Automatica 48 (2012) 2607–2613

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Global regulation of a class of feedforward and non-feedforward nonlinear systems with a delay in the input ☆

Min-Sung Koo [a], Ho-Lim Choi [b,1], Jong-Tae Lim [a]

[a] Department of Electrical Engineering, Korea Advanced Institute of Science and Technology, 373-1 Guseong-dong, Yuseong-gu, Daejeon, 305-701, Republic of Korea

[b] Department of Electrical Engineering, Dong-A University, 840 Hadan2-Dong, Saha-gu, Busan, 604-714, Republic of Korea

ARTICLE INFO

Article history:
Received 25 November 2010
Received in revised form
17 February 2012
Accepted 16 May 2012
Available online 19 July 2012

Keywords:
Time-delay system
Global regulation
Feedforward and non-feedforward

ABSTRACT

In this paper, we propose a new state feedback controller using dynamic gain for input-delayed systems with high-order nonlinearity terms in both feedforward and non-feedforward forms. The controller design is based on a reduction method to remove the input delay and a gain scaling technique involving appropriate powers of high-order nonlinearity. As a result, more generalized systems containing feedforward and nonfeedforward terms with an input delay are regulated when the proposed power order condition of the nonlinear function is satisfied. An example is given to show the generality of our result over existing results.

1. Introduction

Time-delay systems have received much attention with regard to stabilization or regulation issues (Choi & Lim, 2006, 2010a,b; Fang & Lin, 2006; Mazenc, Mondié, & Niculescu, 2003; Yue, 2004; Zhang, 2006, 2009). The stabilization or regulation problems of time-delay nonlinear systems with uncertain nonlinearity and a delay in the input have been the subject of several independent studies, recently. Regarding input-delayed systems, there has been much research on the stabilization or regulation of a chain of integrators where there is a delay in the input. In Fang and Lin (2006), a state feedback stabilizing controller based on a forwarding technique was proposed for a known delay in the input. When a delay in the input is constant but unknown, an output feedback control scheme in which a low-gain parameter is tuned online was proposed by Choi and Lim (2006). When the input delay is time-varying, a dynamic gain approach was developed by Choi and Lim (2010a). In addition, some low-gain feedback laws were established by Zhou, Duan, and Li (2009, 2011). Under input saturation, some bounded control approaches were established by Fang and Lin (2006), Teel (1993) and Zhou, Duan, and Li (2008). When a chain of integrator systems has a perturbed nonlinearity under an input-matching condition, it has been shown that the system stability is strongly affected by the suggested four types of nonlinearity in Choi and Lim (2010a).

Meanwhile, the stabilization or regulation problems of the feedforward systems have been researched and many related studies in either state or output feedback forms have been published in recent years (Chen & Huang, 2009; Jankovic, 2009; Karafyllis, 2006; Krishnamurthy & Khorrami, 2007; Krstic, 2002, 2010; Ye, 2003, 2005; Ye & Unbehauen, 2004). It is of note that a chain of integrator systems with a delayed input can be viewed as a special class of feedforward systems (see Choi & Lim, 2006, 2010a). Global stabilization under the condition that the nonlinearity satisfies a certain feedforward condition is solved in Zhang (2006, 2009) when there is a delay in the input. In Zhang (2006), feedforward nonlinear systems characterized by a linear growth condition are handled. In Zhang (2009), an input-delayed chain of power integrators with high order feedforward nonlinearity is considered. However, when the order of the power integrator is one, the condition of nonlinearity is same as the one in Zhang (2006). However, in most of these studies, the systems considered and control methods are naturally limited to a certain class of feedforward systems only. Thus, if the systems contain some additional 'non-feedforward' terms, most of the existing results become non-applicable. Even though, in Krishnamurthy and Khorrami (2007), the authors developed a dynamic-gain state

☆ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0007325). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Michael Malisoff under the direction of Editor Andrew R. Teel.

E-mail addresses: goose@kaist.ac.kr (M.-S. Koo), hlchoi@dau.ac.kr (H.-L. Choi), jtlim@stcon.kaist.ac.kr (J.-T. Lim).

[1] Tel.: +82 051 200 7734; fax: +82 051 200 7743.

0005-1098/$ – see front matter
doi:10.1016/j.automatica.2012.06.062

feedback controller which allows some non-feedforward and non-triangular terms in the nonlinearity, the problem of the time-delay in the input was not considered.

In this paper, we develop a new regulating state feedback controller with an adaptive dynamic gain inspired by Choi and Lim (2010a) and Krishnamurthy and Khorrami (2007). More specifically, the controller design is mainly based on the reduction method to remove the input delay and a gain scaling technique involving appropriate powers of high-order nonlinearity. Regarding the system nonlinearity, we propose a new condition which becomes quite extensible via a nonlinear function under the power order condition. With the proposed controller, we show that (i) a class of nonlinear systems that have both feedforward and non-feedforward nonlinear terms are regulated with a delay in the input; (ii) the feedforward and non-feedforward terms are not limited to linear growth conditions, but they include some high-order terms; (iii) the proposed controller has an adaptive gain such that the growth rate of the nonlinearity does not need to be known. Moreover, the proposed controller is continuous unlike the discontinuous switching-type controller in Krishnamurthy and Khorrami (2007). With these new conditions and our proposed controller, the global regulation problem of time-delayed nonlinear systems which has not been treated by any of the aforementioned existing work is solved. An example is given for illustration.

2. Problem formulation

We consider the global regulation problem by state feedback for the following class of nonlinear systems with a delay in the input

$$\dot{x} = Ax + Bu(t-\tau) + \delta(t, x, u) \tag{1}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}$ is the input of the system, and $\tau \in [0, \infty)$ is the finite delay time. The initial condition is given as $u(\theta) = \nu(\theta), \theta \in [-\tau, 0]$, $\nu(\theta)$ is a continuous function, and the origin is the unique equilibrium point of the zero input system of (1). The matrices (A, B) are a Brunovsky canonical pair ($A = [a_{ij}]$, $i = 1, \ldots, n, j = 1, \ldots, n$ with $a_{ij} = 1$ if $i = 1, \ldots, n-1$, $j = i+1$ and $a_{ij} = 0$ if $j \neq i+1$ and $B = [0, \ldots, 0, 1]^T$) and the nonlinearity is an $n \times 1$ vector such that $\delta(t, x, u) = [\delta_1(t, x, u), \ldots, \delta_n(t, x, u)]^T$.

Notations: Throughout the paper, $\|x_t\| = \sup_{-\tau \le \theta \le 0} \|x(t+\theta)\|$ denotes the sup norm where we let $x_t \in \mathcal{C}([-\tau, 0], \mathcal{R}^n)$ be defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$, Analogously, we mean that $z_t(\theta) = z(t+\theta)$ and $u_t(\theta) = u(t+\theta), \theta \in [-\tau, 0]$. Also, $\|x\|$ denotes the Euclidean norm and other norms are denoted by subscripts.

Define a matrix $E_{\gamma(t)} = \mathrm{diag}[\frac{1}{\gamma(t)^{n-1}}, \ldots, \frac{1}{\gamma(t)}, 1], \gamma(t) \ge 1$. Then, the nonlinear functions $\delta_i(t, x, u) : \mathcal{R} \times \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}$, $i = 1, \ldots, n$ are $\mathcal{C}^1$ and satisfy the following feedforward and non-feedforward conditions.

Assumption 1. There exist unknown constants $L_1 \ge 0, L_2 \ge 0$ and a nonnegative function $\phi(x, u, \gamma(t))$ such that

$$\|E_{\gamma(t)}\delta(t, x, u)\|_1 \le (L_1 + L_2\phi(x, u, \gamma(t))) \times \left(\sum_{i=1}^{n-2} \frac{|x_{i+2}|}{\gamma(t)^{n-i}} + \frac{|u|}{\gamma(t)}\right) \tag{2}$$

for all $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, and $\gamma(t) \in \mathcal{R}$.

Assumption 2. (a) There exist functions $\phi_i(x, u, \gamma(t))$ such that

$$\phi(x, u, \gamma(t)) \le \sum_{i=1}^{m} \phi_i(x, u, \gamma(t)),$$

$$\phi_i(x, u, \gamma(t)) = \prod_{j=1}^{n} |x_j|^{a_{(i,j)}} |u|^{\mu_i} \gamma(t)^{\nu_i} \tag{3}$$

for all $x \in \mathcal{R}^n$ and $u \in \mathcal{R}$ where $a_{(i,j)}, \mu_i \ge 0$, $\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{(i,j)} + \mu_i\right) > 0\right)$, and ν_i is an any real number for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

(b) The following inequality (power order condition) holds

$$\nu_i + \sum_{j=1}^{n}(n-j)a_{(i,j)} - \mu_i < 1, \quad i = 1, \ldots, m. \tag{4}$$

Roughly speaking, L_1 represents the linear growth rate and L_2 represents the growth rate of high-order terms. Consider the following linear growth feedforward condition (Koo, Choi, & Lim, 2010)

$$|\delta_i(t, x, u)| \le \bar{L}(|x_{i+2}| + \cdots + |x_n| + |u|), \quad i = 1, \ldots, n-2 \tag{5}$$

with $|\delta_{n-1}(t, x, u)| \le |u|, |\delta_n(t, x, u)| = 0, \bar{L} \ge 0$. It is easy to see that if (5) holds, then the condition (2) always hold with $L_1 = \bar{L}n(n-1)$ and $L_2 = 0$, but not vice versa. Thus, the extension to high-order terms and non-feedforward terms are obtained via a $\phi(\gamma(t), x, u)$. We illustrate the extension concept via a function $\phi(x, u, \gamma(t))$ in the following example.

Example A. Let $n = 2, \delta_1(t, x, u) = x_1x_2u^2$ and $\delta_2(t, x, u) = x_1u^3$. In this case, it clearly contains both feedforward and non-feedforward terms. Applying Assumption 1, we obtain

$$\begin{aligned}\|E_{\gamma(t)}\delta(t, x, u)\|_1 &\le \gamma(t)^{-1}|x_1|\,|x_2|\,|u|^2 + |x_1|\,|u|^3 \\ &\le (|x_1|\,|x_2|\,|u| + \gamma(t)|x_1|\,|u|^2)\gamma(t)^{-1}|u|.\end{aligned} \tag{6}$$

Then, we have $\phi(x, u, \gamma(t)) = |x_1|\,|x_2|\,|u| + \gamma(t)|x_1|\,|u|^2$ from (6). By Assumption 2(a), we can obtain that $\phi_1(x, u, \gamma(t)) = |x_1|\,|x_2|\,|u|$ and $\phi_2(x, u, \gamma(t)) = \gamma(t)|x_1|\,|u|^2$. Then, Assumption 2(b) is satisfied by taking $a_{(1,1)} = 1, a_{(1,2)} = 1, \mu_1 = 1, \nu_1 = 0, a_{(2,1)} = 1, a_{(2,2)} = 0, \mu_2 = 2$, and $\nu_2 = 1$.

We note that in order to check Assumptions 1 and 2, we need to go through some algebraic manipulations as shown in the above example. In Remark 1, we provide a more direct form for easy and quick checking. Notably, the direct form in Remark 1 represents only a subset of conditions from Assumptions 1 and 2, but it is still useful when the nonlinearity is in this form.

Remark 1 (*Direct Form*). (i) The nonlinear functions $\delta_i(t, x, u)$ satisfy the following form

$$|\delta_i(t, x, u)| \le \left(\bar{L}_{1i} + \bar{L}_{2i}\prod_{j=1}^{n}|x_j|^{a_{ij}}|u|^{b_i}\right) \times \left(\sum_{j=i}^{n-2}|x_{j+2}| + |u|\right), \quad i = 1, \ldots, n-2 \tag{7}$$

with $|\delta_{n-1}(t, x, u)| \le \left(\bar{L}_{1n-1} + \bar{L}_{2n-1}\prod_{j=1}^{n}|x_j|^{a_{n-1j}}|u|^{b_{n-1}}\right)|u|$ and $|\delta_n(t, x, u)| \le \bar{L}_{2n}\left(\prod_{j=1}^{n}|x_j|^{a_{nj}}|u|^{b_n}\right)|u|$, $\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} + b_i\right) > 0\right)$ where unknown constants $\bar{L}_{1i} \ge 0, \bar{L}_{2i} \ge 0,\ i = 1, \ldots, n-1$, and $\bar{L}_{2n} \ge 0$.

(ii) The following inequalities (power order conditions) hold

$$\sum_{j=1}^{n}(n-j)a_{ij} - b_i < 1, \quad i = 1, \ldots, n-1$$

$$\sum_{j=1}^{n}(n-j)a_{nj} - b_n < 0. \tag{8}$$

Here, we show that this direct form indeed satisfies Assumptions 1 and 2. Applying Assumption 1, it is easy to obtain that

$$\|E_{\gamma(t)}\delta(t,x,u)\|_1 \le \left(L_1 + L_2\left(\sum_{i=1}^{n-1}\prod_{j=1}^{n}|x_j|^{a_{ij}}|u|^{b_i} + \prod_{j=1}^{n}|x_j|^{a_{nj}}|u|^{b_n}\gamma(t)\right)\right) \times \left(\sum_{i=1}^{n-2}\frac{|x_{i+2}|}{\gamma(t)^{n-i}} + \frac{|u|}{\gamma(t)}\right) \quad (9)$$

where $L_1 = \max_{i=1,\dots,n-1}\{\bar{L}_{1i}\}$, and $L_2 = \max_{i=1,\dots,n}\{\bar{L}_{2i}\}$. From Assumption 2(a), we obtain $\phi_i(t,x,\gamma(t)) = \prod_{j=1}^{n}|x_j|^{a_{ij}}|u|^{b_i}$, $i = 1,\dots,n-1$, and $\phi_n(t,x,\gamma(t)) = \prod_{j=1}^{n}|x_j|^{a_{nj}}|u|^{b_n}\gamma(t)$. Then, we have $a_{(i,j)} = a_{ij}$, $\mu_i = b_i$, and $\nu_i = 0$, $i = 1,\dots,n-1$, $a_{(n,j)} = a_{nj}$, $\mu_n = b_n$, and $\nu_n = 1$, $j = 1,\dots,n$. Thus, we can see that (8) satisfies Assumption 2(b). Using the direct form (i) and (ii) in Remark 1, we can easily see that Example A satisfies the proposed condition. Taking $a_{11} = 1$, $a_{12} = 1$, and $b_1 = 1$ from $|\delta_1(t,x,u)| \le |x_1|\,|x_2|\,|u(t)|^2$ and $a_{21} = 1$, $a_{22} = 0$, and $b_2 = 2$ from $|\delta_2(t,x,u)| \le |x_1|\,|u(t)|^3$ by (7), we can show that the power order condition (8) holds.

Finally, to show that this Direct Form is a subset of Assumptions 1 and 2, we give a simple example:

Let $n = 2$, $\delta_1(t,x,u) = x_1^2u^3 + x_2^3u^4$ and $\delta_2(t,x,u) = x_1^3x_2^5u^5$. Since the form of $\delta_1(t,x,u)$ is not satisfied with (7), this case violates Direct Form. However, applying Assumption 1, we obtain

$$\begin{aligned}\|E_{\gamma(t)}\delta(t,x,u)\|_1 &\le \gamma(t)^{-1}(|x_1|^2|u|^3 + |x_2|^3|u|^4) + |x_1|^3|x_2|^5|u|^5\\ &\le (|x_1|^2|u|^2 + |x_2|^3|u|^3 + \gamma(t)|x_1|^3|x_2|^5|u|^4)\\ &\quad\times\gamma(t)^{-1}|u|. \end{aligned} \quad (10)$$

Then, we have $\phi(x,u,\gamma(t)) = |x_1|^2|u|^2 + |x_2|^3|u|^3 + \gamma(t)|x_1|^3|x_2|^5|u|^4$ from (10). By Assumption 2(a), we can obtain that $\phi_1(x,u,\gamma(t)) = |x_1|^2|u|^2$, $\phi_2(x,u,\gamma(t)) = |x_2|^3|u|^3$ and $\phi_3(x,u,\gamma(t)) = \gamma(t)|x_1|^3|x_2|^5|u|^4$. Then, by taking $a_{(1,1)} = 2$, $a_{(1,2)} = 0$, $\mu_1 = 2$, $\nu_1 = 0$, $a_{(2,1)} = 0$, $a_{(2,2)} = 3$, $\mu_2 = 3$, $\nu_2 = 0$, $a_{(3,1)} = 3$, $a_{(3,2)} = 5$, $\mu_3 = 4$, and $\nu_3 = 1$, Assumption 2(b) is satisfied. Then, it is obvious that Direct Form of Remark 1 is a subset of Assumptions 1 and 2.

3. Main result

To solve our control problem, we introduce an adaptive controller with a dynamic gain as follows.

Controller:

$$u = K(\gamma(t))\left[x_1,\dots,x_{n-1},x_n + \int_{t-\tau}^{t}u(s)ds\right]^T \quad (11)$$

where $K(\gamma(t)) = [k_1/\gamma(t)^n,\dots,k_n/\gamma(t)]$. Note that u is well defined and continuous for some suitably defined initial condition $u(\theta) = \nu(\theta)$, $\theta \in [-\tau, 0]$. The following are explanations: From (11), it is easy to see that at $t = 0$, we have $u(0) = K(\gamma(0))x(0) + \frac{k_n}{\gamma(0)}\int_{-\tau}^{0}\nu(\theta)d\theta$. Then, we have $\int_{-\tau}^{0}\nu(\theta)d\theta = \frac{\gamma(0)}{k_n}(u(0) - K(\gamma(0))x(0)) = \frac{\gamma(0)}{k_n}(\nu(0) - K(\gamma(0))x(0))$. Thus, any continuous function $\nu(\theta)$ that satisfies $\int_{-\tau}^{0}\nu(\theta)d\theta = \frac{\gamma(0)}{k_n}(\nu(0) - K(\gamma(0))x(0))$ is valid (Choi & Lim, 2010a).

Dynamic gain:

$$\begin{aligned}\dot{\gamma}(t) &= w\gamma(t)^{\alpha+c-1}\sum_{j=1}^{m}\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \int_{t-\tau}^{t}u(s)ds\right|\right)^{\beta_j}\\ &\quad + w\gamma(t)^{\alpha+c-1}\sum_{j=1}^{m}\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \int_{t-\tau}^{t}u(s)ds\right|\right)^{\beta_j+2}\\ &\quad + \gamma(t)^{\alpha+c-1}\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \int_{t-\tau}^{t}u(s)ds\right|\right)^2 \end{aligned} \quad (12)$$

with $\gamma(0) = 1$ where

$$w = \begin{cases}0, & \text{if } L_2 = 0\\ 1, & \text{if } L_2 \neq 0\end{cases} \quad (13)$$

and a positive constant $0 < c < 1 - \left(\nu_i + \sum_{j=1}^{n}(n-j)a_{(i,j)} - \mu_i\right)$ for all $i = 1,\dots,m$, $\alpha = \max_{i=1,\dots,m}\{\nu_i + \sum_{j=1}^{n}(n-j)a_{(i,j)} - \mu_i\}$, and $\beta_i = \sum_{j=1}^{n}a_{(i,j)} + \mu_i$, $i = 1,\dots,m$.

Note that the positive constant c always exists because of Assumption 2(b).

The dynamic gain (12) is continuous and differentiable and tuned online through an unknown growth rate of the nonlinearity. Note that Assumption 1 implies that the norm bound structure of nonlinearity is known. Thus, if the nonlinearity contains only linear growth terms, we trivially have $\phi(x,u,\gamma(t)) = 0$, which means we can set $L_2 = 0$. If $L_2 = 0$, the dynamic gain (12) is virtually the same as the dynamic gain of Koo et al. (2010) (a linear growth rate version). If $L_2 \neq 0$, the terms in the first and second lines of (12) are designed to adaptively cancel the effect of the high-order nonlinearity. The parameters α, β_i, and c are chosen from the powers of the high-order nonlinearity.

Here, we address some mathematical notations and setups for Theorem 1 and its proof. Let $A_{K(\gamma(t))} = A + BK(\gamma(t))$. Then, we define $K = K(1)$ and $A_K = A_{K(1)}$. If A_K is Hurwitz, from Choi and Lim (2006), we can obtain a Lyapunov equation of $A_{K(\gamma(t))}^TP_{K(\gamma(t))} + P_{K(\gamma(t))}A_{K(\gamma(t))} = -\gamma(t)^{-1}E_{\gamma(t)}^2$ with $P_{K(\gamma(t))} = E_{\gamma(t)}P_KE_{\gamma(t)}$ from $A_K^TP_K + P_KA_K = -I$ where I denotes an $n \times n$ identity matrix.

Lemma 1. *For the constants $\eta \ge 0$, $\varepsilon > 0$, and $0 < \theta < 1$, the following inequality holds $\int_0^t \eta e^{-\varepsilon s^\theta}ds < +\infty$ on $t \in [0, +\infty)$.*

Proof. The proof is in the Appendix. □

Lemma 2. *Under Assumptions 1 and 2, the dynamic controller (11)–(12) is applied to the system (1). Then, all states of the closed-loop system (1) are globally regulated if dynamic gain $\gamma(t)$ is bounded for $t \in [0, +\infty)$.*

Proof. The proof is in the Appendix. □

Theorem 1. *Under Assumptions 1 and 2, select K such that A_K is Hurwitz. Then, the dynamic controller (11)–(12) globally regulates the closed-loop system (1). Also, the dynamic gain $\gamma(t)$ is bounded for $t \in [0, +\infty)$.*

Proof. We divide the proof into two parts for easy reading.

Part 1: (The closed-loop system and the norm-bound of the derivative of the Lyapunov equation.)

Consider the stability preserving reduction-type change of variables used in Choi and Lim (2010a) given by

$$[z_1,\dots,z_{n-1},z_n]^T = \left[x_1,\dots,x_{n-1},x_n + \int_{t-\tau}^{t}u(s)ds\right]^T. \quad (14)$$

Then, via (14), the system (1) is transformed into

$$\dot{z} = Az + Bu - B_1\int_{t-\tau}^{t}u(s)ds + \bar{\delta}(t,z,u_t) \quad (15)$$

where $B_1 = [0, \ldots, 1, 0]^T$ and $\bar{\delta}(t, z, u_t) = \delta(t, x, u)$ where x is replaced by z and the input u in the time interval $[t-\tau, t]$ using (14). With (14), the controller (11) is represented by

$$u = K(\gamma(t))z. \tag{16}$$

With the controller (16), we have the closed-loop system

$$\dot{z} = A_{K(\gamma(t))}z - B_1 \int_{t-\tau}^{t} K(\gamma(s))z(s)ds + \bar{\delta}(t, z, u_t). \tag{17}$$

There exists $P_K = P_K^T > 0$ such that

$$A_K^T P_K + P_K A_K = -I, \qquad \pi_1 I \le P_K D + DP_K \le \pi_2 I \tag{18}$$

where $D = \mathrm{diag}[\frac{2n-1}{2}, \ldots, \frac{2(n-i)+1}{2}, \ldots, \frac{1}{2}]$, $i = 1, \ldots, n$, $\pi_1, \pi_2 > 0$. With this, we set a Lyapunov function as $V(z) = \gamma(t)^{-1}z^T P_{K(\gamma(t))}z$. Since $\gamma(t)$ is nondecreasing (see (12)), it is obvious that $\gamma(t) \ge 1$ for all t. Note that we have

$$\gamma(t)^{-1}\lambda_1 \|E_{\gamma(t)}z\|^2 \le V(z) \le \gamma(t)^{-1}\lambda_2 \|E_{\gamma(t)}z\|^2 \tag{19}$$

where $\lambda_1 = \lambda_{\min}(P_K)$ and $\lambda_2 = \lambda_{\max}(P_K)$, which denotes the minimal and maximal eigenvalues of P_K.

Along the trajectory of (17), we obtain

$$\begin{aligned}\dot{V}(z) = &-\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 + 2\gamma(t)^{-1}z^T P_{K(\gamma(t))}\bar{\delta}(t, z, u_t) \\ &- 2\gamma(t)^{-1}z^T P_{K(\gamma(t))}B_1 \int_{t-\tau}^{t} u(s)ds \\ &- \dot{\gamma}(t)\gamma(t)^{-2}z^T E_{\gamma(t)}(P_K\bar{D} + \bar{D}P_K)E_{\gamma(t)}z \\ &- \dot{\gamma}(t)\gamma(t)^{-2}z^T P_{K(\gamma(t))}z\end{aligned} \tag{20}$$

where $\bar{D} = \mathrm{diag}[n-1, \ldots, 1, 0]$.

Noting that $K(\gamma(t))z = \gamma(t)^{-1}KE_{\gamma(t)}z$, we can equivalently express u as $u = \gamma(t)^{-1}KE_{\gamma(t)}z$. Then, we have

$$\begin{aligned}\left|\int_{t-\tau}^{t} u(s)ds\right| &\le \tau \sup_{-\tau\le\theta\le 0} \gamma(t+\theta)^{-1} \sup_{-\tau\le\theta\le 0} |KE_{\gamma(t+\theta)}z(t+\theta)| \\ &= \tau\gamma_t^{-1}\|K\| \, \|E_{\gamma_t}z_t\|.\end{aligned} \tag{21}$$

From (21), $E_{\gamma(t)}B_1 \int_{t-\tau}^{t} u(s)ds \le \gamma(t)^{-1}\left|\int_{t-\tau}^{t} u(s)ds\right| \le \gamma(t)^{-1}\gamma_t^{-1}\tau\|K\| \, \|E_{\gamma_t}z_t\|$. Note that $z^T E_{\gamma(t)}(P_K\bar{D} + \bar{D}P_K)E_{\gamma(t)}z + z^T P_{K(\gamma(t))}z = z^T E_{\gamma(t)}(P_K D + DP_K)E_{\gamma(t)}z$. Substituting these inequalities into (20) and using (18),

$$\begin{aligned}\dot{V}(z) \le &-\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 - \pi_1\dot{\gamma}(t)\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 \\ &+ 2\gamma(t)^{-1}\|P_K\| \, \|E_{\gamma(t)}z\| \, \|E_{\gamma(t)}\bar{\delta}(t, z, u_t)\|_1 \\ &+ 2\gamma(t)^{-2}\gamma_t^{-1}\tau\|K\| \, \|P_K\| \, \|E_{\gamma(t)}z\| \, \|E_{\gamma_t}z_t\|.\end{aligned} \tag{22}$$

Using (13), (21), and Assumption 1, $\|E_{\gamma(t)}\bar{\delta}(t, z, u_t)\|_1$ of (22) is derived as

$$\begin{aligned}\|E_{\gamma(t)}\bar{\delta}(t, z, u_t)\|_1 &= \|E_{\gamma(t)}\delta(t, x, u)\|_1 \\ &\le (L_1 + L_2\phi(x, u, \gamma(t)))\gamma(t)^{-2} \\ &\quad \times \left(\sum_{i=1}^{n-2} \frac{|x_{i+2}|}{\gamma(t)^{n-i-2}} + \gamma(t)|u|\right) \\ &\le (L_1 + L_2\bar{\phi}(z, u_t, \gamma(t)))\gamma(t)^{-2}\sqrt{n} \\ &\quad \times \left(\|E_{\gamma(t)}z\| + \left|\int_{t-\tau}^{t} u(s)ds\right| + \|K\| \, \|E_{\gamma(t)}z\|\right) \\ &\le L(1 + w\bar{\phi}(z, u_t, \gamma(t)))\gamma(t)^{-2}(\|E_{\gamma_t}z_t\| + \|E_{\gamma(t)}z\|)\end{aligned} \tag{23}$$

where $L = (L_1 + L_2)\sqrt{n}(1 + \|K\| + \tau\|K\|)$ and $\bar{\phi}(z, u_t, \gamma(t)) = \phi(x, u, \gamma(t))$.

Using $|x_i| \le \gamma(t)^{n-i}\|E_{\gamma(t)}x\|_1$, we have, for $i = 1, \ldots, m$,

$$\begin{aligned}\prod_{j=1}^{n} |x_j|^{a_{(i,j)}} &\le \gamma(t)^{\sum_{j=1}^{n}(n-j)a_{(i,j)}} \\ &\quad \times \left(\|E_{\gamma(t)}z\|_1 + \left|\int_{t-\tau}^{t} u(s)ds\right|\right)^{\sum_{j=1}^{n} a_{(i,j)}}.\end{aligned} \tag{24}$$

By Assumption 2(a) and (24), the upper bound of the term $\bar{\phi}(z, u_t, \gamma(t))$ is obtained as

$$\begin{aligned}\bar{\phi}(z, u_t, \gamma(t)) &= \phi(x, u, \gamma(t)) \\ &\le \sum_{i=1}^{m}\prod_{j=1}^{n} |x_j|^{a_{(i,j)}}|u|^{\mu_i}\gamma(t)^{\nu_i} \\ &\le \sum_{i=1}^{m} \gamma(t)^{q_i}\left(\sqrt{n}\|E_{\gamma(t)}z\| + \left|\int_{t-\tau}^{t} u(s)ds\right|\right)^{r_i} \\ &\quad \times \gamma(t)^{-\mu_i}\|K\|^{\mu_i}\|E_{\gamma(t)}z\|^{\mu_i} \\ &\le \sum_{i=1}^{m} b_i\gamma(t)^{q_i-\mu_i}\left(\|E_{\gamma(t)}z\| + \|E_{\gamma_t}z_t\|\right)^{r_i}\|E_{\gamma(t)}z\|^{\mu_i}\end{aligned} \tag{25}$$

where $q_i = \nu_i + \sum_{j=1}^{n}(n-j)a_{(i,j)}$, $r_i = \sum_{j=1}^{n} a_{(i,j)}$, and $b_i = n^{\frac{r_i}{2}}(1+\tau)^{r_i}\|K\|^{\mu_i}$, $i = 1, \ldots, m$.

Now, we apply the Razumikhin theorem (Choi & Lim, 2010a; Hale & Verduyn Lunel, 1993) to (22)–(25). We set a function $p(s) = hs$, $h > 1$ such $p(s) > s$ for $s > 0$. Then, by using $p(s)$, we can set $V(z(t+\theta)) < p(V(z(t))) = hV(z(t))$ if $\theta \in [-\tau, 0]$, which leads to $\lambda_1\gamma_t^{-1}\|E_{\gamma_t}z_t\|^2 < h\lambda_2\gamma(t)^{-1}\|E_{\gamma(t)}z(t)\|^2$ using (19). Thus, using $\gamma(t)^{-1} < \gamma_t^{-1}$ and $(\gamma_t^{-1})^2 < \gamma_t^{-1}$, we can assure that there exists $h_1, h_2 > 1$ such that

$$\begin{aligned}&\|E_{\gamma_t}z_t\| < h_1\|E_{\gamma(t)}z(t)\|, \\ &\gamma_t^{-1}\|E_{\gamma_t}z_t\| < h_2\gamma(t)^{-\frac{1}{2}}\|E_{\gamma(t)}z(t)\|.\end{aligned} \tag{26}$$

From (22)–(26), we have

$$\begin{aligned}\dot{V}(z) \le &-\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 - \pi_1\dot{\gamma}(t)\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 \\ &+ \gamma(t)^{-3}\sigma\left(\gamma(t)^{\frac{1}{2}} + w\sum_{i=1}^{m}\bar{b}_i\gamma(t)^{q_i-\mu_i}\|E_{\gamma(t)}z\|^{r_i+\mu_i}\right) \\ &\times \|E_{\gamma(t)}z\|^2\end{aligned} \tag{27}$$

where $\sigma = 2\|P_K\|((1+h_1)L + \tau h_2\|K\|)$ and $\bar{b}_i = b_i(1+h_1)^{r_i}$, $i = 1, \ldots, m$.

From (12), it is clear that $w\gamma(t)^{\alpha+c-1}\sum_{i=1}^{m}\|E_{\gamma(t)}z\|^{\beta_i} \le \dot{\gamma}(t)$. Substituting (12) into (27) and using that $\sum_{i=1}^{m}\bar{b}_i\gamma(t)^{q_i-\mu_i}\|E_{\gamma(t)}z\|^{r_i+\mu_i} \le m\bar{b}\gamma(t)^{\alpha}\sum_{i=1}^{m}\|E_{\gamma(t)}z\|^{\beta_i}$, $\bar{b} = \max_{i\in[1,m]}\{\bar{b}_i\}$, we obtain

$$\begin{aligned}\dot{V}(z) \le &-\frac{1}{2}\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 - \frac{1}{2}\gamma(t)^{-3}(\gamma(t) \\ &- 2\sigma\gamma(t)^{\frac{1}{2}})\|E_{\gamma(t)}z\|^2 \\ &\times w\pi_1\gamma(t)^{-3+\alpha}\left(\gamma(t)^c - \pi_1^{-1}m\sigma\bar{b}\right) \\ &\times \sum_{i=1}^{m}\|E_{\gamma(t)}z\|^{\beta_i+2}.\end{aligned} \tag{28}$$

It is clear that the right-hand side of (28) is negative if $\gamma(t) \ge \max\{\sqrt{2\sigma+1}, (\pi_1^{-1}m\sigma\bar{b}+1)^{1/c}\}$. If we consider a case where $\gamma(t)$ converges to a value less than $\max\{2\sigma+1, (\pi_1^{-1}m\sigma\bar{b}+1)^{1/c}\}$, then global regulation follows trivially by Lemma 2. Note that Lemma 2 is proved without using the Razumikhin theorem. Thus, we only need to consider a case that $\gamma(t) \geq \max\{\sqrt{2\sigma+1}, (\pi_1^{-1}m\sigma\bar{b}+1)^{1/c}\}$.

Part 2: (Boundedness of $\gamma(t)$ and z and the system regulation.)

The closed-loop system (17) has a unique solution $(z(t), \gamma(t))$ on the maximally extended interval $[0, T_f)$ for some $T_f \in (0, \infty]$. We show that the $\gamma(t)$ and z are bounded on $[0, T_f)$. We first show that $\gamma(t)$ cannot escape at $t = T_f$. To prove that, we suppose that $\lim_{t\to T_f}\gamma(t) = +\infty$. Since $\gamma(t)$ is nondecreasing, there exists a finite time $t^* \in (0, T_f)$, such that

$$\gamma(t) \geq \max\left\{\sqrt{2\sigma+1}, (\pi_1^{-1}m\sigma\bar{b}+1)^{1/c}\right\} \tag{29}$$

for $t \in [t^*, T_f)$. From (28) and (29), it follows that

$$\dot{V}(z) \leq -\frac{1}{2}\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2. \tag{30}$$

From (19) and (30), we obtain, for $t \in [t^*, T_f)$

$$\|E_{\gamma(t)}z\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|E_{\gamma(t^*)}z(t^*)\|e^{-\frac{1}{4\lambda_2}\int_{t^*}^t \gamma(s)^{-1}ds}. \tag{31}$$

Note that, from (12),

$$\begin{aligned}\gamma(t)^\rho - \gamma(t^*)^\rho &= \rho\int_{t^*}^t \dot{\gamma}(s)\gamma(s)^{1-\alpha-c}ds \\ &= \rho\int_{t^*}^t w\sum_{i=1}^m\left(\|E_{\gamma(s)}z(s)\|_1^{\beta_i} + \|E_{\gamma(s)}z(s)\|_1^{\beta_i+2}\right)ds \\ &\quad + \rho\int_{t^*}^t \|E_{\gamma(s)}z(s)\|_1^2 ds\end{aligned} \tag{32}$$

where $\rho = 2-\alpha-c > 1$ because of $0 < c < 1-\alpha$. From (31), we have $\|E_{\gamma(t)}z\|_1 \leq \sqrt{n}\|E_{\gamma(t)}z\| \leq \sqrt{\frac{n\lambda_2}{\lambda_1}}\|E_{\gamma(t^*)}z(t^*)\|$. Using this and (32), we have

$$\gamma(t) \leq \left(\rho_1(t-t^*) + \gamma(t^*)^\rho\right)^{\frac{1}{\rho}} \tag{33}$$

where $\rho_1 = w\rho\sum_{i=1}^m((\frac{n\lambda_2}{\lambda_1})^{\beta_i/2}\|E_{\gamma(t^*)}z(t^*)\|^{\beta_i} + (\frac{n\lambda_2}{\lambda_1})^{(\beta_i+2)/2}\|E_{\gamma(t^*)}z(t^*)\|^{\beta_i+2}) + \rho(\frac{n\lambda_2}{\lambda_1})\|E_{\gamma(t^*)}z(t^*)\|^2$. Then, from (33), we obtain

$$\int_{t^*}^t \gamma(s)^{-1}ds \geq \rho_2\left(\left(\rho_1(t-t^*)+\gamma(t^*)^\rho\right)^{1-\frac{1}{\rho}} - \gamma(t^*)^{\rho-1}\right) \tag{34}$$

where $\rho_2 = \rho_1^{-1}(1-1/\rho)^{-1} > 0$ because of $\rho > 1$.

Using (31), (34), and Lemma 1, we obtain

$$\begin{aligned}&\int_{t^*}^t w\sum_{i=1}^m\left(\|E_{\gamma(s)}z(s)\|_1^{\beta_i} + \|E_{\gamma(s)}z(s)\|_1^{\beta_i+2}\right)ds \\ &\quad + \int_{t^*}^t \|E_{\gamma(s)}z(s)\|_1^2 ds \\ &\leq \int_{t^*}^t \rho_1 e^{-\frac{\rho_3\rho_2}{4\lambda_2}((\rho_1(s-t^*)+\gamma(t^*))^{1-\frac{1}{\rho}} - \gamma(t^*)^{\rho-1})}ds < +\infty\end{aligned} \tag{35}$$

where $\rho_3 = \min_{i\in[1,m]}\{\beta_i, 2\}$. From (32) and (35), we have

$$\begin{aligned}+\infty &= \gamma(T_f)^\rho - \gamma(t^*)^\rho \\ &\leq \rho\int_{t^*}^{T_f} w\sum_{i=1}^m\left(\|E_{\gamma(s)}z(s)\|_1^{\beta_i} + \|E_{\gamma(s)}z(s)\|_1^{\beta_i+2}\right)ds \\ &\quad + \rho\int_{t^*}^{T_f}\|E_{\gamma(s)}z(s)\|_1^2 ds \\ &\leq \rho n^{\frac{\rho_4+2}{2}}\int_{t^*}^{T_f} w\sum_{i=1}^m\left(\|E_{\gamma(s)}z(s)\|^{\beta_i} + \|E_{\gamma(s)}z(s)\|^{\beta_i+2}\right)ds \\ &\quad + \rho n^{\frac{\rho_4+2}{2}}\int_{t^*}^{T_f}\|E_{\gamma(s)}z(s)\|^2 ds < +\infty\end{aligned} \tag{36}$$

where $\rho_4 = \max_{i=1,\ldots,m}\{\beta_i\}$. Then, (36) yields a contradiction. Thus, the dynamic gain $\gamma(t)$ is well defined and bounded on $[0, T_f)$.

Next, we show that z is well defined and bounded on the interval $[0, T_f)$. From (28), we have

$$\begin{aligned}V(z) - V(z(0)) \leq &-\int_0^t \gamma(T_f)^{-3}(\gamma(T_f)-\sigma)\|E_{\gamma(s)}z(s)\|^2 ds \\ &-\int_0^t w\pi_1\gamma(T_f)^{-3+\alpha}\left(\gamma(T_f)^c - \pi_1^{-1}m\sigma\bar{b}\right) \\ &\times\sum_{i=1}^m\|E_{\gamma(s)}z(s)\|^{\beta_i+2}ds.\end{aligned} \tag{37}$$

The boundedness of $\gamma(t)$ and (32) implies that $\int_0^t\sum_{i=1}^m\|E_{\gamma(s)}z(s)\|^{\beta_i+2}ds < \infty$ and $\int_0^t\|E_{\gamma(s)}z(s)\|^2 ds < \infty$ on $[0, T_f)$. Using these inequalities and noting that $\frac{\lambda_1}{\gamma(T_f)}\|E_{\gamma(t)}z\|^2 \leq V(z)$, from (37), we obtain $\|E_{\gamma(t)}z\|^2 < +\infty$ on $[0, T_f)$. This, with the boundedness of $\gamma(t)$, implies that z is well defined and bounded on the interval $[0, T_f)$. In summary, we have shown that $\gamma(t)$ and $\|z\|$ are well defined and all bounded on the maximally extended interval $[0, T_f)$.

Finally, we show the global regulation of system (1) when there exists a finite time t^* such that $\gamma(t) \geq \max\{\sqrt{2\sigma+1}, (\pi_1^{-1}m\sigma\bar{b}+1)^{1/c}\}$ for $t > T_f$. Letting $T_f = +\infty$, we obtain (30) for $t \in [t^*, \infty)$. Thus, we reach that

$$\dot{V}(z) \leq -\frac{1}{2}\gamma(t)^{-2}\|E_{\gamma(t)}z\|^2 \tag{38}$$

for $t \in [t^*, +\infty)$. In view of (38) and the Razumikhin theorem (Choi & Lim, 2010a; Hale & Verduyn Lunel, 1993), $\|E_{\gamma(t)}z\| \to 0$ as $t \to \infty$. This yields $\|z\| \to 0$ as $t \to \infty$ by boundedness of $\gamma(t)$. Since (14) is a stability preserving transformation for a finite τ (Choi & Lim, 2010b), we have $|x_1| + \cdots + |x_n| \to 0$ as $t \to +\infty$ as well. Therefore, the global regulation is achieved. □

Remark 2. For practical implementation of the distributed controller (11) with dynamic gain (12), an approximation of the integral term using a quadrature formula can be applied (Michiels, Mondie, Roose, & Dambrine, 2004). The approximated controller is presented as

$$u = K(\gamma(t))\left[x_1, \ldots, x_{n-1}, x_n + \sum_{j=1}^n d_{j,1}u(t-\theta_{j,1})\right]^T \tag{39}$$

$$\begin{aligned}\dot{\gamma}(t) &= w\gamma(t)^{\alpha+c-1} \\ &\times\sum_{j=1}^m\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \sum_{l=1}^n d_{l,2}u(t-\theta_{l,2})\right|\right)^{\beta_j} \\ &+ w\gamma(t)^{\alpha+c-1} \\ &\times\sum_{j=1}^m\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \sum_{l=1}^n d_{l,3}u(t-\theta_{l,3})\right|\right)^{\beta_j+2} \\ &+ \gamma(t)^{\alpha+c-1} \\ &\times\left(\sum_{i=1}^{n-1}\frac{|x_i|}{\gamma(t)^{(n-i)}} + \left|x_n + \sum_{l=1}^n d_{l,4}u(t-\theta_{l,4})\right|\right)^2\end{aligned} \tag{40}$$

where $d_{j,h} > 0$ is a coefficient of the quadrature formula and $\theta_{j,h} \in [0, \tau]$ for $j = 1, \ldots, n$ and $h = 1, \ldots, 4$. See Michiels et al. (2004) for further discussion.

Remark 3. The proposed conditions in Assumptions 1 and 2 may have some room for robustness in the sense that we may only need to know that the order of the nonlinearities belong to certain ranges in checking the power order condition. However, if the order of the nonlinearities is unknown completely, then further research should be followed to solve such a problem.

4. Illustrative example

Comparison example: We reconsider Example A

$$\dot{x}_1 = x_2 + \eta_1(t)x_1x_2u^2$$
$$\dot{x}_2 = u(t-\tau) + \eta_2(t)x_1u^3 \quad (41)$$

where $\tau = 1$ and $\eta_1(t)$ and $\eta_2(t)$ are known to be finite.

Due to the nonlinearity structure and an input-delay, the methods of Chen and Huang (2009), Choi and Lim (2006), Choi and Lim (2010a), Fang and Lin (2006), Krishnamurthy and Khorrami (2007), Krstic (2002), Mazenc et al. (2003), Ye (2003), Ye (2005), Ye and Unbehauen (2004), Yue (2004) and Zhang (2006, 2009) are not applicable. We can set the controller parameters as $K = [-2.25, -3], \alpha = 0, \beta_1 = \beta_2 = 3, c = 0.01$. For the initial function $\nu(\theta)$, we can set $\nu(\theta) = \bar{c}\theta + \nu(0)$ where $\bar{c} = \frac{2}{\tau^2}\left(\nu(0)\tau - \frac{\gamma(0)}{k_2}(\nu(0) - K(\gamma(0))x(0))\right)$.

Application example: We consider a cart–pole system with small length/strong gravity effects (Olfari-Saber, 2001)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 + \kappa\frac{x_3x_4^2}{(1+x_3^2)^{\frac{3}{2}}}$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = u(t-\tau). \quad (42)$$

As addressed in Olfari-Saber (2001), this is an underactuated system. New features are there is an additional delay in the input and the parameter κ does not need to be known with our control scheme. According to the viewpoint about the upper bound of the nonlinearity, we can design and set the parameters of our controller differently. Regarding the bound of nonlinearity, we consider two choices: (i) $\frac{x_3x_4^2}{(1+x_3^2)^{\frac{3}{2}}} \le |x_3|^{\frac{1}{2}}|x_4|^2$ and (ii) $\frac{x_3x_4^2}{(1+x_3^2)^{\frac{3}{2}}} \le |x_4|^2$. For the choice (i), applying Assumptions 1 and 2(a), we have $\phi(x, u, \gamma(t)) = \phi_1(x, u, \gamma(t)) = |x_3|^{\frac{1}{2}}|x_4|$. Then, by taking $\mu_1 = \nu_1 = a_{(1,1)} = a_{(1,2)} = 0, a_{(1,3)} = 0.5, a_{(1,4)} = 1$, Assumption 2(b) is satisfied. For the choice (ii), applying Assumptions 1 and 2(a), we have $\phi(x, u, \gamma(t)) = \phi_1(x, u, \gamma(t)) = |x_4|$. Then, by taking $\mu_1 = \nu_1 = a_{(1,1)} = a_{(1,2)} = a_{(1,3)} = 0, a_{(1,4)} = 1$, Assumption 2(b) is satisfied. We set the controller parameters as $K = [-0.0256, -0.2560, -0.9600, -1.6], \alpha = 0.5, \beta_1 = 1.5, c = 0.01$ for the choice (i), and $\alpha = 0, \beta_1 = 1, c = 0.01$ for the choice (ii). For simulation, we set $\kappa = 0.2$ and $\tau = 0.1$. The simulation results are shown in Fig. 1 with the initial states $x_1(0) = 0.2002, x_2(0) = x_4(0) = 0, x_3(0) = 0.1003$. From the simulation results, the overshoot of the choice (i) is smaller than that of the choice (ii). Thus, we can see that not only the proposed method is applicable to input-delayed practical systems, but also there is a flexibility in designing a controller by utilizing the order of nonlinearities.

5. Conclusions

We have presented a state feedback controller for a class of nonlinear systems with uncertain nonlinearity and a delay in the input. The newly proposed condition is quite extensible via a nonlinear function containing the full states and the input.

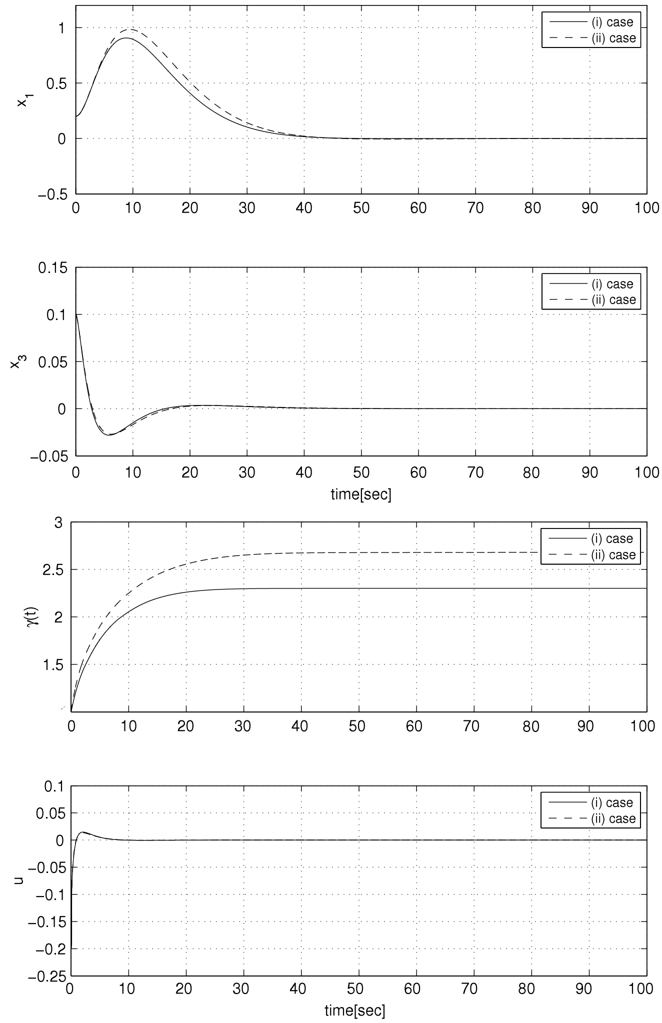

Fig. 1. The responses for a cart–pole system.

As a result, not only nonlinear feedforward terms but also some nonlinear non-feedforward terms are included in the system. Moreover, the included feedforward and non-feedforward terms are not limited to linear growth conditions, but contain some nontrivial high-order terms. The proposed control scheme has an adaptive function which was designed so that the growth rate of nonlinearity does not need to be known. The principle of our controller design is based on a gain scaling technique involving appropriate powers of the high-order nonlinearity. In short, we solve the global regulation problem of nonlinear systems that have feedforward and non-feedforward nonlinear terms with unknown growth rates while there is a delay in the input. Through the examples, we show the improved and generalized features of our result over existing ones.

Appendix

Proof of Lemma 1. Let $s^\theta = k$. Then, we have $\int_0^t \eta e^{-\varepsilon s^\theta} ds = \int_0^{t^\theta} \frac{\eta}{\theta} k^{\frac{1-\theta}{\theta}} e^{-\varepsilon k} dk$. Let ω be the minimum integer such that $\omega \ge \frac{1-\theta}{\theta}$. Note that $k^{\frac{1-\theta}{\theta}} \le 1 + k^\omega$ for $k \ge 0$ and $\int k^\omega e^{-\varepsilon k} dk = e^{-\varepsilon k} \sum_{j=0}^{\omega} (-1)^j \frac{\omega! k^{\omega-j}}{(\omega-j)!(-\varepsilon)^{j+1}}$ (Zwillinger, 2003). With these inequalities, we obtain $\int_0^{t^\theta} \frac{\eta}{\theta} k^{\frac{1-\theta}{\theta}} e^{-\varepsilon k} dk \le \int_0^{t^\theta} \frac{\eta}{\theta}(1 + k^\omega) e^{-\varepsilon k} dk = -\frac{\eta}{\varepsilon\theta} e^{-\varepsilon k}\Big|_{k=0}^{t^\theta} + \int_0^{t^\theta} \frac{m_1}{\theta} k^\omega e^{-\varepsilon k} dk = -\frac{\eta}{\varepsilon\theta} e^{-\varepsilon k}\Big|_{k=0}^{t^\theta} + \frac{\eta}{\theta} e^{-\varepsilon k} \sum_{j=0}^{\omega}$

$(-1)^j \frac{\omega! k^{\omega-j}}{(\omega-j)!(-\varepsilon)^{j+1}} |_{k=0}^{t^\theta}$. Then, it is obvious that $\int_0^t \eta e^{-\varepsilon s^\theta} ds < +\infty$. □

Proof of Lemma 2. Let $z_i = x_i$, $i = 1, \ldots, n-1$, and $z_n = x_n + \int_{t-\tau}^t u(s)ds$. The system (1) is transformed into $\dot{z} = Az + Bu - B_1 \int_{t-\tau}^t u(s)ds + \bar{\delta}(t, z, u_t)$ where $B_1 = [0, \ldots, 1, 0]^T$ and $\bar{\delta}(t, z, u_t) = \delta(t, x, u)$ where x is replaced by z and the input u in the time interval $[t-\tau, t]$ and the controller (11) is represented as $u = K(\gamma(t))z$. From (12), we have

$$\gamma(t)^\rho - \gamma(t^*)^\rho = \rho \int_0^t \dot{\gamma}(s)\gamma(s)^{1-\alpha-c} ds \tag{43}$$

where $\rho = 2 - \alpha - c > 1$. Substituting (12) into the right-hand side of (43) and using the boundedness of $\gamma(t)$, we have $\int_0^t \|E_{\gamma(t)}z(s)\|_1^2 ds < +\infty$ and $\|z\| < +\infty$ for $t \in [0, +\infty)$. Thus, there is no finite escape phenomenon.

Since $|z_n| = \left| x_n + \int_{t-\tau}^t u(s)ds \right| < +\infty$ on $t \in [0, +\infty)$ from the boundedness of $\gamma(t)$ and $\|E_{\gamma(t)}z\|^2$ on $t \in [0, +\infty)$, we have $\left| \int_{t-\tau}^t u(s)ds \right| < +\infty$ for $t \in [0, +\infty)$. Using the boundedness of $\|E_{\gamma(t)}z\|$ and $\left| \int_{t-\tau}^t u(s)ds \right| < +\infty$ for $t \in [0, +\infty)$, we have

$$\begin{aligned} \left\| \frac{d(E_{\gamma(t)}z)}{dt} \right\| &\le \gamma(t)^{-1}\|A_K\| \, \|E_{\gamma(t)}z\| + \|E_{\gamma(t)}\bar{\delta}(t, z, u_t)\|_1 \\ &\quad + \gamma(t)^{-1} \left| \int_{t-\tau}^t u(s)ds \right| \\ &\quad + \dot{\gamma}(t)\gamma(t)^{-1}\|\bar{D}\| \, \|E_{\gamma(t)}z\| < +\infty. \end{aligned} \tag{44}$$

Then, we obtain that $\|E_{\gamma(t)}z\| < +\infty$, $\int_0^t \|E_{\gamma(s)}z(s)\|^2 ds < +\infty$, and $\left\| \frac{d(E_{\gamma(t)}z)}{dt} \right\| < +\infty$ on $[0, +\infty)$. This yields $z \to 0$ as $t \to +\infty$ by Lemma 7 (Fontes, 2001) and boundedness of $\gamma(t)$. From $x_1 = z_i$, $i = 1, \ldots, n-1$ and $|x_n| \le |z| + \tau|u_t|$, for a finite τ, we have $|x_1| + \cdots + |x_n| \to 0$ as $t \to +\infty$ as well. Therefore, the global regulation is achieved. □

References

Chen, T., & Huang, J. (2009). Global robust output regulation by state feedback for strict feedforward systems. *IEEE Transactions on Automatic Control*, *54*(9), 2157–2163.

Choi, H.-L., & Lim, J.-T. (2006). Stabilization of a chain of integrators with an unknown delay in the input by adaptive output feedback. *IEEE Transactions on Automatic Control*, *51*(8), 1359–1363.

Choi, H.-L., & Lim, J.-T. (2010a). Asymptotic stabilization of an input-delayed chain of integrators with nonlinearity. *Systems & Control Letters*, *59*(6), 374–379.

Choi, H.-L., & Lim, J.-T. (2010b). Output feedback regulation of a chain of integrators with an unknown time-varying delay in the input. *IEEE Transactions on Automatic Control*, *55*(1), 363–368.

Fang, H., & Lin, Z. (2006). A further result on global stabilization of oscillators with bounded delayed input. *IEEE Transactions on Automatic Control*, *51*(1), 121–128.

Fontes, F. A. C. C. (2001). A general framework to design stabilizing nonlinear model predictive controllers. *Systems & Control Letters*, *42*(2), 127–143.

Hale, J. K., & Verduyn Lunel, S. M. (1993). *Introduction to functional differential equations*. New York: Springer-Verlag.

Jankovic, M. (2009). Cross-term forwarding for systems with time delay. *IEEE Transactions on Automatic Control*, *54*(3), 498–511.

Karafyllis, I. (2006). Finite-time global stabilization by means of time-varying distributed delay feedback. *SIAM Journal on Control and Optimization*, *45*(1), 320–342.

Koo, M.-S., Choi, H.-L., & Lim, J.-T. (2010). Global regulation of a class of uncertain nonlinear systems by switching adaptive controller. *IEEE Transactions on Automatic Control*, *55*(12), 2822–2827.

Krishnamurthy, P., & Khorrami, F. (2007). Generalized state scaling and applications to feedback, feedforward, and nontriangular nonlinear systems. *IEEE Transactions on Automatic Control*, *52*(1), 102–108.

Krstic, M. (2002). Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems. *IEEE Transactions on Automatic Control*, *49*(10), 1668–1682.

Krstic, M. (2010). Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, *55*(2), 287–303.

Mazenc, F., Mondié, S., & Niculescu, S. (2003). Global asymptotic stabilization for chains of integrators with a delay in the input. *IEEE Transactions on Automatic Control*, *48*(1), 57–63.

Michiels, W., Mondie, S., Roose, D., & Dambrine, M. (2004). The effect of approximating distributed delay control laws on stability. In *Lecture notes in comput. sci. eng.*: *Vol. 38*. *Advances in time-delay systems* (pp. 207–222). Berlin: Springer.

Olfari-Saber, R. (2001). Nonlinear control of underactuated mechanical systems with application to robotics and aerospace vehicles. "Ph.D. Dissertation". MIT.

Teel, A. R. (1993). Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, *18*(3), 165–171.

Ye, X. (2003). Universal stabilization of feedforward nonlinear systems. *Automatica*, *39*(1), 141–147.

Ye, X. (2005). Adaptive output feedback control of nonlinear systems with unknown nonlinearities. *Automatica*, *41*(8), 1367–1374.

Ye, X., & Unbehauen, H. (2004). Global adaptive stabilization for a class of feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, *49*(5), 786–792.

Yue, D. (2004). Robust stabilization of uncertain systems with unknown input delay. *Automatica*, *40*(2), 331–336.

Zhang, X. (2006). Global asymptotic stabilization of feedforward nonlinear systems with a delay in the input. *International Journal of Systems Science*, *37*(3), 141–148.

Zhang, X. (2009). Asymptotic stabilization of high-order feedforward systems with delays in the input. *International Journal of Robust and Nonlinear Control*, *20*(12), 1395–1406.

Zhou, B., Duan, G.-R., & Li, Z.-Y. (2008). On improving transient performance in global control of multiple integrators system by bounded feedback. *Systems & Control Letters*, *57*(10), 867–875.

Zhou, B., Duan, G.-R., & Li, Z.-Y. (2009). Properties of the parametric Lyapunov equation based low-gain design with applications in stabilization of time-delay systems. *IEEE Transactions on Automatic Control*, *54*(7), 1698–1704.

Zhou, B., Duan, G.-R., & Li, Z.-Y. (2011). Stabilization of a class of linear systems with input delay and the zero distribution of their characteristic equations. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *58*(2), 388–401.

Zwillinger, D. (2003). *Standard mathematical tables and formulae* (31st ed.). A CRC Press.

Min-Sung Koo received the B.S.E. degree in 2004 and M.S. degree in 2006 and Ph.D. degree in 2011 from the Department of Electrical Engineering, KAIST (Korea Advanced Institute of Science and Technology), Daejeon, Korea, respectively. She was a postdoctoral researcher at KAIST. Currently, she works at Korea Technology Finance Corporation. Her research interests include nonlinear system, switching system, high-order system and time-delay systems.

Ho-Lim Choi received the B.S.E. degree from the Department of Electrical Engineering, The University of Iowa, USA in 1996, and M.S. degree in 1999 and Ph.D. degree in 2004, from KAIST (Korea Advanced Institute of Science and Technology), respectively. Currently, he is an associate professor at the Department of Electrical Engineering, Dong-A University, Busan. His research interests are in the nonlinear control problems with an emphasis on feedback linearization, gain scheduling, singular perturbation, output feedback, time-delay systems. He is a member of IEEE, IEICE, KIEE, ICROS.

Jong-Tae Lim received the B.S.E.E. degree from Yonsei University, Seoul, Korea, in 1975, the M.S.E.E. degree from the Illinois Institute of Technology, Chicago, in 1983, and the Ph.D. degree in Computer, Information and Control Engineering from the University of Michigan, Ann Arbor, in 1986. He is currently a professor in the Department of Electrical Engineering and Computer Science, Korea Advanced Institute of Science and Technology. His research interests are in the areas of system and control theory, communication networks, and discrete event systems. He is a member of IEEE, IEICE, KIEE, and KITE.