Diversity-Multiplexing Tradeoff for Practical MIMO Channels

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Abstract— **In this paper, we analyze the outage performance and diversity-multiplexing tradeoff (DMT), originally introduced by Zheng and Tse, for certain classes of multiple antenna systems. In particular, three types of practical channel models such as Rician model, rank-deficient model, and spatially correlated model are considered. The asymptotic behaviors of the outage are analyzed in the limit of high SNR, large Rician factor, and large antenna array. The corresponding DMT curves are also investigated.**

I. Introduction

Recent work of Zheng and Tse characterized a fundamental tradeoff between the diversity and multiplexing gains [1] assuming independent and identically distributed (i.i.d.) Rayleigh fading channels without channel state information at the transmitter (CSIT). Such assumptions can be pessimistic. More realistic scenarios include having a partial CSIT. Such a model is similar to Rician fading channel models. In this paper, we analyze the diversity-multiplexing tradeoff (DMT) for Rician channels. In addition, we investigate several rank-deficient channel models in poor scattering environments. The channel model with spatial correlation among multiple antennas is also examined. We compare and analyze how these three types of practical channel models affect the outage behaviors and DMT. Most of our work is based on recently refined results on DMT called the throughput-reliability tradeoff (TRT) [2].

II. Main results

We assume a flat fading channel. Let n_T and n_R denote the number of transmit and receive antennas, respectively. $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ is the channel matrix. ρ corresponds to the average SNR at each receive antenna. For notational convenience, we define $m = \min\{n_T, n_R\}$, $n = \max\{n_T, n_R\}$ and all logarithms are assumed to be to the base 2 throughout the paper.

For a given target rate R, the outage probability with a diagonal input covariance matrix with equal power allocation can be written by

$$P_{\text{out}}(R,\rho) = \Pr\left\{\log\det\left(\mathbf{I}_{n_R} + \frac{\rho}{n_T}\mathbf{H}\mathbf{H}^{\dagger}\right) < R\right\}. \quad (1)$$

A. MIMO Rician channels and channels with partial CSI

For Rician channels, the channel matrix $\mathbf{H}$ is decomposed into

$$\mathbf{H} = \sqrt{\frac{K}{K+1}}\bar{\mathbf{H}} + \sqrt{\frac{1}{K+1}}\mathbf{H}_w \quad (2)$$

where $K \geq 0$ denotes the Rician factor, $\bar{\mathbf{H}}$ is deterministic, and the entries of $\mathbf{H}_w$ are independent zero-mean unit-variance complex Gaussian. This can also be considered as a channel model with partial CSIT where the transmitter knows the mean $\bar{\mathbf{H}}$ of the channel $\mathbf{H}$.

In this section, we show the several distinguishing features of the outage behavior and analyze the DMT in MIMO Rician channels. It is first shown that the diversity order, which is the slope of the outage probability (on a log-log scale), remains the same but the asymptotic SNR gap (ASG) between Rayleigh and Rician fading channels exists at high SNR.

Let $\phi_1, \phi_2, \cdots, \phi_m$ denote the m eigenvalues of $\bar{\mathbf{H}}$. We define the diversity order for given R, ρ, K, and $\mathbf{\Phi}$ as

$$d_o(R,\rho,K,\mathbf{\Phi}) = -\frac{d\log P_{\text{out}}(R,\rho,K,\mathbf{\Phi})}{d\log\rho}$$

where $\mathbf{\Phi} = \text{diag}\{\phi_i\}$.

Definition 1: In the limit of high SNR, the ASG $\Delta_a(R,K,\mathbf{\Phi})$, if it exists, is defined as

$$\Delta_a(R,K,\mathbf{\Phi}) \triangleq \lim_{\rho\to\infty}\frac{P_o^I(R, P_{\text{out}}(R,\rho,K,\mathbf{\Phi}), 0, \mathbf{\Phi})}{\rho} \quad (3)$$

where P_o^I is the inverse function of P_{out} on the second argument, i.e., $P_o^I(R, P_{\text{out}}(R,\rho,K,\mathbf{\Phi}), K, \mathbf{\Phi}) = \rho$.

Theorem 1: The ASG is given by

$$\Delta_a(R,K,\mathbf{\Phi}) = \frac{e^{K\,\text{tr}(\mathbf{\Phi})/mn}}{K+1}. \quad (4)$$

Note that this gap is irrelevant to the target data rate R. The outage performance is improved by a constant dB gap in high SNR.

The following outage behavior of the Rician channel exhibits the maximum diversity order (MDO) at a certain SNR. Since such an operating condition can give us the best diversity order, it is of practical interest to find when and how much gain is obtained.

Definition 2: Let β denote $\beta = \log\rho$. The MDO $d_{\max}(R,K,\mathbf{\Phi})$ is defined as

$$d_{\max}(R,K,\mathbf{\Phi}) \triangleq d_o(R,\rho^*,K,\mathbf{\Phi}) \quad (5)$$

at the corresponding desired SNR

$$\beta_{\text{op}}(R,K,\mathbf{\Phi}) \triangleq \log\operatorname*{argmax}_{\rho} d_o(R,\rho,K,\mathbf{\Phi}) \quad (6)$$

where $\log\rho^* = \beta_{\text{op}}(R,K,\mathbf{\Phi})$.

In AWGN channels, i.e., deterministic channels, there exists a minimum SNR threshold denoted as $\beta_G(R)$ in the log scale, above which the outage probability in (1) tends to zero. It would be meaningful to examine the gap between $\beta_{\text{op}}(R,K,\mathbf{\Phi})$ and $\beta_G(R)$, which is defined as the finite-SNR gap (FSG).

Definition 3: As the Rician factor K tends to infinity, the FSG is defined as

$$\Delta_f(R) \triangleq \lim_{K\to\infty}\left[2^{\beta_{\text{op}}(R,K,\mathbf{\Phi})-\beta_G(R)}\right]. \quad (7)$$

Theorem 2: In the limit of large K for MISO or SIMO systems, we get

$$\beta_{\text{op}}(R,K) = \log\left(\frac{4n_T(2^R-1)}{n}\right) + O\left(\frac{1}{K}\right) \quad (8)$$

$$d_{\max}(R,K) = \frac{n}{4}K + \frac{1+2n}{4} + O\left(\frac{1}{K}\right). \quad (9)$$

Theorem 3: In MISO or SIMO systems, the FSG is given by

$$\Delta_f(R) = 4 \quad (10)$$

which is equal to 6dB.

Note that it is irrelevant to the target data rate R.

In addition to the above outage analysis, we also anlyze the DMT for Rician. Approaches based on [1], [2] fail to explain the finite-SNR behavior of Rician channels such as the MDO and FSG. In the following work, we formulate the differential DMT (DDMT) that can capture such finite-SNR behaviors.

Definition 4: The differential diversity gain $\bar{d}_D(\bar{r}_D, R, K)$ of MISO or SIMO systems is defined by

$$\bar{d}_D(\bar{r}_D, R, K) \triangleq \lim_{\delta \log \rho \to 0} \left[\frac{\log P_{\text{out}}(R, \rho^*, K)}{\delta \log \rho} - \frac{\log P_{\text{out}}(R + \bar{r}_D \delta \log \rho, 2^{\log \rho^* + \delta \log \rho}, K)}{\delta \log \rho} \right] \quad (11)$$

at asymptotically large K where $\bar{r}_D$ is the differential multiplexing gain.

Theorem 4: Let $r_{\max}(R)$ denote the differential multiplexing gain satisfying $\bar{d}_D(\bar{r}_D, R, K) = 0$. For the operating region in the vicinity of the MDO, the DDMT of the Rician fading channels is obtained as

$$\bar{d}_D(\bar{r}_D, R, K) = d_{\max}(R, K) - \frac{d_{\max}(R, K)}{r_{\max}(R)} \bar{r}_D \quad (12)$$

where $r_{\max}(R) = 1 - 2^{-R}$.

B. Rank-deficient MIMO channels

We first consider the poor scattering environment where scatterers are located far enough from both the transmitter and the receiver so that the angular spread is vanishingly small. Assuming scatters are grouped into P clusters where there are L scatterers in each cluster, the channel gain $\mathbf{H}$ is given by

$$\mathbf{H} = \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \mathbf{H}_p = \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \mathbf{u}_p \cdot h_p \cdot \mathbf{v}_p^\dagger \quad (13)$$

whose rank is upper bounded by $\min\{P, n_T, n_R\}$. $\mathbf{u}_p$ and $\mathbf{v}_p$ are the column vectors whose elements are i.i.d. complex random variables uniform on the unit circle. As $L \to \infty$, h_p is approximated as a complex Gaussian random variable (Rayleigh fading). Based on the TRT in [2], we can estimate $d^*(r)$ from the simulation results of the diversity order in rank-deficient channels [3] where DMT curves are lowered due to rank-deficiency and approach to that of i.i.d. Rayleigh fading channels as $P \to \infty$.

Theorem 5: For $n_T \times n_R$ rank-deficient Rayleigh fading MIMO channels with P clusters, the optimal tradeoff curve $d^*(r)$ in the limit of $n_T, n_R \to \infty$ is given by

$$d^*(r) = P - r. \quad (14)$$

where $d^*_{\max} = r_{\max} = P$.

For $P = 1$, there is a slightly different rank-deficient model called the keyhole model. In this model, the channel $\mathbf{H}$ is given by

$$\mathbf{H} = \boldsymbol{\beta} \boldsymbol{\alpha}^\dagger \quad (15)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the column vectors whose elements are complex Gaussian random variables.

Theorem 6: For $n_T \times n_R$ keyhole model, the optimal tradeoff curve $d^*(r)$ is given by

$$d^*(r) = m(1 - r). \quad (16)$$

where $d^*_{\max} = m$ and $r_{\max} = 1$.

Although not physically motivated, the following model provides an alternative model for rank-deficient channels where rank-deficiency in channel matrix is introduced directly by just forcing some eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ to zero.

Theorem 7: If randomly chosen $m - k$ eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ are forced to zero in the $n_T \times n_R$ MIMO channel $\mathbf{H} = [h_{ij}]$, $h_{ij} \sim \mathcal{CN}(0, 1)$, the optimal tradeoff curve is same as that of the $(n_T + n_R - k) \times k$ MIMO channel $\mathbf{G} = [g_{ij}]$, $g_{ij} \sim \mathcal{CN}(0, 1)$.

C. Spatially correlated MIMO channels

When there exists the spatial correlation among the receive antennas, $\mathbf{H}$ can be represented by

$$\mathbf{H} = \boldsymbol{\Sigma}^{\frac{1}{2}} \mathbf{H}_w \quad (17)$$

where $\boldsymbol{\Sigma} = E[\mathbf{h}_j \mathbf{h}_j^\dagger]$, $\mathbf{h}_j$ is the jth column of $\mathbf{H}$.

Lemma 1: For the $n_T \times n_R$ correlated MIMO channel with correlation matrix $\boldsymbol{\Sigma}$, the optimal tradeoff curve $d^*(r)$ is same as that of $\boldsymbol{\Sigma} = \mathbf{I}$.

Hence, DMT is not changed by spatial correlation. The only thing worth being evaluated here is the amount of such a gap.

Definition 5: In the limit of high SNR, $\tilde{\Delta}_a(R, \boldsymbol{\Sigma})$, if it exists, is defined as

$$\tilde{\Delta}_a(R, \boldsymbol{\Sigma}) \triangleq \lim_{\rho \to \infty} \frac{P_o^I(R, P_{\text{out}}(R, \rho, \mathbf{I}), \boldsymbol{\Sigma})}{\rho} \quad (18)$$

where P_o^I is the inverse function of P_{out} on the second argument, i.e., $P_o^I(R, P_{\text{out}}(R, \rho, \mathbf{I}), \mathbf{I}) = \rho$.

Theorem 8: The $\tilde{\Delta}_a(R, \boldsymbol{\Sigma})$ is given by

$$\tilde{\Delta}_a(R, \boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{-\frac{1}{m}} \quad (19)$$

Note that this gap is irrelevant to the target data rate R. The degradation appears only as a penalty in SNR gap in dB.

III. Discussions and Conclusion

It is shown that the Ricianness and spatial correlation can change the outage performance by a constant dB gap in SNR, but it cannot change the DMT, i.e., diversity order at high SNR. For Rician channels, the MDO shows that there exists an SNR where the diversity order is maximized, which can be a desired operating point. We also analyze the DDMT that is suitable for capturing the DMT for Rician. We next show DMT curves for several rank-deficient channels are lowered. It is verified that DMT curves of rank-deficient channels approach to that of the i.i.d. Rayleigh fading channels as the scattering becomes rich. We refer readers to [3], [4] for more details and proofs.

References

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