A Two-Phase Algorithm for Frequency Assignment in Cellular Mobile Systems

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***Abstract*—In this paper, we consider the frequency assignment problem (FAP) in a cellular mobile communication system under the assumption that there is no channel interference between two cells separated by more than a certain distance. This special structure is observed in most cellular systems. To handle the considered FAP, we use the pattern approach which fits naturally to the problem. Based on this approach, we are able to formulate the considered FAP into a manageable optimization problem and propose a two-phase heuristic algorithm for the problem. Computational experiments show that our algorithm performs much better in both solution quality and computational time than the recently developed algorithms for FAP. Since the considered FAP well reflects most cellular systems, our algorithm can be applied to many practical situations.**

I. Introduction

RECENTLY, there has been a tremendous increase in traffic demand for communication services in cellular mobile communication systems [7], [8]. However, the frequency spectrum allocated to the systems is limited; hence, how to use frequency channels in the most economical way whether the systems are based on the *frequency division multiple access* (FDMA) or the *time division multiple access* (TDMA) [7], [8], [11] becomes more important. The *frequency assignment problem* (FAP) deals with this problem.

Inspired by the *graph coloring techniques*, many algorithms have been provided for FAP (see [1], [4], [6], [12], [14] and the literature quoted therein). Since FAP is an NP-complete optimization problem [6], most of these algorithms have a heuristic nature, and their results are still far from being satisfactory [2]. Furthermore, some researchers pointed out that any of the known heuristic algorithms could provide a solution whose quality deviates from the optimal value by more than 100% in certain cases [5]. This is mainly because their algorithms are targeted to FAP in general form and, therefore, cannot be effective in all the special cases of the problem.

In this paper, unlike the existing works, we consider the FAP in a cellular system with a special structure, which has a *maximal distance of channel interference* [2]–[4], [7], [8]. In this system, two cells whose mutual distance is greater than a certain number do not have any channel interference. Hence, it is possible to assign the same frequency channels to those distant cells. For this specially structured FAP, some theoretical results are provided when the required number of frequency channels is homogeneously distributed over the cells [3], [8]. For the nonhomogeneous case, there is none. We formulate the considered FAP into a manageable optimization problem by using the *pattern approach* [7], [8], [13] which fits naturally to the problem. The resulting formulation has a specific structure and this enables us to develop an excellent heuristic algorithm for the considered FAP. With randomly generated problems in a hexagonal cellular system, computational experiments of our algorithm show a significant improvement in both solution quality and computational time over the recent general-purpose algorithms for FAP. Since the considered FAP well reflects most cellular systems, our algorithm would be applicable to many practical situations.

In the next section, FAP is described formally. Section III explains the FAP in a cellular system with a maximal distance of channel interface. The formulation of this FAP based on the pattern approach is also provided in this section. In Section IV, we present a two-phase heuristic algorithm and some theoretical results about the algorithm. Section V shows the computational experiments of our algorithm. Finally, our discussion is contained in Section VI.

II. Frequency Assignment Problem (FAP)

In a cellular mobile communication system, in order to treat the traffic demand at the required grade of service, some number of frequency channels should be assigned to each cell of the system under the electromagnetic constraints such as *co-channel constraint*, *adjacent channel constraint*, and *co-site constraint* [2], [4]. The number of frequency channels allocated to a cellular system is finite because of the limited frequency spectrum. However, in recent years, demand for frequency channels has been increased drastically. Hence, how to assign the required number of frequency channels to each cell under the above constraints becomes an important problem. This problem is called FAP. It has been widely used as a decision criterion of FAP to minimize the maximum span of the frequency channels used in a cellular system [1]–[4], [6], [12], [14], and is also adopted in this paper. This allows the system to use the available frequency spectrum most efficiently (see [6] for a discussion).

The FAP to minimize the maximum span can be formalized as follows [2], [4]. Let $X = \{1, 2, \cdots, n\}$ denote the set of cells in a cellular system. A *requirement vector* on X is an n-vector $M = (m_i)$ with nonnegative integer components, where the component m_i represents the number of frequency channels required by ith cell. A *compatibility matrix* on X is

Manuscript received March 5, 1993; revised April 8, 1993 and October 11, 1993.

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IEEE Log Number 9403011.

a symmetric $n \times n$-matrix $C = (c_{ij})$ with nonnegative integer entries. Frequency channels are assumed to be evenly spaced, so they can be identified with positive integers. Assume that $F = \{1, 2, \cdots, f\}$ corresponds to the set of frequency channels which could be assigned to the system. Let f_{ki} be a binary variable which is equal to one if kth frequency channel is assigned to ith cell and, equal to zero, otherwise. Then, $z = \max_{k \in F, i \in X} \{k | f_{ki} = 1\}$ is the maximum span of the frequency channels used in the system and, for given X, M, C, and F, we need to find $\{f_{ki}\}$ to minimize z.

$$
\begin{aligned}
\text{(P)} \quad & \min \ z = \max_{k \in F, i \in X} \{k | f_{ki} = 1\} \\
& \text{s.t.} \sum_{k \in F} f_{ki} \geq m_i, \quad \text{for all } i \in X, \qquad (1) \\
& |k - l| \geq c_{ij}, \quad \text{for all } k,\ l \in F \text{ and } i, j \in X \\
& \qquad \text{such that } f_{ki} = f_{lj} = 1, \qquad (2) \\
& f_{ki} = 0 \text{ or } 1, \quad \text{for all } k \in F \text{ and } i \in X. \qquad (3)
\end{aligned}
$$

In the problem (P), the compatibility matrix $C = (c_{ij})$ of the constraint (2) represents all three types of the channel constraints. For example, if cells i and j are *co-channel cells*, then $c_{ij} = 0$. If there is a co-channel constraint between cells i and j, then $c_{ij} = 1$. If there is an adjacent channel constraint between cells i and j, then $c_{ij} \geq 2$. The co-site constraint is expressed by the diagonal element c_{ii}.

A special case of (P) whose compatibility matrix $C = (c_{ij})$ has only zero and one entries is reduced to a *graph coloring problem* [9], a well-known NP-complete problem. From this reason, (P) is often called the *generalized graph coloring problem* [6].

III. FAP With Maximal Distance of Channel Interference

Consider a cellular system, where two cells whose mutual distance is greater than a certain number, say d, do not have any channel interference. The number d is called the maximal distance of channel interference. The FAP in such a cellular system is a specific case of (P) with a compatibility matrix $C = (c_{ij})$ given by

$$
c_{ij} \begin{cases} = 0 & \text{if the distance between } i\text{th cell and} \\ & \quad j\text{th cell is greater than } d, \\ \geq 0 & \text{otherwise.} \end{cases} \qquad (4)
$$

In (4), following the assumptions made in [1], [2], [4], [12], we assume that $c_{ii} > c_{ij}$ for all $i, j \in X$, and $c_{ii} = \alpha > 0$ for all $i \in X$ throughout this paper.

For solving the FAP with the given structure of the compatibility matrix, we suggest using the pattern approach which fits naturally to this problem. To explain our approach more conveniently, we will consider a regular hexagonal cellular system. The generalization of our approach to irregular non-hexagonal cellular systems is discussed in the last section of this paper.

Fig. 1 shows a system of 49 regular hexagonal cells and the 19 cells of the system which may have channel interference with the cell "A" when the number d is given by $\sqrt{4} = 2$ (i.e., $d = \sqrt{Q(i,j)} = \sqrt{i^2 + ij + j^2}$, i and j are *shift parameters* [3], [4], [7], [8]). In this figure, the frequency channels assigned to the cell A can be reused in the cells other than the 19 cells.

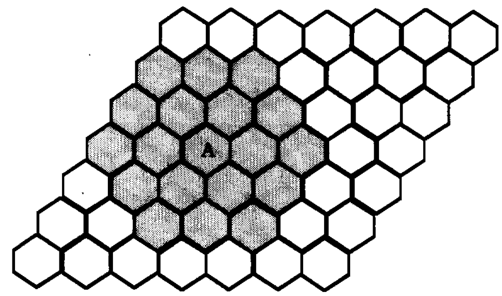


Fig. 1. Cells having interference with A.

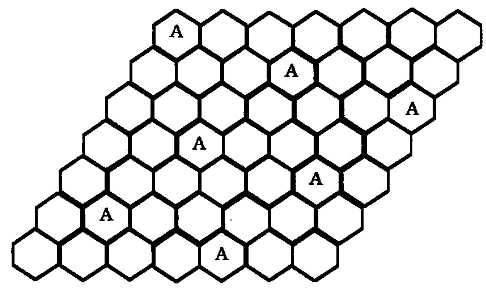
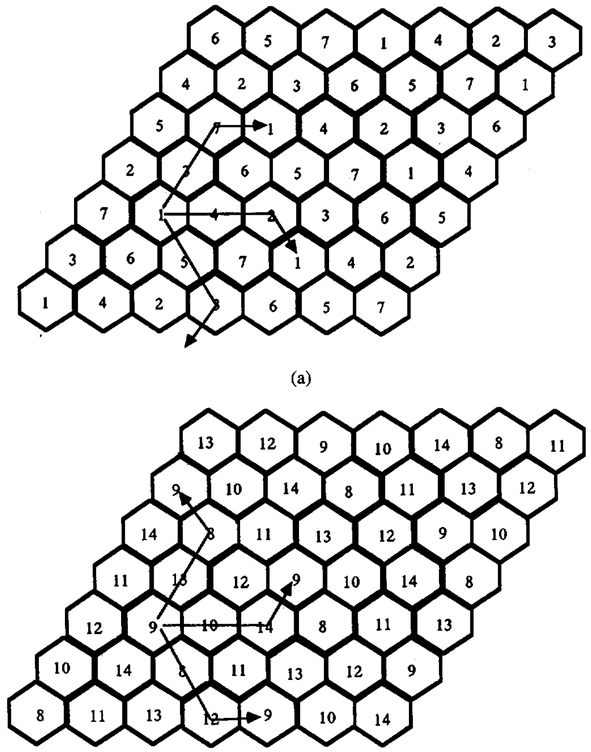


Fig. 2. Pattern.

For the considered FAP in a regular hexagonal cellular system, we use the *compact* pattern approach [13]. In assigning frequency channels to a cellular system, the set of co-channel cells forms a pattern. Fig. 2 shows a pattern where co-channel cells are labeled "A." A compact pattern is a pattern such that the average distance between co-channel cells is minimal. For example, Fig. 3 shows 14 compact patterns when the minimum distance between co-channel cells, say $\overline{d}$, is set to $\sqrt{7}$. These 14 compact patterns are composed of seven *clockwise* compact patterns and seven *anticlockwise* compact patterns. Each cell belongs to exactly two compact patterns—clockwise and anticlockwise. Fig. 4 depicts a compact pattern approach which is a frequency reuse scheme when there are 14 compact patterns.

By comparing Fig. 1 to Fig. 4, it is easily seen that the compact pattern approach is naturally applicable to the considered FAP in a regular hexagonal cellular system. This can be done by adjusting the number $\overline{d}$ of the compact patterns. For a valid compact pattern approach, $\overline{d}$ should be greater than the number d. Now suppose that $\overline{d}\,(>d)$ is given. Then we can construct clockwise and anticlockwise compact patterns by finding two solution pairs of positive integers i and j which satisfy the equation $\overline{d}^2 = Q(i,j) = i^2 + ij + j^2$. For example, in Fig. 3, the solution pair (i,j) = (1, 2) is for clockwise compact patterns, and (i,j) = (2, 1) is for anticlockwise (when one solution pair for clockwise compact patterns is obtained, another for anticlockwise in straightforward). However, in the cases such as $\overline{d} = \sqrt{3}, \sqrt{4}, \sqrt{9}, \sqrt{12}$, etc., only one solution



(a)

(b)

Fig. 3. Compact patterns. (a) Clockwise. (b) Anticlockwise. (The number in each cell denotes the compact pattern to which the cell belongs)

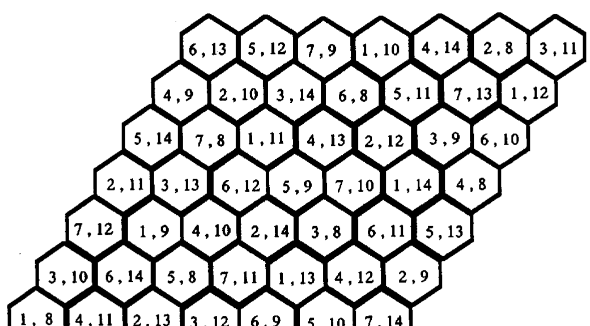


Fig. 4. Compact pattern approach. (The numbers in each cell denote the compact patterns to which the cell belongs)

pair for the above equation exists and, therefore, it makes no difference between clockwise and anticlockwise compact patterns. Hence, it is desirable to adjust $\overline{d}$ to the smallest value such that $\overline{d} > d$ and $\overline{d}^2 = Q(i,j)$ has two different solution pairs at least. When there are more than two different solution pairs, e.g., $\overline{d} = \sqrt{49}$, any of them can be used.

Now, based on the compact pattern approach, we formulate the FAP in a regular hexagonal cellular system with a maximal distance of channel interference. Our formulation focuses on how to assign frequency channels optimally to each compact pattern, not to each cell. After frequency channels are assigned to each compact pattern, reassignment of these channels to each cell is done according to the scheme similar to Fig. 4. Let the set $G_1 = \{1, \cdots, r\}$ represent clockwise compact patterns, and $G_2 = \{r+1, \cdots, 2r\}$ represent anticlockwise. Then, each of G_1 and G_2 is a set of disjoint patterns and covers all cells in the system. Let $G = G_1 \cup G_2 = \{1, \cdots, 2r\}$ be the set of all the compact patterns (the number r is equal to the number $\overline{d}^2$ of the compact patterns [8]). For each $p \in G$, let X_p, a subset of X, be the set of cells which belong to pth compact pattern. For convenience, we assume that $X_p \cap X_q$ is nonempty for all $p \in G_1$ and $q \in G_2$. Let f_{kp} be a binary variable which is equal to one if kth frequency channel is assigned to pth compact pattern and, equal to zero, otherwise.

Consider the cells which belong to both pth and qth compact patterns, where $p \in G_1$ and $q \in G_2$. Then, to cover the requirement of these cells, it should be satisfied that $\Sigma_{k \in F}(f_{kp} + f_{kq}) \geq m_i$ for all $i \in X_p \cap X_q$. Now, define $\overline{m}_{pq} = \max\{m_i | i \in X_p \cap X_q\}$ for all $p \in G_1$ and $q \in G_2$. Then, the number of frequency channels assigned to pth and qth compact patterns should satisfy the following requirement constraint

$$\sum_{k \in F}(f_{kp} + f_{kq}) \geq \overline{m}_{pq}, \quad \text{for all } p \in G_1 \text{ and } p \in G_2. \quad (5)$$

The separation between frequency channels assigned to pth and qth compact patterns should be greater than or equal to c_{ij} for all $i \in X_p$ and $j \in X_q$. For all $p, q \in G$, let $\overline{c}_{pq} = \max\{c_{ij} | i \in X_p \text{ and } j \in X_p\}$. Then, we have the following channel separation constraint of our formulation

$$|k - l| \geq \overline{c}_{pq}, \quad \text{for all } k, l \in F \text{ and } p, p \in G \text{ such that } f_{kp} = f_{lq} = 1. \quad (6)$$

With the constraints (5) and (6), our formulation is given by

$$(\mathrm{P_c}) \qquad \min z = \max_{k \in F, p \in G}\{k | f_{kp} = 1\}$$
$$\text{s.t.} \quad (5) \text{ and } (6),$$
$$f_{kp} = 0 \text{ or } 1, \quad \text{for all } k \in F \text{ and } p \in G. \quad (7)$$

Since $(\mathrm{P_c})$ is a rather simplified formulation of the considered FAP in a regular hexagonal cellular system, the optimal span of $(\mathrm{P_c})$ may be greater than that of (P) with the compatibility matrix given by (4). However, the formulation of $(\mathrm{P_c})$ has some merits. In (P), the problem size depends on the number of cells in the system. On the other hand, in $(\mathrm{P_c})$, the problem size is determined by the number of compact patterns. Since the number of compact patterns is much smaller than the number of cells in most cases, $(\mathrm{P_c})$ is much smaller than (P). Moreover, $(\mathrm{P_c})$ has some special structure and it is well utilized in our algorithm for $(\mathrm{P_c})$. Our interest is how efficiently the formulation $(\mathrm{P_c})$ approximates the considered FAP in a regular hexagonal system, and it is empirically verified by the computational experiments in Section V.

Example 1: Consider a regular hexagonal cellular system of 49 cells which has the same cell structure as the system in Fig. 1. A requirement vector $M = (m_i)$ of size 49 is given. A 49×49 compatibility matrix $C = (c_{ij})$ is also given according to (4) with $d = \sqrt{4} = 2$. To formulate $(\mathrm{P_c})$ for the system, we can construct 14 compact patterns of Fig. 3 (i.e., $\overline{d} = \sqrt{7}$). With relation to the constraints (5) and (6) in $(\mathrm{P_c})$, we can obtain $\overline{m}_{pq} = m_i$, where $i \in X_p \cap X_q$, and the 14×14 symmetric matrix $\overline{C} = (\overline{c}_{pq})$ given by

$$\bar{C} = \begin{pmatrix} \alpha & * & * & * & * & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & \alpha & * & * & * & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & * & \alpha & * & * & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & * & * & \alpha & * & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & * & * & * & \alpha & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & * & * & * & * & \alpha & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ * & * & * & * & * & * & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & * & * & * & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & \alpha & * & * & * & * & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & \alpha & * & * & * & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & * & \alpha & * & * & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & * & * & \alpha & * & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & * & * & * & \alpha & * \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & * & * & * & * & * & * & \alpha \end{pmatrix} \tag{8}$$

where α is the number given in (4), and $*$ dentoes a nonnegative number less than α. In $(\mathrm{P_c})$, the matrix $\overline{C}$ generally has the same structure as (8) except the matrix size.

IV. Two-Phase Heuristic Algorithm

We suggest an efficient algorithm for $(\mathrm{P_c})$ in this section. Since $(\mathrm{P_c})$ is still a large problem, our algorithm is basically a heuristic algorithm. However, it becomes an exact algorithm for $(\mathrm{P_c})$ when certain conditions are satisfied.

A. Phase I: Obtaining a Good Satisfying Vector

First, we make the following definition in relation to the constraint (5) of $(\mathrm{P_c})$.

Definition 1: For the problem $(\mathrm{P_c})$, a $|G|$-sized vector $\hat{M} = (\hat{m}_p)$ of nonnegative integer components is called *satisfying vector* of $(\mathrm{P_c})$ if $\hat{m}_p + \hat{m}_q \geq \overline{m}_{pq}$ for all $p \in G_1$ and $q \in G_2$, where $|G|$ is the number of elements in the set G (i.e., $|G| = 2r$).

For a given satisfying vector $\hat{M}$ of $(\mathrm{P_c})$, we construct the following problem

$$(\mathrm{P_c}(\hat{M})) \qquad \min z = \max_{k\in F, p\in G} \{k | f_{kp} = 1\}$$

$$\text{s.t.} \sum_{k\in F} f_{kp} = \hat{m}_p, \quad \text{for all} \quad p \in G, \tag{5'}$$

$$(6) \quad \text{and} \quad (7).$$

The constraint $(5')$ says that the number of frequency channels assigned to pth compact pattern should be equal to $\hat{m}_p$. Hence, pth component of the satisfying vector $\hat{M}$ means the number of frequency channels that should be assigned to pth compact pattern.

Now, from Definition 1 and the formulation of $(\mathrm{P_c}(\hat{M}))$, we have the following lemmas.

Lemma 1: $v(\mathrm{P_c}) = \min\{v(\mathrm{P_c}(\hat{M}))|$ satisfying vector $\hat{M}$ of $(\mathrm{P_c})\}$, where $v(\cdot)$ denotes the optimal object function value of the problem $(\cdot)$.

Proof: For any satisfying vector $\hat{M}$ of $(\mathrm{P_c})$, every feasible solution of $(\mathrm{P_c}(\hat{M}))$ is also feasible in $(\mathrm{P_c})$ by Definition 1. Hence, $v(\mathrm{P_c}) \leq v(\mathrm{P_c}(\hat{M}))$. Let an optimal solution of $(\mathrm{P_c})$ be $\{f^*_{kp}\}$. Then, we can construct a satisfying vector of $(\mathrm{P_c}), \hat{M}^* = (\Sigma_{k\in F} f^*_{k1}, \Sigma_{k\in F} f^*_{k2}, \cdots, \Sigma_{k\in F} f^*_{k|G|})$. It is obvious that an optimal solution of $(\mathrm{P_c}(\hat{M}^*))$ is also $\{f^*_{kp}\}$. Therefore, $v(\mathrm{P_c}) = v(\mathrm{P_c}(\hat{M}^*))$, and thus the proof is complete. ■

Lemma 2: $v(\mathrm{P_c}(\hat{M})) \geq \mathrm{LB}(\mathrm{P_c}(\hat{M})) = 1 + \beta(\Sigma_{p\in G} \hat{m}_p - 2) + \alpha$, where α is the number given in (4) and $\beta = \min\{\overline{c}_{pq} | p, q \in G\}$ in the constraint (6) of $(\mathrm{P_c}(\hat{M}))$.

Proof: Consider a relaxed problem of $(\mathrm{P_c}(\hat{M}))$, where the matrix $\overline{C} = (\overline{c}_{pq})$ in the constraint (6) of $(\mathrm{P_c}(\hat{M}))$ is replaced by

$$\bar{C} = \begin{pmatrix} \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta & \beta & \beta & \beta \end{pmatrix} \tag{9}$$

Without loss of generality, we assume that $\overline{C}$ is a 14×14 matrix (compare this matrix to (8) in Example 1). In this relaxed problem, we can aggregate the clockwise compact patterns as one pattern, say y_1, and the anticlockwise compact patterns as y_2. Since α greater than β from the assumption in (4), an optimal frequency assignment of the relaxed problem is $\{1, 1+\beta, \cdots, 1+\beta(\Sigma_{p\in G_1} \hat{m}_p - 1)\}$ for the pattern y_1, and $\{1+\beta(\Sigma_{p\in G_1} \hat{m}_p - 1)+\alpha, 1+\beta\Sigma_{p\in G_1} \hat{m}_p + \alpha, \cdots, 1+\beta(\Sigma_{p\in G} \hat{m}_p - 2)+\alpha\}$ for the pattern y_2. Therefore, the proof is complete. ■

If we can have a good satisfying vector $\hat{M}$ of $(\mathrm{P_c})$, then, from Lemma 1, it is likely that an accurate solution of $(\mathrm{P_c})$ can be obtained by solving the problem $(\mathrm{P_c}(\hat{M}))$ which is smaller sized than $(\mathrm{P_c})$. In this paper, we suggest the following problem to obtain a good satisfying vector of $(\mathrm{P_c})$.

Phase I:

$$(\mathrm{P^1_c})$$

$$\min \left(\sum_{p\in G_1} m_p + \sum_{q\in G_2} m_q \right)$$

$$\text{s.t.} \quad m_p + m_q \geq \overline{m}_{pq}, \quad \text{for all} \quad p \in G_1 \text{ and } q \in G_2, \tag{10}$$

m_p and m_q are nonnegative integers for all $p \in G_1$

$$\text{and} \quad q \in G_2. \tag{11}$$

Consider an optimal solution of (P_c^1) denoted by a vector $\tilde{M} = (\tilde{m}_p)$ of size $|G|$. Then, $\tilde{M}$ is a satisfying vector of (P_c) such that the summation value of its components is minimized. Therefore, in view of $(\mathrm{P}_c(\tilde{M}))$, the vector $\tilde{M}$ has the property that the total number of frequency channels assigned to the compact patterns is minimized.

The Phase I problem (P_c^1) has integerness condition. However, the constraint matrix of (10) satisfies the *totally unimodular property*, and $\overline{m}_{pq}$ are integers; hence, we can delete the integerness condition [9], [10]. Then, (P_c^1) is a simple linear programming problem. Furthermore, the dual of the LP problem (P_c^1) is a well-known *assignment problem* which has a very efficient algorithm called the *Hungarian method* [10]. From a dual solution of (P_c^1), we can easily obtain an optimal solution of the problem (P_c^1). The computational complexity of this approach is $O(|G_1|^3)$ or $O(|G_2|^3)$ [10].

Once we have obtained an optimal solution $\tilde{M} = (\tilde{m}_p)$ of the Phase I problem (P_c^1), then we solve the problem $(\mathrm{P}_c(\tilde{M}))$ in Phase II. If we could obtain an optimal solution of the problem $(\mathrm{P}_c(\tilde{M}))$, then this solution is also optimal for (P_c) under a certain condition.

Theorem 1: If $v(\mathrm{P}_c(\tilde{M})) = \mathrm{LB}(\mathrm{P}_c(\tilde{M})) = 1 + \beta(\Sigma_{p\in G}\tilde{m}_p - 2) + \alpha$, then an optimal solution of $(\mathrm{P}_c(\tilde{M})), \{\tilde{f}_{kp}\}$, is also an optimal solution of (P_c).

Proof: Since $\tilde{M}$ is satisfying vector, $\{\tilde{f}_{kp}\}$ is a feasible solution of (P_c) by Definition 1. Therefore, $v(\mathrm{P}_c(\tilde{M})) \geq v(\mathrm{P}_c)$. From Lemma 1, consider a satisfying vector $\hat{M}^*$ such that $v(\mathrm{P}_c(\hat{M}^*)) = v(\mathrm{P}_c)$. From the property of $\tilde{M}$ and Lemma 2, $v(\mathrm{P}_c(\tilde{M})) = \mathrm{LB}(\tilde{M}) \leq \mathrm{LB}(\hat{M}^*) \leq v(\mathrm{P}_c(\hat{M}^*)) = v(\mathrm{P}_c)$. Therefore, the proof is complete. ■

B. Phase II: Exact Algorithm for a Special Case of (P_c)

In this subsection, we consider a slightly more special case of the problem (P_c), where the matrix $\overline{C} = (\overline{c}_{pq})$ in the constraint (6) is given as follows

$$\bar{C} = \begin{pmatrix}
\alpha & \gamma & \gamma & \gamma & \gamma & \gamma & \gamma & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\gamma & \alpha & \gamma & \gamma & \gamma & \gamma & \gamma & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\gamma & \gamma & \alpha & \gamma & \gamma & \gamma & \gamma & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\gamma & \gamma & \gamma & \alpha & \gamma & \gamma & \gamma & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\gamma & \gamma & \gamma & \gamma & \alpha & \gamma & \gamma & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
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\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \gamma & \gamma & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \alpha & \gamma & \gamma & \gamma & \gamma & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \alpha & \gamma & \gamma & \gamma & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \alpha & \gamma & \gamma & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \gamma & \alpha & \gamma & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \gamma & \gamma & \alpha & \gamma \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \gamma & \gamma & \gamma & \alpha
\end{pmatrix} \tag{12}$$

In (12), γ is a positive integer less than α, and we assume $\overline{C}$ is 14×14 without loss of generality. In this case, we can provide an efficient Phase II algorithm which can generate an exact optimal solution of (P_c) under certain conditions. Let the vector $\tilde{M} = (\tilde{m}_p)$ be an optimal solution of the Phase I problem (P_c^1). Now, for the problem $(\mathrm{P}_c(\tilde{M}))$, define $r_1 = \max\{\tilde{m}_p | p \in G_1\}$ and $r_2 = \max\{\tilde{m}_q | q \in G_2\}$. Let p^* be the number of components of $\tilde{M}$ such that $\tilde{m}_p = r_1$ for $p \in G_1$, and let q^* be the number of components of $\tilde{M}$ such that $\tilde{m}_q = r_2$ for $q \in G_2$. Then, the following algorithm for $(\mathrm{P}_c(\tilde{M}))$ generates an optimal solution of $(\mathrm{P}_c(\tilde{M}))$, and this solution is also an optimal solution of (P_c) provided that $p^* > \lfloor \alpha/\gamma \rfloor$ and $q^* > \lfloor \alpha/\gamma \rfloor$, where $\lfloor \alpha/\gamma \rfloor$ is the integer part of the number α/γ.

Algorithm A: Phase II Algorithm for $(\mathrm{P}_c(\tilde{M}))$ when $\overline{C}$ is given by (12).

Step 1: (Initialize): Set $\delta = 1$.

1) Assign index $1, \cdots, |G_1|$ to clockwise compact patterns in decreasing order of $\tilde{m}_p$ for all $p \in G_1$.
2) Assign index $|G_1|+1, \cdots, |G|$ to anticlockwise compact patterns in decreasing order of $\tilde{m}_q$ for all $q \in G_2$.

Step 2: (Clockwise compact patterns): If $\tilde{m}_p = 0$ for all $p \in G_1$, then go to Step 3.

1) Set $p = 1$.
2) If $\tilde{m}_p = 0$, then go to Step 2.4.
3) Assign frequency $1+\gamma(\delta-1)$ to pth clockwise compact pattern. Set $\tilde{m}_p = \tilde{m}_p - 1$ and $\delta = \delta + 1$.
4) Set $p = p+1$. If $p > |G_1|$, then go to Step 2. Otherwise, go to Step 2.2.

Step 3: (Anticlockwise compact patterns): If $\tilde{m}_q = 0$, for all $q \in G_2$, then stop and take the solution.

1) Set $q = |G_1| + 1$.
2) If $\tilde{m}_q = 0$, then go to Step 3.4.
3) Assign frequency $1 + \gamma(\delta - 2) + \alpha$ to qth anticlockwise compact pattern. Set $\tilde{m}_q = \tilde{m}_q - 1$ and $\delta = \delta + 1$.
4) Set $q = q+1$. If $q > |G|$, then go to Step 3. Otherwise, go to Step 3.2.

Theorem 2: If $p^* > \lfloor \alpha/\gamma \rfloor$ and $q^* > \lfloor \alpha/\gamma \rfloor$, then Algorithm A generates an optimal solution of (P_c) with the frequency span $1 + \gamma(\Sigma_{p\in G}\tilde{m}_p - 2) + \alpha$, when the matrix $\overline{C} = (\overline{c}_{pq})$ in the constraint (6) is given by (12).

Proof: In the problem $(\mathrm{P}_c(\tilde{M}))$, from Step 2 and Step 3 of Algorithm A, it is evident that the solution generated by Algorithm A satisfies the constraint (5). From Steps 2.3 and 3.3, the solution satisfies the constraint (6) when $p \neq q$. Let $f(u, v)$ denote the frequency channel which Algorithm A assigns to uth compact pattern in vth turn $(v = 1, \cdots, \tilde{m}_u)$. If $u \in G_1$, then $f(u,v) - f(u, v-1) \geq p^*\gamma > \lfloor \alpha/\gamma \rfloor \gamma \geq \alpha$ for all $2 \leq v \leq \tilde{m}_u$. If $u \in G_2$, then $f(u,v) - f(u,v-1) \geq q^*\gamma > \lfloor \alpha/\gamma \rfloor \gamma \geq \alpha$ for all $2 \leq v \leq \tilde{m}_u$. Therefore, $f(u,v) - f(u,v-1) \geq \alpha$ for all $u \in G$ and v, and this means that the solution generated by Algorithm A satisfies the constraint (6) when $p = q$. From Step 3.3, we can easily see that the frequency span of the solution is $1 + \gamma(\Sigma_{p\in G}\tilde{m}_p - 2) + \alpha$. Therefore, from Theorem 1, Algorithm A generates an optimal solution of (P_c). ■

C. Phase II: General Heuristic Algorithm

Theorem 2 says that we can obtain an optimal solution of a special case of (P_c). However, for a general case of problem (P_c), it is desirable to produce a good feasible solution of

TABLE I
COMPUTATIONAL RESULTS

Prob. No.		1	2	3	4	5	6	7	8	9
c_{ij}		1	2,3	3,4	1	2,3	3,4	1	2,3	3,4
c_{ii}		3	5	7	3	5	7	3	5	7
2Phase[a]	SPAN[b]	**98**	**279**[c]	**368**	**127**	**366**[c]	**488**	**173**	**546**[c]	**672**[c]
	TIME[d]	2.08	0.046	13.75	3.24	0.047	23.67	6.15	0.052	0.054
CRF	SPAN	127	437	583	168	470	624	234	769	837
	TIME	6.04	33.83	52.83	7.14	46.35	74.42	11.04	83.32	104.68
CRR	SPAN	127	319	457	168	381	520	234	610	705
	TIME	8.62	23.67	33.83	14.39	35.81	47.73	25.81	80.41	86.39
CCF	SPAN	110	325	445	143	393	545	194	580	724
	TIME	8.18	42.51	69.26	10.76	57.17	92.93	15.59	92.05	104.27
CCR	SPAN	156	332	473	176	432	556	255	594	702
	TIME	10.98	24.33	34.60	15.10	39.76	50.03	27.73	74.48	85.07
DRF	SPAN	134	400	531	161	495	635	236	764	846
	TIME	5.71	32.35	49.92	6.97	46.52	65.08	10.10	74.69	109.35
DRR	SPAN	134	303	448	161	384	540	236	586	692
	TIME	9.28	23.12	34.27	14.00	36.74	50.64	27.40	78.37	85.51
DCF	SPAN	110	318	445	133	391	548	180	560	727
	TIME	10.49	42.89	66.89	13.18	55.91	86.83	18.34	88.21	130.28
DCR	SPAN	134	313	440	193	391	558	237	582	704
	TIME	9.22	22.29	32.51	16.53	36.19	50.14	26.74	77.06	86.34

[a] Two-phase algorithm. [b] Frequency span. [c] Optimal solutions of (P_c) have been obtained. [d] Computational time, in seconds of an IBM-PC 486 machine.

($P_c(\tilde{M})$), where the vector $\tilde{M}$ is an optimal solution of the Phase I problem. Although the problem ($P_c(\tilde{M})$) is smaller than (P_c), it still has many binary integer variables. Hence, we suggest a heuristic algorithm for ($P_c(\tilde{M})$). This algorithm is a slight generalization of the *Largest First Method* of graph coloring which appears in [1], [4], [12], [14]. The algorithm is as follows.

Algorithm B: General Phase II algorithm for ($P_c(\tilde{M})$).

1) Set $f^* = 1$.
2) Compute $d_q = \tilde{m}\,\Sigma_{p\in G}\,\bar{c}_{pq}$ for all $q \in G$.
3) Assign index $1, \cdots, |G|$ to all compact patterns in decreasing order of d_q.
4) Take the frequency channel f^* and assign it to the first *assignable* compact pattern, say q_1.
5) Set $\tilde{m}_{q_1} = \tilde{m}_{q_1} - 1$.
6) If $\tilde{m}_q = 0$, for all $q \in G$, then stop and take the solution. Otherwise, set $f^* = f^* + 1$ and go to Step 2.

In Step 2 of Algorithm B, the value d_q is called the *degree of difficulty*, which can be used as a heuristic measure of the difficulty of assigning a frequency channel to qth compact pattern. In Step 4, if we say kth frequency channel is assignable to pth compact pattern, then this means that, considering only the assignment of frequency channels 1 to $k-1$, we can assign kth frequency channel to pth compact pattern without violating any constraints of the problem ($P_c(\tilde{M})$). In Step 6, when the algorithm terminates, we take the value f^* as the frequency span of ($P_c(\tilde{M})$) and, therefore, that of (P_c).

V. COMPUTATIONAL EXPERIMENTS

In this section, we compare our two-phase algorithm to the eight general-purpose algorithms for FAP which are based on the graph coloring techniques and called CRF, CRR, CCF, CCR, DRF, DRR, DCF, and DCR, respectively [12]. These algorithms are extensions of the sequential heuristic algorithms summarized in [1], [4], [14]. Our algorithm and the other eight algorithms have been coded in PASCAL and run on an IBM-PC 486 machine (33 MHz).

Computational results are summarized in Table I. Test problems are from the regular hexagonal cellular system of Example 1 in Section III. In the experiments, we set the channel separation between each pair of nonco-channel cells by 1 (i.e., absence of the adjacent constraint), 2 or 3, and 3 or 4. For the problems 1, 2, and 3, we nonhomogeneously generate the required number of frequency channels in each cell from the uniform distribution $U(10, 15)$; for the problems 4, 5, and 6, from $U(10, 20)$; and for the problems 7, 8, and 9, from $U(10, 30)$. The problems 2, 5, 8, and 9 satisfy the conditions of Theorem 2. Therefore, in Phase II, we use Algorithm A for these problems and obtain optimal solutions of (P_c). For the other problems, Algorithm B is used in Phase II.

In the experiments, our algorithm produces much better solutions. Furthermore, our algorithm requires a smaller computational burden (in particular, the cases when Algorithm A is used in Phase II require very small computational time). This says that our algorithm is appropriate for the cellular systems where the traffic demand of each cell varies at short intervals. In conclusion, the computational results empirically verify that the formulation (P_c) efficiently approximates the FAP in a regular hexagonal cellular system with a maximal distance of channel interference, and our algorithm for (P_c) performs quite well.

VI. DISCUSSION

In this paper, we considered the FAP in a cellular mobile communication system with a maximal distance of channel interference which appears in many practical situations. In the case of a regular hexagonal cellular system, we adopted the compact pattern approach and presented a two-phase heuristic algorithm for the problem. Computational results reported are quite encouraging.

We can easily generalize our approach to an irregular nonhexagonal system with a maximal distance of channel interference. In this case, the compact patterns are not applicable. Instead, we need to generate two sets, G_1 and G_2, of patterns such that each set has disjoint patterns and covers all cells in the system. We can easily geneate such sets of patterns. However, if we generate patterns more compactly (i.e., in such a way that the average distance between co-channel cells is minimal), then our algorithm would be more efficient. Generating such patterns in an irregular nonhexagonal system is an interesting subject for future research work.

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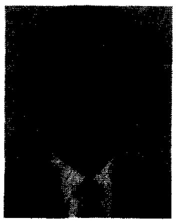

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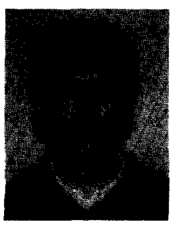

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