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외생변수가 급변하는 상황에서의 확산과정 모형화*

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Modeling Diffusion Process Under Abrupt Changes of External Factors*

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■ Abstract ■

In reality, we can observe anomalous diffusion patterns of cycle-recycle or rejuvenation. Abrupt changes in the market environment such as sudden currency devaluation or change in government policy or those in marketing strategies such as drastic repositioning can lead to such atypical diffusion patterns. The authors present extended Bass models that incorporate effects of such abrupt changes of external factors into the hazard rate and the market potential. Using a set of compact-car data affected by a drastic change in the government policy, they illustrate the strengths of the proposed models.

Keyword : Diffusion Process, External Factors, Hazard Rate

1. Introduction

New product diffusion patterns are typically of bell shape. In reality, however, we can observe anomalous diffusion patterns ([9], p.148 ; [10], p.515] such as cycle-recycle [5] or scalloped patterns [4]. Abrupt and drastic changes in the market environment or in corporate strategies can lead to such atypical diffusion patterns as those of rejuvenation after a slow down or decline in

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growth. For example, sales of the Boeing 727 and 747 were regenerated after a redesign and those of The Crying Game, directed by Neil Jordan in 1992, were surged up after the Academy Award nomination ([9], p.149 and p.196).

In marketing, the Bass model [2] has been extended to incorporate various types of diffusion patterns [11]. One stream of extensions focused on incorporating or analyzing the influence of marketing strategies on the diffusion (e.g., [2, 7]). Another stream focused on developing flexible diffusion models that can fit more general diffusion patterns (e.g., [6]). However, the extended Bass models mainly focus on describing or forecasting effects of continuous or gradual changes of external factors. Thus, the models tend to show limitations in describing anomalous sales patterns resulting from abrupt changes of external factors. In this note, we propose extensions of the Bass model that incorporate effects of external changes that are discontinuous in nature. The models incorporate discontinuous impacts in the hazard rate and the market potential of the Bass model.

An abrupt change may occur in the market environment. The currency crisis that happened in many countries in 1997 is an example. An abrupt change may also come from the firm. When a firm drastically changes basic marketing strategies such as targeting or positioning, anomalous diffusion patterns can be observed.

In the next session, the models are proposed. After discussing some properties of the models, we apply them to diffusion data of compact-cars in the Korean market. The sales of compact-cars were regenerated after an announcement of a new government policy that encourages the sales of gas saving compact-cars.

2. Model Development

Three models are presented to describe effects of exogenous and abrupt changes on new product diffusion. While abrupt impacts are incorporated in the hazard rate in the first model, they are incorporated in the market potential in the second model. The third is a general model in which effects of abrupt changes are modeled in both the market potential and the hazard rate.

2.1 When Abrupt Changes Influence the Hazard Rate : M1 Model

Let $f(t)$ be the density function describing the time of adoption of a population and $F(t)$ the cumulative function. Then, the hazard rate is defined as :

$$\lambda(t)=\frac{f(t)}{1-F(t)}. \tag{1}$$

The hazard rate of the Bass Model is specified as :

$$\lambda_B(t)=p+qF(t). \tag{2}$$

We generalize the Bass hazard function as follows :

$$\lambda(t)=\lambda_B(t)\Big[1+\sum_j \lambda_D(\tau_j)\delta(t-\tau_j)\Big], \tag{3}$$

where $\tau_1<\tau_2<\cdots<\tau_n$, and $\delta(t-\tau_j)$ is a Dirac delta function defined as :

$$\delta(x)=\begin{cases}\infty, & x=0\\ 0, & otherwise.\end{cases} \tag{4}$$

The model assumes that an abrupt external shock appears at time τ_j. $\lambda_D(t)$ is the amount of impact that the shock at τ_j brings into the hazard rate. Without abrupt changes, the hazard rate is $\lambda_B(t)$. For parsimony, we further assume that

$\lambda_D(\tau_j)$ is ϕ_j for all j's. Then, the hazard rate is :

$$\lambda(t) = \frac{f(t)}{1-F(t)} = [p+qF(t)]\Big[1+\sum_j \phi_j \delta(t-\tau_j)\Big]. \quad (5)$$

From equation (5), we can derive :

$$F(t) = F_B\Big(t + \sum_{j|\tau_j \le t} \phi_j\Big) \text{ and} \quad (6)$$

$$F_B(x) = \frac{1-e^{-(p+q)x}}{(q/p)e^{-(p+q)x}+1} \quad (7)$$

where $F_B(x)$ is the cumulative function of the original Bass model.

When an abrupt shock occurs at τ_j, the normal Bass diffusion process stops and jumps to the state that is normally observed at $\tau_j + \phi_j$. Then, the normal Bass diffusion process continues from the new state until another shock occurs at τ_{j+1}. Note that, because the diffusion process jumps forward or backward as external shocks occur, the condition of $\int_0^\infty F'(x)dx = 1$ is not satisfied in the model. Thus, we introduce an adjustment factor, A_j, into the survival function, $1-F(t)$, of equation (8). Our adjusted distribution function at time t is expressed as : 1)

$$G(t) = 1-[1-F(t)]\prod_{j|\tau_j \le t} A_j,$$

$$\text{where } A_j = \frac{1-F_B\Big(\tau_j + \sum_{k=0}^{k=j-1}\phi_k\Big)}{1-F_B\Big(\tau_j + \sum_{k=0}^{k=j}\phi_k\Big)} \quad (8)$$

If there is no external shock, the equation (8) is identical to the distribution function of the Bass Model, implying that the Bass model is a special case of the proposed model.

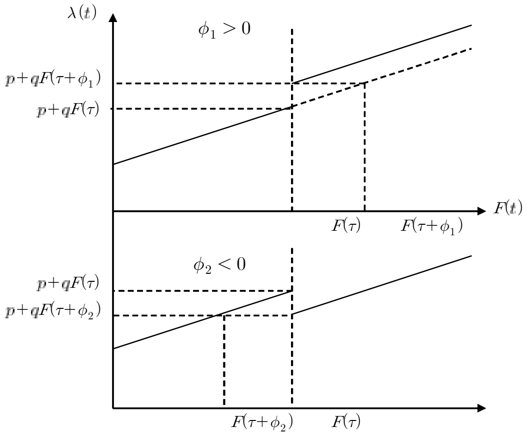
When there is only one abrupt change, the hazard rate becomes :

1) The hazard rate is not affected by such an adjustment factor. See Kalbfleish and Prentice ([3], p.8).

$$\lambda(t) = \frac{F'(t)}{1-F(t)} = \frac{G'(t)}{1-G(t)} = p+qF(t)$$

$$= \begin{cases} p+qF_B(t) = p+qG(t), & t < \tau, \\ p+qF_B(t+\phi) = p+q\Big[1-\dfrac{1-G(t)}{A}\Big], & t \ge \tau. \end{cases} \quad (9)$$

If the diffusion process jumps forward at τ, i.e., $\phi > 0$, then the adjustment factor A in equation (9) is greater than one. Therefore, the word of mouth effect after the shock becomes greater than that before the shock. In this case, the hazard rate is increased after the shock. [Figure 1] shows how the hazard rate changes after the external shock. When $\phi > 0$, some potential customers who are supposed to adopt the new product in the future are buying the product earlier due to the shock. On the other hand, if ϕ is negative, the diffusion process is decelerated compared to that expected by the Bass model, implying that some potential adopters are postponing the purchase of the new product.



[Figure 1] Illustration of the Hazard Rate under an Abrupt Change of External Factor

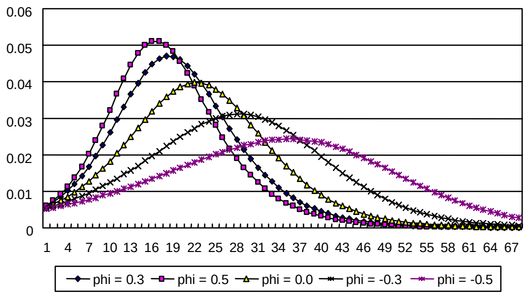
To compare the properties of the proposed models with those of the Generalized Bass Model (GBM) developed by Bass, Krishnan and Jain [2], suppose that advertising and price influence the hazard rate. Denote advertising and price at time j as $ADV(j)$ and $\Pr(j)$, respectively. The impact of advertising and price can be incorporated in ϕ_j as :

$$\sum_{j|\tau_j \le t} \phi_j = \sum_{j|\tau_j \le t} \phi_{Aj} \sum_{j|\tau_j \le t} \phi_{Fj} = \sum_{j|\tau_j \le t} \beta_A \frac{\Delta ADV(j)}{ADV(j-1)} + \sum_{j|\tau_j \le t} \beta_P \frac{\Delta \Pr(j)}{\Pr(j-1)}. \tag{10}$$

where $\Delta ADV(j) = ADV(j) - ADV(j-1)$ and $\Delta \Pr(j) = \Pr(j) - \Pr(j-1)$.

From equation (10), we know that equation (8) is very similar to that of the GBM. Let us assume the regularity in the behavior of advertising, i.e., the ratio between net change in advertising and total advertising level does not change over time. Similarly, let us assume the regularity in price. Then ϕ_j can be expressed as a constant ϕ for all j leading to equation (6).

Bass, Krishnan and Jain [2] showed that, when there is regularity of changes in the levels of decision variables over time, the Bass Model works quite well. Even if the impact of external factors on the hazard rate is unknown, if it occurs consistently over time, the diffusion pattern could be still bell-shaped as shown in [Figure 2]. The Bass Model can produce a good description of actual diffusion pattern in such situations. Like the GBM model, the proposed models can describe and forecast the situations when external effects do not meet the regularity. The focus of the GBM is to model the effects of marketing variables that vary period to period. On the other hand, the model proposed, equation (8), incorporate the effects of shocks that break out abruptly.



[Figure 2] Sales Patterns for Different Impacts of External Factor Influencing Hazard Rate

Now, let us consider such situations that the impact of external factors on the hazard rate is loosely connected to time. We can observe such situations when the impact of external factors is not linear over the time t : for example, the shape of impact of external factors (impact function) is a step function or a trigonometry function like sines and cosines.

2.2 When Abrupt Changes Influence the Market Potential : M2 Model

The adoption function of the Bass model is :

$$f(t) = [1 - F(t)][p + qF(t)]. \tag{11}$$

Suppose that the market potential of the new product is changed abruptly by an external shock. Then, $f(t)$ is affected by the shock because it is a function of the market potential. Our generalized $f(t)$ is expressed as :

$$f(t) = [1 - F(t)][p + qF(t)]\left[1 + \sum_{j|\tau_j \le t} \lambda_D(\tau_j)\delta(t - \tau_j)\right]. \tag{12}$$

$\lambda_D(t)$ is the amount of impact that the discontinuous changes of market potential at τ_j brings into the density function. For parsimony, we further assume that $\lambda_D(\tau_j)$ is μ_j for all j's. Then, the hazard function is :

$$\lambda(t) = \frac{f(t)}{1-F(t)}[p+qF(t)]\Big[1+\sum_{j|\tau_j \le t}\mu_j\delta(t-\tau_j)\Big]. \quad (13)$$

From equation (13), we can derive :

$$F(t) = F_B\Big(t+\sum_{j|\tau_j \le t}\mu_j\Big). \quad (14)$$

The mathematical derivation of (14) is presented in Appendix A. Because the condition of $\int_0^\infty F'(x)dx = 1$ is not satisfied in the model, we introduce an adjustment factor to get the adjusted distribution function at t as :

$$G(t) = 1-[1-F(t)]\prod_{j|\tau_j \le t}A_j, \text{ where}$$

$$A_j = \frac{1-F_B\Big(\tau_j+\sum_{k=0}^{k=j-1}\mu_k\Big)}{1-F_B\Big(\tau_j+\sum_{k=0}^{k=j}\mu_k\Big)} \quad (15)$$

To apply the model to sales data, effects of changing market potentials should be incorporated into the sales function. Suppose there is an external shock at τ changing the market potential from m_0 to $m_0[1/(1+\psi)]$. Then, the following equation holds at time τ :

$$\frac{F(\tau-\epsilon)}{F(\tau)} = \frac{F_B(\tau)}{F_B(\tau+\mu)} = \frac{Y(\tau-\epsilon)/m_0}{Y(\tau)/(m_0/(1+\psi))} = \frac{1}{1+\psi} \quad (16)$$

where $Y(t)$ is a cumulative sales at time t. Note that, we are assuming $Y(\tau) = T(\tau-\epsilon)$ because the market potential changes at τ. Thus, we know the ratio between the market potentials before and after the shock, i.e., $1/(1+\psi)$, if we know $F(\tau-\epsilon)$ and $F(\tau)$. The cumulative sales function with multiple shocks is described as :

$$Y(t) = G(t)m(t), \text{ where}$$

$$m(t) = m_0\prod_{j|\tau_j \le t}\frac{F_B\Big(\tau_j+\sum_{k=0}^{k=j-1}\mu_k\Big)}{F_B\Big(\tau_j+\sum_{k=0}^{k=j}\mu_k\Big)}. \quad (17)$$

From equation (17), we can observe that when the market potential is increased (decreased), the parameter μ_j is negative (positive).

M1 model implies that sales pattern can be systematically getting shorter (longer) over time if the pace of business activities is accelerated (decelerated) and it increases (decrease) the adoption probability so that the cumulative sales are increased. From M2 model expressed equation (17), however, we know that sales curve is shifted to the right and the upper side (level up) when the market potential increases. Inversely, the sales curve is shifted to the left and the lower side (level down) when the market potential decreases. That is, if the impact of external factor increases (decrease) the market potential, the sales pattern can be systematically getting longer (shorter) and the sales curve can be systematically shifted to the upper side (lower side). Thus, even if marketing activity is increased, we can not conclude the product life cycle (PLC) is always shorter because it can increase the market potential. The market potential can be changed by steep price decline, epoch-making technology, etc..

2.3 When Abrupt Changes Influence Both the Market Potential and the Hazard Rate

Combining the models suggested in 2.1 and 2.2, we propose a general model that can be used when abrupt external changes influence both the market potential and the hazard rate

$$G(t) = 1-[1-F(t)]\prod_{j|\tau_j \le t}A_j \text{ , where}$$

$$A_j = \frac{1-F_B\Big(\tau_j+\sum_{k=0}^{k=j-1}\phi_k+\sum_{k=0}^{k=j-1}\mu_k\Big)}{1-F_B\Big(\tau_j+\sum_{k=0}^{k=j}\phi_k+\sum_{k=0}^{k=j}\mu_k\Big)}. \quad (18)$$

The cumulative sales function can be described as :

$Y(t) = G(t)m(t)$, where

$$m(t) = m_0 \prod_{j|\tau_j \leq t} \frac{F_B\left(\tau_j + \sum_{k=0}^{k=j-1} \phi_j + \sum_{k=0}^{k=j-1} \mu_k\right)}{F_B\left(\tau_j + \sum_{k=0}^{k=j-1} \phi_j + \sum_{k=0}^{k=j} \mu_k\right)}. \quad (19)$$

Note that market potential in equation (17) and (19) can be changed over time due to external shocks unlike that in the Bass model. Therefore, we need to estimate the number of ultimate adopters after observing the shocks in the diffusion process. Appendix B shows the calculation procedure of the ultimate market potential. Also, we note that, when carry-over effects are expected for abrupt external shocks, we introduce additional dummies into the model to reflect various carryover patterns such as exponentially increasing or decreasing patterns over time.

On the hazard function or the market potential, the impact of external factor can be transposed into a function of time. If the carryover effects of the external factor must be considered in the diffusion process, it can be considered as multiple external factors, whose impacts have a particular pattern (for example, exponentially decreasing or decreasing like a step function) over time. The dummy type's impact of external factor can also be expressed as a function of time. If we calibrate the proposed models with monthly, bimonthly, and annual data, we should think that the estimated parameters' unit becomes monthly, bimonthly, and annual time, respectively. If the effects of external factor average out during the unit time, we can not extract it from the diffusion process. We can intuitively think that our approach is more useful to characterize the external factor and shows better fitting results if we choose the proper unit time to the extent that the effects of external factor do not average out.

3. Empirical Applications

To estimate the model parameters, we rely on the nonlinear least square method (NLS) proposed by Srinivasan and Mason [12] and used by Bass, Krishnan, and Jain [2]. (See Appendix C.) The models are estimated for a set of diffusion data demonstrating a sales rejuvenating pattern.

One of Korean automobile companies introduced a compact-car of 800cc called "T-car" in June 1991. The sales of the car increased gradually reaching its peak in the third quarter of 1993. Then, its sales declined until it started to show an explosive growth from the third quarter of 1996. In the period, the Korea government publicized its new policy of boosting sales of gas-saving compact cars that included a drastic tax reduction,[2] a reduction of insurance premiums, 50% discounts of highway tolls, and 50% discount of parking fees.

To investigate the diffusion pattern of the T-car, we collected the quarterly data covering from 1991 to 1997. The T-car was only a compact-car during the periods in Korea. We assumed that the abrupt change in the government policy increases the market potential. We also

2) As of January 1 in 1999, the registration tax was 2% of total vehicle price for a compact car and 5% for the other car. All the new car buyers in big cities of South Korea had to buy public bonds which amount to 9% of total vehicle price for a compact car and more than 9% for the other car.

<Table 1> Parameter Estimates for T-car Adoption in Korea

		R^2	SSE[f]	p	q	k_1	k_2	m_0	c
Sales-Domain	BM[a]	0.554	395,935,821	0.039 (0.0124)[d]	0.061 (0.4579)			319,327	0. 896 (0.0006)
	NUI[b]	0.642	317,541,126	0.081 (0.0001)	0.763 (0.0221)	4.687 (0.0007)		199,772 (0.0001)	0. 788 (0.0001)
Time-Domain	PM[c]	0.877	109,388,196	0.041 (0.0001)	0.149 (0.0001)	-15.401 (0.0001)	2.047 (0.0054)	244,709 (0.0001)	

Note) [a] denotes the Bass Model with changed market potential :
sales=$[p+qY/\{m_0(1+cD)\}][Y-m_0(1+cD)]$ where D=0 if $t<\tau$ and D=1 if $t \geq \tau$.
[b] denotes the Non-uniform Influence Model with changed market potential :
sales= $[p+q[Y/\{m_0(1+cD)\}]^{k1}][Y-m_0(1+cD)]$ where D=0 if $t<\tau$ and D=1 if $t \geq \tau$.
[c] denotes the proposed model.
[d] denotes p-value.
[f] denotes Sum of the Squared Errors.

assumed that the policy made the company change its marketing strategies by drastically increasing its marketing efforts. Thus, we used the general model of equation (18) in this case. Our assumption was that

$\phi_h = k_2$ for all j, and $u_1 = k_1$, $u_j = 0$ for $j > 1$,

making the cumulative impacts of the external factor :

$$\sum_{j|t \leq \tau_j} \phi_j + \sum_{j|t \leq \tau_j} \mu_j = k_2(t-\tau) + k_1$$
where $t \geq \tau$ for $t = 1,2,3,\cdots,n$.

While k_1 determines the impact of the new policy on the market potential, k_2 determines its impact on the hazard rate. For empirical comparisons, we selected two models that can reflect changing market potentials - variations of the BM and the NUI model incorporating varying market potentials. The models were estimated by the non-linear least square method. <Table 1> reports the empirical results. [Figure 3-1] graphically shows how the models fit the actual sales data.

The proposed model with $R^2 = 0.877$ fits the data better than the other models. Also, all the estimated parameters of the proposed model are significant ($\alpha < 0.01$). The estimate of parameter k_1 is -15.4 implying that that the market potential was increased by the government policy. The model suggests that the potential was increased from 244,709 to 1,719,443[3]. This implies that the number of ultimate adopters throughout the diffusion process went up from 244,709 to 447,828[4]. In addition, $k_2 = 2.047$ implies that the hazard rate was changed by the new policy. [Figure 3-2] shows that the new policy shortened the diffusion process by making potential adopters buy the compact-car earlier than they would normally do. The model estimates that the policy shortened the diffusion process by the time span of 2.047 quarter.

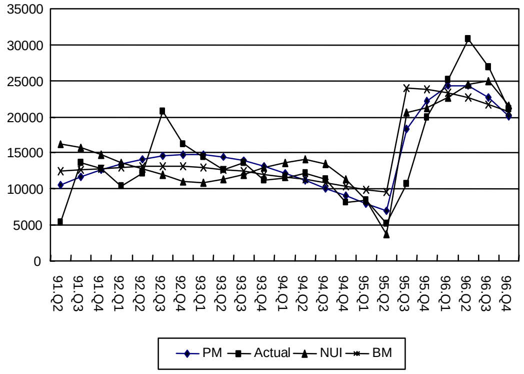
We compared the quality of one step ahead forecasting among the BM, NUI, and proposed model. After estimating each model using data of n periods, we let the model forecast adoptions

3) m_0 = 240,857 and changed market potential = m_0 $[F_B(18)/F_B(18+k_1)]$ = 240,709 * [7.139].
4) 447,828 = $m_0 * F_B(18) + m_0 * [F_B(18)/F_B(18+k_1)] * [1-F_B(18)]$.

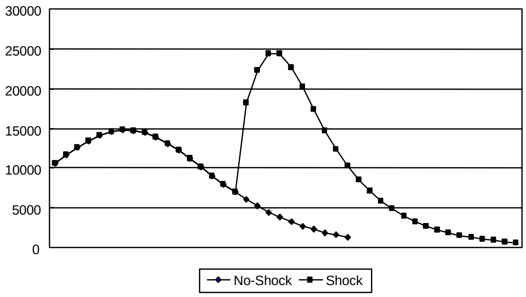
〈Table 2〉 Results of One-step-ahead Forecasting

Mean Squared Error $*10^3$						
Product	With BM	With NUI	With PM	Percentage Reduction in Error by Using PM over BM	Percentage Reduction in Error by Using PM over NUI	Average Over
T-car	107,245.0	93,937.1	20,944.8	80.47%	77.70%	4 forecast-periods

for the $(n+1)$th period. Iterating the procedure, we forecasted sales of 4 different quarters for each model for the data set. <Table 2> reports the mean squared error of the forecasts. As shown in the Table, the proposed model reduced the errors by more than 77.7% compared to other models.



[Figure 3-1] Comparison of Actual and Fitted Sales



[Figure 3-2] Expected Sales of T-car with and without the Shock

[Figure 3] Comparison of Actual and Fitted Sales for T-car in Korea

4. Discussion

In this paper, we proposed and empirically estimated extended Bass models that can incorporate effects of abrupt changes of external factors on the diffusion process. We explicitly modeled such abrupt impacts on the hazard rate and the market potential. There are several advantages of using the proposed models.

First, the proposed models allow us to analyze the diffusion process that shows anomalous patterns such as cycle-recycle or scalloped patterns. Our models have closed-form solutions in the time domain making them easily applicable. Also, they are extended Bass models that maintain basic properties of the Bass model. Unlike other extended Bass models, they can be applied to diffusion data that reflects abrupt changes in the environment or in marketing strategies.

Second, fitting actual data with the proposed model, we can identify the direction and measure the degree of impact of abrupt changes in the external factors. Using the estimated model, we can compare the diffusion processes with and without the external shocks. Also, we can forecast how the diffusion process will change in the future after the occurrence of shocks based on the estimated models.

Third, we can do the simulation analysis using the models under a variety of scenarios regarding the external shocks. We may generate possible scenarios on abrupt changes expected in the future based on meta-analysis of past situations or on consumer surveys. Then, we can build contingency plans after analyzing possible impacts of those potential abrupt changes.

The models presented in this article can be extended further if necessary. They can accommodate other type of diffusion processes than the Bass, if we know a priori that a certain distribution fits the situation better. That is, we can use other specifications than $\lambda_B(t)$ for the hazard rate. For example, Bewley and Fiebig [3] used the distribution of a flexible logistic growth in the telecommunications market. Also, we can incorporate other types of carryover patterns of external shocks than the exponential type used in this note.

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Appendices

Appendix A : Derivation of Equation (13)

The hazard rate of the Bass model is expressed as :

$$\frac{F'(t)}{1-F(t)}=p+qF(t). \tag{A.1}$$

Let us define $R(t)$ such that $m(t)=m_0/R(t)$. Then, a generalized cumulative adoption probability can be expressed as :

$$F(t)=\frac{Y(t)}{m_0}R(t), \tag{A.2}$$

The right-hand-side of equation (A.1) is

$$p+qF(t)=p+q\frac{Y(t)}{m_0}R(t) \text{ and} \tag{A.3}$$

the left-hand-side is

$$\frac{F'(t)}{1-F(t)}=\frac{(Y'(t)/m_0)R(t)}{1-(Y(t)/m_0)R(t)}\left(1+\frac{Y(t)}{Y'(t)}\frac{R'(t)}{R(t)}\right) \text{ if } Y'(t)>0 \quad \text{because} \tag{A.4}$$

$$F'(t)=\frac{d}{dt}\left(\frac{Y(t)}{m_0}R(t)\right)=\frac{Y'(t)}{m_0}R(t)+\frac{Y(t)}{m_0}R'(t). \tag{A.5}$$

Thus, equation (A.1) is :

$$\frac{(Y'(t)/m_0)R(t)}{1-(Y(t)/m_0)R(t)}\left(1+\frac{Y(t)}{Y'(t)}\frac{R'(t)}{R(t)}\right)=p+q\frac{Y(t)}{m_0}R(t) \quad \text{and} \tag{A.6}$$

$$\frac{(Y'(t)/m_0)R(t)}{1-(Y(t)/m_0)R(t)}=\left(p+q\frac{Y(t)}{m_0}R(t)\right)\left(1+\frac{Y(t)}{Y'(t)}\frac{R'(t)}{R(t)}\right)^{-1}. \tag{A.7}$$

Denoting $K(t)=Y(t)/m_0$, equation (A.7) becomes :

$$\frac{K'(t)R(t)}{1-K(t)R(t)}=(p+qK(t)R(t))\left(1-\frac{K(t)R'(t)}{K(t)R'(t)+K'(t)R(t)}\right), \tag{A.8}$$

$$\frac{K'(t)R(t)}{(1-K(t)R(t))(p+qK(t)R(t))}=1-\frac{K(t)R'(t)}{K(t)R'(t)+K'(t)R(t)}, \text{ and} \tag{A.9}$$

$$\frac{K'(t)R(t)}{(1-K(t)R(t))(p+qK(t)R(t))}=1-\frac{[R'(t)/R(t)]}{[R'(t)/R(t)]+[K'(t)/K(t)]}. \tag{A.10}$$

Let us assume that $R'(t)$ is a Dirac function defined as :

$$R'(t)=\Sigma\psi_j\,\delta(t-\tau_j) \tag{A.11}$$

$$\text{where } \delta(x)=\begin{cases}\infty, & x=0,\\ 0, & o/w.\end{cases} \tag{A.12}$$

Furthermore, if we assume that $R(t)=1$ in interval $(0,\tau_1)$, then, the integration constant is 1. Thus, we can derive the step function $R(t)$ as :

$$R(t)=1+\sum_{j|\tau_j\le t}\psi_j. \tag{A.13}$$

The assumption of equation (A.11) drives the dynamic market potential to be expressed as a step function. Let us define $\Psi_j=\sum_{j|\tau_j\le t}\psi_j$. Then, we get the following result from equation (A.10) :

$$\frac{R'(t)}{R(t)}=\Sigma\left(\frac{\psi_j}{1+\Psi_j}\right)\delta(t-\tau_j). \tag{A.14}$$

Using equation (A.14), we can express the right-hand-side of equation (A.10) as

$$1-\frac{[R'(t)/R(t)]}{[R'(t)/R(t)]+[K'(t)/K(t)]}=1-\frac{\Sigma(\psi_j/1+\Psi_j)\delta(t-\tau_j)}{\Sigma(\psi_j/1+\Psi_j)\delta(t-\tau_j)+[K'(t)/K(t)]}=1-\Sigma\sigma(t-\tau_j)\ \text{ where} \tag{A.15}$$

$$\sigma(x)=\begin{cases}1, & x=0,\\ 0, & otherwise.\end{cases} \tag{A.16}$$

Consequently, equation (A.15) is expressed as '0' or '1' because $\infty/\infty=1$. From equation (A.10) and (A.15), we know that the following relation holds :

$$\int_0^t\left[\frac{K'(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)}{\left[1-K(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)\right]\left[p+qK(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)\right]}\right]du=\int_0^t[1-\Sigma\sigma(u-\tau_j)]du. \tag{A.17}$$

To derive $K(t)$ from equation (A.17), let us define :

$$H(t)=\int_0^t\left[\frac{K'(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)}{\left[1-K(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)\right]\left[p+qK(u)\left(1+\sum_{j|\tau_j\le u}\psi_j\right)\right]}\right]du \tag{A.18}$$

and

$$H_j(t)=\int_0^t\left[\frac{K'(u)\left(1+\sum_{i=0}^{j}\psi_i\right)}{\left[1-K(u)\left(1+\sum_{i=0}^{j}\psi_i\right)\right]\left[p+qK(u)\left(1+\sum_{i=0}^{j}\psi_i\right)\right]}\right]du \tag{A.19}$$

where $\psi_0=0,\ j=0,1,2,\cdots,n.$

Then, in the interval $[\tau_j,\tau_{j+1})$ for all j,

$$H(t)=H_j(t)-\sum_{i=1}^{j}(H_i(\tau_j)-H_{i-1}(\tau_i-\epsilon))=t\ \text{ because }\int\sigma(x)dx=0,\ \text{and} \tag{A.20}$$

$$H_j(t)=t+\sum_{i=0}^{j}\mu_i. \tag{A.21}$$

Denoting

$$E_j(t) = \left(1 + \sum_{i=0}^{j} \psi_i\right) K(t), \tag{A.22}$$

equation (A.19) is simplified as

$$\int \left[\frac{E_J'(u)}{[1-E_j(u)][p+qE_j(u)]}\right] du = t + \sum_{i=0}^{j} \mu_i, \tag{A.23}$$

$$E_j(t) = F_B\left(t + \sum_{i=0}^{j} \mu_i\right) \text{ for all } j, \tag{A.24}$$

$$K(t) = \left(1 + \sum_{i=0}^{j} \psi_i\right)^{-1} F_B\left(t + \sum_{i=0}^{j} \mu_i\right). \tag{A.25}$$

For all the intervals,

$$K(t) = \left[1 + \sum_{j|\tau_j \le t} \psi_j\right]^{-1} F_B\left(t + \sum_{j|\tau_j \le t} \mu_j\right). \tag{A.26}$$

Because $F(t) = K(t)R(t)$,

$$F(t) = \left[1 + \sum_{j|\tau_j \le t} \psi_j\right]\left[1 + \sum_{j|\tau_j \le t} \psi_j\right]^{-1} F_B\left(t + \sum_{j|\tau_j \le t} \mu_j\right) = F_B\left(1 + \sum_{j|\tau_j \le t} \mu_j\right). \tag{A.27}$$

Appendix B : Total Number of Ultimate Adopters

In equation (18), let us define $m_i = m(t)$ in the interval $[\tau_i, \tau_{i+1})$ for $i = 1, 2, \cdots, n$. Then, the total number of ultimate adopters ($TNUA$) can be computed by the following equation :

$$TNUA = m_0 G(\tau_1 - \epsilon) + \sum_{i=1}^{n-1} m_i (G(\tau_{i+1} - \epsilon) - G(\tau_i)) + m_n (1 - G(\tau_n)), \tag{B.1}$$

where

$$G(\tau_i - \epsilon) = 1 - \left[1 - F_B\left(\tau_i + \sum_{j|\tau_j \le \tau_{i-1}} \mu_j + \sum_{j|\tau_j \le \tau_{i-1}} \phi_j\right)\right] \prod_{j|\tau_j \le \tau_{i-1}} A_j. \tag{B.2}$$

Appendix C : Estimation Method

In equation (18), let us denote the initial market potential as m_0 and consider the additive error term e_i for the ith time interval. Let the ith time interval to be $[t_{i-1}, t)$ because the adoption probability and market potential are left-continuos functions. Our premise is that the time τ_j when the jth shock occurs corresponds to one of t_i's. This approach of assuming the left continuous intervals can reflect the discontinuous changes of external factors. Then, sales denoted as S_i is :

$$S_i = G(t_i - \epsilon) m(t_i - \epsilon) - G(t_{i-1}) m(t_{i-1}) + e_i, \tag{C.1}$$

for $i = 1, 2, 3, \cdots T$, where

$$G(t_i - \epsilon) = 1 - \left[1 - F_B\left(t_i + \sum_{j|\tau_j \le t_{i-1}} \mu_j + \sum_{j|\tau_j \le t_{i-1}} \phi_j\right)\right] \prod_{j|\tau_j \le t_{i-1}} A_j \quad \text{and} \tag{C.2}$$

$$m(t_i - \epsilon) = m(t_{i-1}). \tag{C.3}$$