ordering of the inputs to a stack filter. If for this ordering the output $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$, with sets $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$ then the output is the same for all input orderings $z_{\hat{k}}, 1 \leq \hat{k} \leq N!$ obtained by permuting the order of the samples within $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$.

Proof: Let the output Y be specified by (3). If $Y = X_j$ then there exists at least one term $\Pi_p, p = 1, 2, \ldots, m$ with $MIN(\Pi_p) = X_j$. This implies that the samples corresponding to the variables in these terms Π_p include X_j and, some or none of the samples in $\mathcal{H}_{ij}$. Also each term Π_p for which $MIN(\Pi_p) \neq X_j$ the we can either have $\Pi_p \in \mathcal{G}_{ij}$ or $\Pi_p \in \mathcal{H}_{ij}$. If $\Pi_p \in \mathcal{H}_{ij}$ then $MAX(MIN(\Pi_1, \Pi_2, \ldots, \Pi_m) \in \mathcal{H}_{ij}$, i.e. $Y \neq X_j$. This is a contradiction. Hence for all terms $MIN(\Pi_p) \neq X_j$, we must have $MIN(\Pi_p) \in \mathcal{G}_{ij}$. Thus $Y = MAX\{\mathcal{G}_{ij} \cup X_j\}$. If the rank of the samples in $\mathcal{H}_{ij}$ are interchanged, X_j still remains the minimum of $X_j \cup \mathcal{H}_{ij}$ and thus all terms Π_p for which we had $MIN(\Pi_p) = X_j$ would still output X_j. If the rank of the samples in $\mathcal{G}_{ij}$ is interchanged then all terms for $MIN(\Pi_p) \neq X_j$ would still have $MIN(\Pi_p) \in \mathcal{G}_{ij}$. Since $Y = MAX\{\mathcal{G}_{ij} \cup X_j\}$ the output will remain the same. ▮

From Lemma 1 we conclude that the number of orderings P_{ij} associated with a given pair of sets $\mathcal{G}_{ij}, \mathcal{H}_{ij}$ such that $Y = X_{(i)} = X_j, 1 \leq i, j \leq N$ is given by

$$P_{ij} = (i-1)!(N-i)!C_{ij} \tag{6}$$

Since each ordering is equally likely we have,

$$P(Y = X_{(i)}, Y = X_j) = \frac{P_{ij}}{N!} \tag{7}$$

and therefore

$$P(Y = X_{(i)}) = \sum_{j=1}^{N} \frac{P_{ij}}{N!} \tag{8a}$$

$$P(Y = X_j) = \sum_{i=1}^{N} \frac{P_{ij}}{N!} \tag{8b}$$

Lemma 1 merely states that if the output Y for a certain ordering is known then the output for several others orderings, obtained by permuting the order of the samples within $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$, is also known. It is important to note that two orderings which have identical outputs do not necessarily have the same sets $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$. For example, the outputs of the median filter — which is a stack filter — of size 3 is same for either of the two inputs, $X_1 \leq X_2 \leq X_3$ and $X_3 \leq X_2 \leq X_1$ but, the corresponding sets $\mathcal{G}_{22}$ and $\mathcal{H}_{22}$ are different. Using Lemma 1 the output distribution of the stack filter with continuous i.i.d. inputs can be obtained.

Theorem 1: The output distribution function $F_Y(.)$ of a stack filter of size N with i.i.d. inputs is given by

$$F_Y(y) = \sum_{i=1}^{N} P(Y = X_{(i)}) F_i(y) \tag{9}$$

where $F_i(.)$ and $f_i(.)$ are the distribution and density functions respectively, of the i^{th} order statistic.

Proof: The output distribution $F_Y(y)$ is given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} P(Y \leq y, Y = X_j, Y = X_{(i)}) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) \end{aligned} \tag{10}$$

The first equality holds because the probability of two continuous valued samples having the same value is equals zero. Thus the events $\{Y = X_j\}$ and $\{Y = X_k\}, j \neq k$ can be treated as mutually exclusive. The last equality holds because the events $\{Y \leq y, Y = X_j, Y = X_{(i)}\}$ is identical to the event $\{X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y\}$ for all values of i and j. Using Lemma 1 we can simplify the expression above as follows

$$\begin{aligned} P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) \\ = \sum_{\text{distinct sets } \mathcal{G}_{ij} \text{ s.t. } Y = X_{(i)} = X_j} P(X_{(i)} \leq y, \mathcal{G}_{ij}) \end{aligned} \tag{11}$$

where $P(X_{(i)} \leq y, \mathcal{G}_{ij})$ is the probability $X_{(i)} \leq y$ for some $\mathcal{G}_{ij}$. Since the input $P(X_{(i)} \leq y, \mathcal{G}_{ij})$ is identical for each distinct $\mathcal{G}_{ij}$ and is given by

$$P(X_{(i)} \leq y, \mathcal{G}_{ij}) = \int_{-\infty}^{y} F_X^{i-1}(\tau)(1 - F_X(\tau))^{N-i} f_X(\tau) d\tau \tag{12}$$

and the summation in (11) boils down to counting all distinct sets $\mathcal{G}_{ij}$ which give $Y = X_{(i)} = X_j$, and has been defined above as C_{ij}. Thus

$$\begin{aligned} P(X_{(i)} \leq y, X_{(i)} = X_j, X_{(i)} = Y) \\ &= C_{ij} \int_{-\infty}^{y} F_X^{i-1}(\tau)(1 - F_X(\tau))^{N-i} f_X(\tau) d\tau \\ &= \frac{(i-1)!\,(N-i)!\,C_{ij}}{N!} F_i(y) \\ &= \frac{P_{ij}}{N!} F_i(y) \\ &= P(Y = X_{(i)}, Y = X_j) F_i(y) \end{aligned} \tag{13}$$

Substituting in (10) and summing over j we get

$$F_Y(y) = \sum_{i=1}^{N} P(Y = X_{(i)}) F_i(y) \; ▮ \tag{14}$$

The following result follows directly from Theorem 1.

Corollary 1: The output distribution function $F_Y(.)$ of a stack filter of size N with i.i.d. inputs is given by

$$F_Y(y) = \sum_{k=1}^{N} c_k F_X^k(y) \qquad (15)$$

$$c_k = \sum_{i=1}^{k} \sum_{r=i}^{k} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{k-r} (-1)^{k-r} \qquad (16)$$

$k = 1, 2, \ldots, N$, where $F_X(.)$ is the distribution function of the input.

Proof: By substituting equation (5) in (9) we get

$$\begin{aligned} F_Y(y) &= \sum_{i=1}^{N} \sum_{r=i}^{N} P(Y = X_{(i)}) \binom{N}{r} F_X^r(y)(1 - F_X(y))^{N-r} \\ &= \sum_{i=1}^{N} \sum_{r=i}^{N} \sum_{m=0}^{N-r} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{m} (-1)^{N-r-m} F_X^{N-m}(y) \\ &= \sum_{i=1}^{N} \sum_{r=i}^{N} \sum_{k=r}^{N} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{k-r} (-1)^{k-r} F_X^k(y) \\ &= \sum_{k=1}^{N} \sum_{i=1}^{k} \sum_{r=i}^{k} P(Y = X_{(i)}) \binom{N}{r} \binom{N-r}{k-r} (-1)^{k-r} F_X^k(y) \end{aligned} \qquad (17)$$

From which (15) and (16) follow. ∎

Notice that $F_X^k(.)$ is also the output distribution of the largest order statistic, denoted $X_{k:k}$, in a window of size k with i.i.d. inputs. $F_Y(.)$ can then be interpreted as a weighted sum of the distributions of the largest order statistic of all filters whose size is less than or equal to N. The example below illustrates Theorem 1 and Corollary 1.

Example 1: The RSP's of the stack filter specified by the Boolean function $f(\mathbf{x}) = x_1x_2 + x_3x_4$ is given by $\mathbf{r} = (0, \frac{2}{3}, \frac{1}{3}, 0)$. Find its output distribution $F_Y(.)$ and density function $f_Y(.)$ for uniform i.i.d. input with distribution $F_X(y) = y, 0 \le y \le 1$.

From Theorem 1 we have $F_Y(y) = \frac{2}{3}F_2(y) + \frac{1}{3}F_3(y)$ with $F_2(y) = 6y^2 - 8y^3 + 3y^4$ and $F_3(y) = 4y^3 - 3y^4$ which gives $F_Y(y) = y^2(2-y)^2$. From Corollary 1 we get $c_1 = 0, c_2 = 4, c_3 = -4$, and $c_4 = 1$ which gives the identical result $F_Y(y) = 4y^2 - 4y^3 + y^4$. ∎

The expressions obtained above for the output distribution of the stack filter are simple, intuitive and easy to implement. Some properties of stack filters are now derived.

Property 1: The n^{th} moments $E(Y^n)$ of the output of a stack filter of size N with i.i.d. inputs is given by

$$E(Y^n) = \sum_{i=1}^{N} P(Y = X_{(i)}) E(X_{(i)}^n) \qquad (18a)$$

$$\text{or,} \qquad E(Y^n) = \sum_{k=1}^{N} c_k E(X_{k:k}^n) \qquad (18b)$$

Proof: $E(Y^n) = \int_{-\infty}^{\infty} y^n f_Y(y)dy = \sum_{i=1}^{N} P(Y = X_{(i)}) \int_{-\infty}^{\infty} y^n f_i(y)dy = \sum_{i=1}^{N} P(Y = X_{(i)}) E(X_{(i)}^n)$. Also, $E(Y^n) = \int_{-\infty}^{\infty} y^n f_Y(y)dy = \sum_{k=1}^{N} c_k \int_{-\infty}^{\infty} k y^n F_X^{k-1}(y) f_X(y)dy$. By definition the the integral equals $E(X_{k:k}^n)$. ∎

Statistical properties of the i^{th} order statistic, in particular the largest order statistic, have been studied in great detail in statistical literature. These results can be used to derive properties of WM filters related to its output moments. When the window size is large (> 9), approximations which are relevant to the expected values can be applied to the expected value of the stack filter output Y [7].

Property 2: The mean $E(Y)$ of the output Y of a stack filter of size N and i.i.d. inputs is given by

$$E(Y) \approx \sum_{i=1}^{N} P(Y = X_{(i)}) F_X(\frac{i}{N+1}) \qquad (19)$$

where $F_X^{-1}(.)$ is the inverse function of the input distribution. ∎

In [7] several bounds for the expected value of the i^{th} order statistics have been obtained. Similar bounds for the expected value for the output of the stack filter can be obtained since it is a linear combination of the expected values of the $i^{th}, i = 1, 2, \ldots, N$ order statistics.

Property 3: The mean of the output of a stack filter of size N with i.i.d. inputs is bounded by

$$\mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \le E(Y) \le \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X \qquad (20)$$

where μ_X and σ_X^2 are the mean and variance of the input. ∎

Proof: From [7, p. 58] we have

$$E(X_{(N)}) \le \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X \qquad (21a)$$

$$E(X_{(1)}) \ge \mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \qquad (21b)$$

i.e. for any $1 \le i \le N$, $\mu_X - \frac{N-1}{\sqrt{2N-1}}\sigma_X \le E(Y) \le \mu_X + \frac{N-1}{\sqrt{2N-1}}\sigma_X$. The result follows after multiplying by $P(Y = X_{(i)})$, each a positive quantity, and adding. ∎

C. Sample Selection Probabilities and the Stack Filter

In an attempt to characterise non-linear filters Mallows [8] hypothesized that a non-linear filter with i.i.d. inputs can be decomposed into a 'linear' and a 'residual' part. The input itself can be decomposed into a sum of processes with Gaussian and non-Gaussian densities respectively. The linear part of the non-linear filter is the linear filter which filters the Gaussian part of the input such that its output is closest to the non-linear filter in the mean-square sense. He also showed that the spectral content of the output of the non-linear filter approximates that of its linear part. Since the frequency response of linear filters is quite well defined, this formulation makes it easy to characterise the output spectrum of a non-linear filter if its linear part is known. In general, finding the linear part of a non-linear filter is rather difficult. However, for filters like the stack filter where the output is always one of the samples from the input window, it was shown that the linear part is a finite impulse response (FIR) filter whose coefficients can be related to the sample selection probabilities. The result is restated for our purposes in Theorem 2 below.

Theorem 2 [8]: The 'linear part' of a stack filter of size N with i.i.d. inputs is a finite impulse response filter whose coefficients $h_j, j = 1, 2, \ldots, N$ are given by

$$h_j = P(Y = X_{N+1-j}) \qquad (22)$$

where $P(Y = X_j)$ denotes the j^{th} sample selection probability. ∎

The frequency response of the FIR filter closely approximates that of the stack filter. Its output spectrum is obtained empirically. A result of simulation is shown in Fig. 1. The examples studied displayed low-pass behaviour, a feature which is characteristic of all stack filters.

Observation 1: All stack filters have low-pass characteristics for i.i.d. inputs. ∎

Proof: Let $h_j, j = 1, 2, \ldots, N$ be the coefficients of the FIR filter which is the linear part of the stack filter. Let $H(j\omega)$ denote the Fourier transform of its impulse response function. We have

$$| H(j\omega) |^2 = \sum_{n=1}^{N} h_n^2 + 2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} h_n h_{n+m} cos\omega m \qquad (23)$$

The coefficient of the cosine terms in the equality are always positive since they are probabilities. The observation follows from the fact that for $|w| \le \frac{\pi}{N}$ all terms in (23) will be additive. ∎

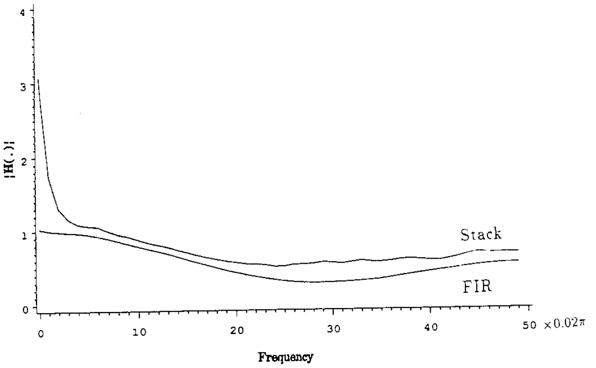

Figure 1: Output spectra of a stack filter with $g(\mathbf{x}) = x_1x_2 + x_4x_5 + x_3$ and the corresponding FIR filter.

D. Stack Filters with Identical Selection Probabilities

For window size 3, stack filters with identical selection probabilities have identical deterministic behaviour. For higher window sizes, however, it is possible to find stack filters with identical selection probability, but different deterministic behaviour. Table 1 below lists some stack filters of size 4 with identical selection probabilities but different output for the same pair of inputs. Filters of this type are of considerable interest and are being studied.

Boolean Function	RSP	SSP	Sample Inputs	Output
$x_1x_2 + x_3x_4$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$X_1 < X_2 < X_3 < X_4$	X_3
			$X_1 < X_3 < X_2 < X_4$	X_3
$x_1x_3 + x_2x_4$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$X_1 < X_2 < X_3 < X_4$	X_2
			$X_1 < X_3 < X_2 < X_4$	X_2
$x_1x_4 + x_2x_3$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$X_1 < X_2 < X_3 < X_4$	X_2
			$X_1 < X_3 < X_2 < X_4$	X_3

Table 1: Stack filters with identical selection probabilities but different outputs.

3. COMPUTATION OF SELECTION PROBABILITIES

Let a stack filter of size N be specified by the positive Boolean function $g(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $x_j \in \{0, 1\}, j = 1, 2, \ldots, N$ are the threshold level binary signals corresponding to the real-valued inputs $X_j, j = 1, 2, \ldots, N$. For each $j = 1, 2, \ldots, N$ the Boolean function $g(\mathbf{x})$ can be rewritten as $g(\mathbf{x}) = x_j f_j(\hat{\mathbf{x}}_j) + h_j(\hat{\mathbf{x}}_j)$, $\hat{\mathbf{x}}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$ and $f_j(.), h_j(.)$ are the Boolean functions in $\hat{\mathbf{x}}$. C_{ij}'s can then be computed as shown in Theorem 3 below.

Theorem 3: The elements $C_{ij}, i, j = 1, 2, \ldots, N$ are given by

$$C_{ij} = \sum_{\hat{\mathbf{x}}_j, w_H(\hat{\mathbf{x}}_j) = N-i} f_j(\hat{\mathbf{x}}_j)\overline{h_j(\hat{\mathbf{x}}_j)} \tag{24}$$

where the product is logical and the summation is arithmetic over the binary values 0 and 1. $w_H(\hat{\mathbf{x}}_j)$ is the number of 1's in the vector $\hat{\mathbf{x}}_j$ and $\overline{h_j(.)}$ denotes logical negation.

Proof: If $f_j(\hat{\mathbf{x}}_j) = 1$ and $h_j(\hat{\mathbf{x}}_j) = 0$ then $g(\mathbf{x}) = x_j$. This is true at each threshold level, hence the output $Y = X_j$. When $w_H(\hat{\mathbf{x}}) = N - i$ then among the elements in $\hat{\mathbf{x}}_j$ there are exactly $(i-1)$ 0's and $(N-i)$ 1's. If $x_j = 0$ then among the variables in $\mathbf{x}$ there are exactly i 0's and $(N-i)$ 1's. If $x_j = 1$ then among the variables in $\mathbf{x}$ there are exactly $i - 1$ 0's and $(N + 1 - i)$ 1's. In either case it implies that x_j has the same value i^{th} ranked sample at each threshold level and thus the output $Y = X_{(i)} = X_j$. By counting all the cases where $g(\mathbf{x}) = x_j$ and $w_H(\hat{x}_j) = N - i$ we get the stated result. ∎

Notice that we are computing C_{ij} and P_{ij} for each $i, j = 1, 2, \ldots, N$. It will therefore be more convenient to refer to them as elements of matrices. Thus $\mathbf{C} = \{C_{ij}\}_{N\times N}$ is called the *combination matrix* or the *C-matrix* and $\mathbf{P} = \{P_{ij}\}_{N\times N}$ is called the *permutation matrix* or the *P-matrix*. The following observation follows from (8).

Observation 2: $P(Y = X_{(i)}) = (i^{th}$ row sum of $\mathbf{P})/N!$ and $P(Y = X_j) = (j^{th}$ column sum of $\mathbf{P})/N!$. ∎

If the variables $x_{j'}$ and $x_{j''}$ are exchanged then the expression $g(\mathbf{x}) = x_j f_j(\hat{\mathbf{x}}_j) + h_j(\hat{\mathbf{x}}_j)$ remains unchanged except for the fact that $f_{j'}(.)$ and $h'_j(.)$ are now exchanged with $f_{j''}(.)$ and $h_{j''}(.)$ respectively. The coefficients C_{ij}'s are affected in a similar manner. As a result we have the following property.

Property 4: For any permutation of the input samples $X_1, X_2, \ldots, X_N$ of a stack filter the $C(P)$-matrix of the new filter are obtained by an identical permutation of the columns of the original $C(P)$-matrix. ∎

The proof is rather straightforward and is therefore omitted. The following result follows from Property 4.

Property 5: For any permutation of the input samples $X_1, X_2, \ldots, X_N$: A. The rank selection probabilities $P(Y = X_{(i)}), i = 1, 2, \ldots, N$ remain unchanged; B. The sample selection probabilities $P(Y = X_{(i)}), i = 1, 2, \ldots, N$ are permuted in the same manner as the input variables. ∎

The property above can be obtained by Observation 2 and Property 5. An important consequence of this property is the following.

Property 6: For any permutation of the i.i.d. input samples $X_1, X_2, \ldots, X_N$ the output distribution function $F_Y(.)$ of a stack filter remain unchanged. ∎

4. CONCLUSIONS

In this paper rank and sample selection probabilities were used for the statistical analysis of stack filters. Rank selection probabilities were used to derive the output distribution By using sample selection probabilities it was shown that the spectral characteristics of stack filters is basically low-pass. A method to compute the selection probabilities was outline and some statistical properties of stack filter were also presented.

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