Analysis and Implementation of Linear Combination of Weighted Order Statistic Filters

Jongkwan Song and Yong Hoon Lee

Dept. of Electrical Engineering, Korea Advanced Institute of Science and Technology
373-1 Kusung-dong, Yusung-Gu, Taejun, 305-701, Korea
TEL : 82-42-869-3437
FAX : 82-42-869-3410

Abstract

Linear combination of WOS(LWOS) filters, that can be thought of as an extension of stack filters, can represent any Boolean function(BF) or its extension, which is called the extened BF(EBF). In this paper, we present a procedure for finding an LWOS filter of the simplest type from LWOS filters which are equivalent to a given BF or EBF. In addition, a property that is useful for implementing an LWOS filter is derived and an algorithm for LWOS filtering is presented.

I. Introduction

The LWOS filter[1] is a digital filter which is a composition of weighted order statistic(WOS)[2][3] and linear combination of order statistic(LOS)[4][5] filters. It has a window moving over an input sequence. At each point, the data inside the window are duplicated according to the weighting parameters and sorted as in WOS filtering. Then the output is obtained as a linear combination of the sorted data(See Fig.1).

LWOS filters obey the threshold decomposition property[6], and can be fully specified in the binary(0–1) domain. Conversely, it has been shown that an arbitrary Boolean function(BF) or its extension, which is called the *extended* Boolean function(EBF), can be represented as an LWOS filter if the given BF or EBF yields zero for a zero input vector. This fact indicates that the class of LWOS filters encompasses a variety of filters which include nonlinear stack filters[2] and linear FIR filters. In [1], a procedure for finding an LWOS filter that is equivalent to a given BF or EBF is presented. The procedure results in an LWOS filter of a specific form, which is referred to as the *canonical* LWOS filter.

While the canonical LWOS filter is useful for representing a BF or EBF, its implementation requires heavy computational load because the dimension of the linear combination vector(l in Fig. 1) of a canonical LWOS filter of span N is $2^N - 1$. In this paper, we shall show that in many cases a BF or an EBF can be expressed as an LWOS filter which is simpler than the canonical one. In addition, an algorithm for implementing LWOS filters is presented.

The organization of this paper is as follows. In Section II, we review some details of LWOS filtering and intorduce our notations. Section III describes a procedure for finding an LWOS filter corresponding to a BF or an EBF. In Section IV, a property that is useful for implementing an LWOS filter is derived and an algorithm for LWOS filtering is presented.

II. LWOS Filters

Let $X(m)$ be an input process which takes on integer values in $\{0, 1, \ldots, M-1\}$. A window of width N slides across the input $X(m)$. Then the output $Y(m)$ of a LWOS filter, with weights w_i, $1 \le i \le N$, and coefficients for linear combination l_j, $1 \le j \le K$ is given by

$$Y(m) = \sum_{j=1}^{K} l_j \cdot [j^{th} Largest\{\overbrace{X_1(m), \ldots, X_1(m)}^{w_1 \text{ times}}, \overbrace{X_2(m), \ldots, X_2(m)}^{w_2 \text{ times}}, \ldots, \overbrace{X_N(m), \ldots, X_N(m)}^{w_N \text{ times}}\}] \quad (2-1)$$

where $X_j(m)$ is the j^{th} sample from the left in the window at m, and $K = \sum_{i=1}^{N} w_i$. In order to simplify notations, the time index m will be dropped from $X_i(m)$ and $Y(m)$. Using vector representations, (1-1) may be rewritten as

$$Y = \sum_{j=1}^{K} l_j \cdot [j^{th} Largest\,\{\mathcal{W}(X)\}] = \boldsymbol{l} \cdot \mathcal{W}_{OS}'(X) \quad (2-2)$$

where $\mathcal{W}(\cdot)$is a duplicating operator defined as

$$\mathcal{W}(X) = \{\overbrace{X_1, \ldots, X_1}^{w_1 \text{ times}}, \overbrace{X_2, \ldots, X_2}^{w_2 \text{ times}}, \ldots, \overbrace{X_N, \ldots, X_N}^{w_N \text{ times}}\} \quad (2-3)$$

$\mathcal{W}_{OS}(X)$ is the vector obtained by ordering the elements of $\mathcal{W}(X)$ and prime($\prime$) means transposition. Since LWOS filters obey the threshold decomposition property, its output Y can be rewritten as

$$Y = \boldsymbol{l} \cdot \mathcal{W}_{OS}'(X) = \sum_{k=1}^{M-1} \boldsymbol{l} \cdot \mathcal{W}_{OS}'\,\{I_k(X)\} \quad (2-4)$$

where $I_k(X)$ is an N-dimensional row vector given by $I_k[X] = \{I_k[X_1], \ldots, I_k[X_N]\}$ and $I_k(x)$ is an indicator function defined as $I_k(x) = 1$ if $x \ge k$ and 0 otherwise. As with stack filtering, LWOS filters can be specified on the binary(0-o-1) domain. If we let $\boldsymbol{x} = (x_1, \ldots, x_N)$ be a binary input vector, then the output y of an LWOS filter corresponding to the input is given by

$$y = \boldsymbol{l} \cdot \mathcal{W}_{OS}'(\boldsymbol{x}) \quad (2-5)$$

Note that $\mathcal{W}_{OS}(\boldsymbol{x})$ is a vector which consists of successive 1's followed by successive 0's. An EBF $f(\boldsymbol{x})$ is defined by

$$f(\boldsymbol{x}^i) = y^i, \quad 1 \le i \le P \quad (2-6)$$

where $P = 2^N - 1$, $\boldsymbol{x}^i$ is a binary vector of length N which is the radix 2 representation of a positive integer i, and y^i is the output of $f(\cdot)$ for the input $\boldsymbol{x}^i$. An LWOS filter identical to a given EBF can be obtained by finding $\boldsymbol{w}$ and $\boldsymbol{l}$ satisfying

$$\boldsymbol{l} \cdot \mathcal{W}_{OS}'(\boldsymbol{x}^i) = y^i, \quad 1 \le i \le P \quad (2-7)$$

or equivalently

$$[A] \cdot l' = y' \qquad (2-8)$$

where

$$[A] = \begin{pmatrix} \overbrace{1 \cdots 1}^{w_N \text{ times}} \; 0 & \cdots & 0 \\ \overbrace{1 \;\cdots\; 1}^{w_{N-1} \text{ times}} \; 0 & \cdots & 0 \\ \overbrace{1 \;\;\cdots\;\; 1}^{w_N + w_{N-1} \text{ times}} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ \overbrace{1 \qquad \cdots \qquad 1}^{K \text{ times}} & & \end{pmatrix}, \quad y' = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^P \end{pmatrix}$$

$K = \sum_{i=1}^{N} w_i$, and the number of 1's in the i^{th} row of $[A]$ is $w' \cdot x^i$, $[A]$ is a $P \times K$ matrix. A sufficient condition regarding the existence of w and l satisfying (2-7) or (2-8) is described as follows:

Property 2-1 [1] : Given an EBF. If the weight vector w of an LWOS filter satisfies the following two conditions, then we can uniquely determine l_i;the LWOS filter with the parameter vectors w and l is identical to the EBF.

Conditions:

$$w' \cdot x^i \neq w' \cdot x^j \quad i \neq j, \quad 1 \leq i, j \leq P \qquad (2-9a)$$

$$K = P. \qquad (2-9b)$$

When these conditions are met, the linear combination vector l can be obtained by solving (2-8). The canonocal LWOS filter is the simplest in the sense that the dimension K of l is equal to P. Specifically, the canonical LWOS filter is one of the LWOS filters that satisfying the conditions in (2-9). It has the weight vector

$$w = \left(2^{N-1}, 2^{N-2}, \cdots, 2^0\right). \qquad (2-10)$$

For this weight vector, we get

$$[A] = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad [A]^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \end{pmatrix} \qquad (2-11)$$

and the elements of l is given by

$$l_i = \begin{cases} y^i, & i = 1 \\ y^i - y^{i-1}, & 2 \leq i \leq P \end{cases} \qquad (2-12)$$

The canonical LWOS filter can represents any EBF $f(x)$, with $f(0) = 0$. In the following section, we shall show that LWOS filters which are equivalent to some EBF's can have K smaller than P.

III. Non-Canonical LWOS Filters Representing an EBF

Suppsos that some outputs y^i and y^j, $i \neq j$, $1 \leq i, j \leq P$, of a given EBF $f(x)$ have the same value, that is, $f(x^i) = f(x^j)$. The weight w of an LWOS filter representing this EBF may *not* have to satisfy the condition in (2-9 a). The reason is that $w \cdot x^i = w \cdot x^j$ will result in $l \cdot W'_{OS}(x^i) = l \cdot W'_{OS}(x^j)$, which is true since $y^i = y^j$ (See (2-7), (2-8)). The condition in (2-9 a) has been obtained under the assumption that y^i, $1 \leq i \leq P$ are different each other. When some of the outputs of a given EBF have identical values, it is often possible to find a non-canonical LWOS filter which is equivalent to the EBF and is simpler than the canonical one. A property which is a basis for finding such an LWOS filter is presented next.

Property 3-1 : Given an EBF $f(x)$. If a weight w satisfies the conditions presented below, then we can uniquely determine the linear combination vector l; the LWOS filter with the parameter vectors w and l is equivalent to the given EBF.

Conditions:

(i) $w \cdot x^i \neq w \cdot x^j$ whenever $f(x^i) \neq f(x^j)$. On the other hand, $w \cdot x^i$ may be the same as $w \cdot x^j$ if $f(x^i) = f(x^j)$.

(ii) $\{w \cdot x^i | 1 \leq i \leq 2^N - 1\} = \{1, 2, \ldots, K\}$ where $K = \sum_{i=1}^{N} w_i$ may be less than P.

Proof : Note that the canonical LWOS filter($K = P$) satisfies these conditions. Suppose that only two output values are equivalent and the rest are different each other, i.e., $f(x^i) = f(x^j)$ for some i, j, $i \neq j$, and $f(x^n) \neq f(x^m)$ unless $\{n, m\} = \{i, j\}$. In addition, assume that we were able to find w that meets the conditions (i) and (ii), with $w \cdot x^i = w \cdot x^j$ and $K = P - 1$. With this w, we can find l of an LWOS filter equivalent to the given EBF by solving (2-8). In this case $[A]$ is a $P \times (P-1)$ matrix, l and y are vectors with dimension $(P-1)$ and P, respectively. Note that i-th and j-th rows of $[A]$ are exactly the same and the i-th and j-th entries of y are also the same. Therefore, we can reduce the dimension of $[A]$ and y to $(P-1) \times (P-1)$ and $(P-1)$, respectively, by removing either i-th or j-th row of $[A]$ and the corresponding entry from y. If we denote the resulting matrix by $[\hat{A}]$ and output vector by $\hat{y}$, then

$$[\hat{A}] \cdot l' = \hat{y}'. \qquad (3-1)$$

Due to condition (ii), which indicates that the numbers of successive 1's at the rows of $[\hat{A}]$ are different each other, the square matrix $[\hat{A}]$ is nonsingular, and l can be determined uniquely. The LWOS filter with the obtained parameters w and l is identical to the given EBF, because (3-1) is essentially the same as (2-8). This proof can be extended directly to the cases where several outputs of a given EBF have identacal values ∎

Consider the problem of finding an LWOS filter with minimum possible value of K among LWOS filters which are equivalent to a given EBF. Such an LWOS filter can be obtained through the following minimization problem

$$\text{Minimize} \quad \sum_{i=1}^{N} w_i,$$
$$\text{Under the conditions in Property 3-1} \qquad (3-2)$$

This problem can be solved by exhaustively searching all possible weight vector, $w = (w_1, \ldots, w_N)$ that meet the following constraints:

w_i are integers, $w_i \geq 1$ for all $1 \leq i \leq N$.

$$\sum_{i=1}^{N} w_i \leq P = 2^N - 1$$

At least one w_i should equal to 1

Conditions in Property 3-1 $\qquad$ (3-3)

The second constraint above comes from the fact that the canonical LWOS filter with $K = P$ can alway be found, and the third is a consequence of the second condition in Property 3-1. Notice that the first three constraints in (3-3) can be checked trivially. For these vectors we examine whether the conditions in Property 3-1 are met, starting with $w = (1,1,\ldots,1)$ and by increasing $\sum_{i=1}^{K} w_i$ one by one until the vector that meets all the constraints is found. After getting the weight w, the linear combination vector l is found by solving (3-1). Due to the second condition in Property 3-1, $[\hat{A}]$ can be converted into a lower triangular matrix by exchanging its rows and the corresponding entries of $\hat{y}$[See (2-11)], and l can be evaluated systematically as in (2-12). The example below illustrates the process for obtaining an LWOS filter corresponding to a BF.

Example 3-1 : Consider a Boolean function $f(x) = x_1 \cdot \overline{x}_2 \cdot \overline{x}_3 + x_1 \cdot x_2 \cdot x_3 + \overline{x}_1 \cdot \overline{x}_2 \cdot x_3 + \overline{x}_1 \cdot x_2 \cdot \overline{x}_3$. For this function, the minimization in (3-3) results in $w = (1,1,1)$. Table 1 shows the truth table associated with this $f(x)$ and $\{w \cdot x^i\}$. It is seen that $w \cdot x^i \neq w \cdot x^j$ whenever $f(x^i) \neq f(x^j)$. The input–output relation of the $f(x)$ is rewirtten by using the matrix–vector notations as in (2-8).

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (3-4)$$

After removing the identical rows from $[A]$ and the corresponding entries from y, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and $(l_1, l_2, l_3) = (1 \;\; -1 \;\; 1)$ It is interesting to note that the LWOS filter with $w = (1 \;\; 1 \;\; 1)$ and $l = (1 \;\; -1 \;\; 1)$ is the LOS filter defined as $Y = X_{(1)} + X_{(3)} - X_{(2)}$ where $X_{(j)}$ is the j–th largest sample inside a window. Thus the given BF is in fact an LOS filter. ∎

Usually the BF expression of a stack filter is long and cumbersome. Following the approach presented in this section, stack filters can be represented as an LWOS filter which is often simpler than the corresponding BF.

IV. LWOS Filters with Multilevel Inputs

In this section, we shall show that the output Y of an LWOS filter for the M–valued inputs $\{X_1, \ldots, X_N\}$ can be expressed as a weighted average of the N input samples, where the weights are determined adaptively depending on the input values. In deriving this result, it is convenient to represent the rank and the time index of each input sample simultaneously. The time index of the input with rank j is denoted by $q(j)$ so that $X_{q(1)} \geq \ldots \geq X_{q(N)}$. For example, for $N = 3$ and input ordering $X_2 \geq X_3 \geq X_1$ we get $q(1) = 2$, $q(2) = 3$, and $q(3) = 1$. Clearly $j \in \{1, \ldots, N\}$, $q(j) \in \{1, \ldots, N\}$ and $q(i) \neq q(j)$ if $i \neq j$. Throughout it is assumed that the ranks of input samples are different each other. When some input samples have the same value, any one of them may be condidered as a larger one; we simply assign different rank indices to them. For example, if $X_2 = X_3 > X_1$ then we may choose either $\{q(1) = 2, q(2) = 3, q(3) = 1\}$ or $\{q(1) = 3, q(2) = 2, q(3) = 1\}$. Using these notations, our major result is described in the following theorem.

Theorem 4-1 : The output of a canonical LWOS filter with span N can be expressed as

$$Y = \sum_{j=1}^{N} \left(y^{s(j)} - y^{s(j-1)} \right) \cdot X_{q(j)} \qquad (4-1)$$

where $s(j) = \sum_{m=1}^{j} w_{q(m)}$, $s(0) = 0$, $w_{q(m)}$ is the weight corresponding to $X_{q(m)}$, $y^{s(j)}$, and $y^{s(j-1)}$ are the outputs of the LWOS filter for the binary inputs that are the radix–2 representations of $s(j)$ and $s(j-1)$, respectively.

Proof :

$$\begin{aligned} Y &= l \cdot W'_{OS}(X) \\ &= \left(\sum_{m=1}^{w_{q(1)}} l_m \right) \cdot X_{q(1)} + \left(\sum_{m=w_{q(1)}+1}^{w_{q(1)}+w_{q(2)}} l_m \right) \cdot X_{q(2)} + \cdots \\ &\quad + \left(\sum_{m=w_{q(1)}+\cdots+w_{q(N-1)}+1}^{w_{q(1)}+\cdots+w_{q(N)}} l_m \right) \cdot X_{q(N)}. \qquad (4-2) \end{aligned}$$

Using (2-12),

$$\sum_{m=i}^{j} l_m = \sum_{m=i}^{j} \left(y^m - y^{m-1} \right) = y^j - y^{i-1}, \quad i \leq j.$$

Therefore

$$\begin{aligned} Y &= \left(y^{w_{q(1)}} \right) \cdot X_{q(1)} + \left(y^{w_{q(1)}+w_{q(2)}} - y^{w_{q(1)}} \right) \cdot X_{q(2)} + \cdots \\ &\quad + \left(y^{w_{q(1)}+\cdots+w_{q(N)}} - y^{w_{q(1)}+\cdots+w_{q(N-1)}} \right) \cdot X_{q(N)} \\ &= \sum_{j=1}^{N} \left(y^{w_{q(1)}+\cdots+w_{q(j)}} - y^{w_{q(1)}+\cdots+w_{q(j-1)}} \right) \cdot X_{q(j)}. \qquad (4-3) \end{aligned}$$

∎

In this theorem, $s(j)$ is determined according to the input ordering. As an example, consider the following.

Example 4-1 : If $N = 3$ and input ordering is $X_2 \geq X_3 \geq X_1$, then $q(1) = 2, q(2) = 3, q(3) = 1$. Thus $s(1) = w_2 = 2, s(2) = w_2 + w_3 = 3, s(3) = w_2 + w_3 + w_1 = 7$ and $Y = (y^2 - y^0) \cdot X_2 + (y^3 - y^2) \cdot X_3 + (y^7 - y^3) \cdot X_1$. Since $N = 3$, there are 3! possible input orderings; $s(j)$'s and filter outputs for the orderings are summerized in Table 2.

$s(j)$ and $s(j-1)$ of a canonical LWOS filter which are the sum of the weights $w_m = 2^{N-m}$, $1 \leq m \leq N$, have some properties useful for evaluating the outputs of LWOS filters.

Lemma 4-1 : Let s_j and s_{j-1} be binary sequences of length N that are the radix–2 representations of $s(j)$ and $s(j-1)$, respectively. Then s_j and s_{j-1} have the following properties:

(A) $q(j)$–th significant bits of s_j and s_{j-1} are 1 and 0, respectively.

(B) k–th significant bits, $k \neq q(j)$, of s_j and s_{j-1} are identical; they are 1 if $k \in \{q(1), \ldots, q(j)\}$ and 0, otherwise.

Proof : Since $w_m = 2^{N-m}$, only the m–th significant bit of the radix 2 representation of w_m is 1 and all other bits are 0. Therefore the k–th significant bit of s_j is 1 only when

$k \in \{q(1), \ldots, q(j)\}$. Clearly, the $q(j)$-th significant bit of $\mathbf{s}_j$ is 1 since $q(j) \in \{q(1), \ldots, q(j)\}$. But the $q(j)$-th significant bit of $\mathbf{s}_{j-1}$ is 0 because $s(j-1) = s(j) - w_{q(j)}$: the k-th significant bit of $\mathbf{s}_{j-1}$ is 1 only when $k \in \{q(1), \ldots, q(j-1)\}$ and it has the same value as that of $\mathbf{s}_j$ for all $k \neq q(j)$. ∎

Since the output of a stack filter is always one of the input data $(X_1, \ldots, X_N)$, $y^{s(j)} - y^{s(j-1)}$ in (4-1) is 1 for a j and 0 for the others.

Property 4-1 : For stack filters, $y^{s(j)} - y^{s(j-1)} = 1$ for a j, and $y^{s(i)} - y^{s(i-1)} = 0$ for all i, $i \neq j$, $1 \leq i \leq N$.

The proof for this is rather trivial, and is omitted. For an LWOS filter representing an arbitrary Boolean function, which is not a PBF, $y^{s(j)} - y^{s(j-1)}$ may be -1, as shown below.

Property 4-2 : Consider an LWOS filter represented by a Boolean function $f_{BF}(\mathbf{x})$ which is not a PBF. For this filter, $y^{s(j)} - y^{s(j-1)}$ in (4-1) is either 1 or 0 or -1. If $y^{s(j)} - y^{s(j-1)} = -1$ for some j, then $f_{BF}(\mathbf{x})$ includes the complement of $x_{q(j)}$, say $\overline{x}_{q(j)}$.

Proof: Since the output of $f_{BF}(\mathbf{x})$ is either 0 or 1, it is obvious that $y^{s(j)} - y^{s(j-1)}$ takes one of 0, 1, and -1. Suppose that $y^{s(j)} - y^{s(j-1)} = -1$ for some j. Let us express $f_{BF}(\mathbf{x})$ in the following sum of product form: $f_{BF}(\mathbf{x}) = x_{q(j)} \cdot g_1(\mathbf{x}) + \overline{x}_{q(j)} \cdot g_2(\mathbf{x}) + g_3(\mathbf{x})$, where $g_3(\mathbf{x})$ represents all product terms which include neigher $x_{q(j)}$ nor $\overline{x}_{q(j)}$, and $x_{q(j)} \cdot g_1(\mathbf{x})$ and $\overline{x}_{q(j)} \cdot g_2(\mathbf{x})$ represent all product terms which include $x_{q(j)}$ and $\overline{x}_{q(j)}$, respectively. Of course, neither $x_{q(j)}$ nor $\overline{x}_{q(j)}$ are included in $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$. From Lemma 4-1 (A), $y^{s(j)} = f_{BF}(\mathbf{s}_j) = g_1(\mathbf{s}_j) + g_3(\mathbf{s}_j)$ and $y^{s(j-1)} = f_{BF}(\mathbf{s}_{j-1}) = g_2(\mathbf{s}_{j-1}) + g_3(\mathbf{s}_{j-1})$. But $g_3(\mathbf{s}_j) = g_3(\mathbf{s}_{j-1})$ because $g_3(\mathbf{x})$ includes neither $x_{q(j)}$ nor $\overline{x}_{q(j)}$[See Lemma 4-1 (B)]. Thus $y^{s(j)} - y^{s(j-1)} = g_1(\mathbf{s}_j) - g_2(\mathbf{s}_{j-1})$. Therefore $y^{s(j)} - y^{s(j-1)} = -1$ indicates that $g_1(\mathbf{s}_j) = 0$ and $g_2(\mathbf{s}_{j-1}) = 1$. Following from this we can see that $g_2(\mathbf{x}) \neq 0$ in $f_{BF}(\mathbf{x}) = x_{q(j)} \cdot g_1(\mathbf{x}) + \overline{x}_{q(j)} \cdot g_2(\mathbf{x}) + g_3(\mathbf{x})$, and that $f_{BF}(\mathbf{x})$ includes $\overline{x}_{q(j)}$. ∎

For FIR filters, $y^{s(j)} - y^{s(j-1)}$ should equal to the impulse response $h_{q(j)}$. This is illustrated in the following property.

Property 4-3 : For an FIR filter with impulse response $\mathbf{h} = (h_1, \ldots, h_N)$, whose output is given $Y = \mathbf{X} \cdot \mathbf{h}'$, $y^{s(j)} - y^{s(j-1)} = h_{q(j)}$.

The proof is omitted. Although Theorem 4-1 is derived for canonical LWOS filters, it is also true for general LWOS filters. This is because an LWOS filter can always be expressed as a canonical LWOS filter. For a given LWOS filter, we can evaluate (4-1) using $\mathbf{w}$ in (2-1-) after obtaining the EBF corresponding to this filter. Hence (4-1) can be thought of as an expression of the output of general LWOS filters.

The result in Theorem 4-1 would be useful for evaluating the outputs of LWOS filters. An algorithm for LWOS filtering, based on Theorem 4-1, is described as follows.

An algorithm for LWOS filtering:

Step 1 : Sort the input samples $X_1, \ldots, X_N$.
Step 2 : Evaluate $s(j)$, $1 \leq j \leq N$.
Step 3 : Evaluate $y^{s(j)}$, $1 \leq j \leq N$.
Step 4 : Evaluate Y, using (4-1).

In this algorithm, Step 1 may be implemented by using the running sorting algorithm in [5] which requires $O(N)$ comparisons. In Step 2, $s(j)$, $1 \leq j \leq N$, can be obtained by j swapping operations. Step 3 may be implemented by storing the extended truth table. Finally, Step 4 requires $O(N)$ multiplications and $O(N)$ additions. If we implement LWOS filtering directly according to its definition, $O(N)$ comparisons, $O(K)$ additions and $O(K)$ multiplications are required. Since K is the dimension of the linear combination vector $\mathbf{l}$ and $K \geq N$, the algorithm would be more efficient than the direct implementaion when $K \gg N$.

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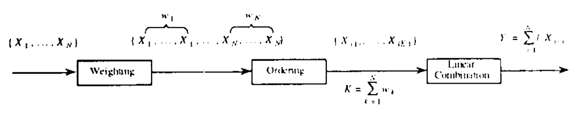

Fig. 1. LWOS filtering. Here X_i is the i-*th* sample from the left of the window. $X_{(j)}$ is the j-*th* largest sample. $\mathbf{w} = (w_1, w_2, \ldots, w_N)$, $\mathbf{l} = (l_1, l_2, \ldots, l_K)$.

i	$\mathbf{x}^i$	y^i	$\mathbf{w} \cdot \mathbf{x}^i$
1	0 0 1	1	1
2	0 1 0	1	1
3	0 1 1	0	2
4	1 0 0	1	1
5	1 0 1	0	2
6	1 1 0	0	2
7	1 1 1	1	3

Table 1. Truth Table of $f(\mathbf{x}) = x_1 \overline{x}_2 \overline{x}_3 + x_1 x_2 x_3 + \overline{x}_1 \overline{x}_2 x_3 + \overline{x}_1 x_2 \overline{x}_3$

Ordering	{ s(1) s(2) s(3) }	Y
$X_1 \geq X_2 \geq X_3$	{ 4 6 7 }	$(y^4 - y^0)X_1 + (y^6 - y^4)X_2 + (y^7 - y^6)X_3$
$X_1 \geq X_3 \geq X_2$	{ 4 5 7 }	$(y^4 - y^0)X_1 + (y^5 - y^4)X_3 + (y^7 - y^5)X_2$
$X_2 \geq X_1 \geq X_3$	{ 2 6 7 }	$(y^2 - y^0)X_2 + (y^6 - y^2)X_1 + (y^7 - y^6)X_3$
$X_2 \geq X_3 \geq X_1$	{ 2 3 7 }	$(y^2 - y^0)X_2 + (y^3 - y^2)X_3 + (y^7 - y^3)X_1$
$X_3 \geq X_1 \geq X_2$	{ 1 5 7 }	$(y^1 - y^0)X_3 + (y^5 - y^1)X_1 + (y^7 - y^5)X_2$
$X_3 \geq X_2 \geq X_1$	{ 1 3 7 }	$(y^1 - y^0)X_3 + (y^3 - y^1)X_2 + (y^7 - y^3)X_1$

Table 2. $s(j)$ and Y corresponding to each ordering when $N = 3$