Cavity-QED-based scheme for verification of the photon commutation relation

2010 New J. Phys. 12 033019

(http://iopscience.iop.org/1367-2630/12/3/033019)

New Journal of Physics
The open–access journal for physics

Cavity-QED-based scheme for verification of the photon commutation relation

Jiyong Park[1], **Su-Yong Lee**[2], **Ho-Joon Kim**[1] **and Hai-Woong Lee**[1,3]

[1] Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
[2] Department of Physics, Texas A&M University at Qatar, Education City, PO Box 23874, Doha, Qatar
E-mail: hwlee@kaist.edu

New Journal of Physics **12** (2010) 033019 (12pp)
Received 4 December 2009
Published 12 March 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/3/033019

Abstract. We propose an experimental scheme to prove the photon commutation relation, $[a, a^\dagger] = 1$. The scheme exploits interaction between a single-mode cavity field and three two-level atoms, two of which are prepared in an entangled state. The success of the scheme is subject to observation of the atoms in predetermined states after the interaction. We show, in particular, that a reasonably high success probability can be obtained with our scheme by preparing the cavity field in a superposition of two Fock states and choosing the interaction times appropriately.

Contents

[3] Author to whom any correspondence should be addressed.

1. Introduction

Recent progress in quantum state engineering has made the experimental realization of single-photon creation operation $a^\dagger$ and annihilation operation a possible [1]. The field states generated by such operations are usually termed photon-added and photon-subtracted states, respectively. Photon-added coherent and thermal states were first considered by Agarwal and Tara [2, 3], who showed that these states exhibit non-classical properties that are absent in the original coherent and thermal states. The photon creation operation $a^\dagger$ on coherent and thermal states was performed experimentally by Zavatta *et al* [4]–[6]. The same group achieved sequences of photon creation and annihilation operations, namely photon-annihilation-then-creation operation $a^\dagger a$ and photon-creation-then-annihilation operation $aa^\dagger$ [7]. They showed, in particular, that operations $a^\dagger a$ and $aa^\dagger$ applied to the same field state produce states exhibiting different photon statistics, proving experimentally the non-commutativity of operators a and $a^\dagger$. This, however, does not prove the exact commutation relation $[a, a^\dagger] = 1$. The exact commutation relation can only be proven if one constructs a scheme that performs a coherent superposition of the two operations $a^\dagger a$ and $aa^\dagger$. One such scheme was recently proposed [8] and realized experimentally [9], in which a coherent superposition of the two operations is achieved in a single-photon interferometer setup employing beam splitters and a parametric amplifier.

The commutation relation $[a, a^\dagger] = 1$ is the most fundamental relation that defines quantum optics, and it is thus of fundamental importance to prove it experimentally. In this paper, we propose a cavity-quantum electrodynamics (QED)-based scheme to prove the commutation relation $[a, a^\dagger] = 1$. A theoretical analysis of cavity-QED-based schemes that perform individual photon creation and annihilation operations and their combinations, including the commutation operation, has been given recently [10]. A cavity-QED-based scheme to realize the photon-annihilation-then-creation operation $a^\dagger a$ and the creation-then-annihilation operation $aa^\dagger$ has been proposed [11, 12], exploiting interaction between two atoms and a cavity field. The scheme we propose here exploits interaction between three atoms, two of which are entangled initially, and a cavity field. A coherent superposition of the two operations, $a^\dagger a$ and $aa^\dagger$, results from the initial entanglement of the two atoms and post-selection of the states of the atoms that erases information on whether photon subtraction occurs before or after photon addition. Like the scheme proposed by Kim *et al* [8], our scheme is probabilistic as it relies on post-selection of the states of the atoms after their interaction with the cavity field. We pay particular attention to the issue of the success probability of the scheme.

In section 2, we briefly review the fundamentals of the interaction between a two-level atom and a monochromatic radiation field. The scheme we propose is described in section 3. It is shown that the final state of the cavity field after its interaction with the three atoms is approximately equal to the state that results when the commutation operation $aa^\dagger - a^\dagger a$ is applied to the initial cavity state, if the interaction times are chosen sufficiently short. By showing that the final state of the cavity field is the same as the initial cavity state, the commutation relation can thus be proven. The success probability of the scheme, however, is generally very small. In section 4, we show that the success probability of the scheme can significantly be enhanced if consideration is restricted to the case when the cavity field is prepared initially in a superposition of two Fock states. In section 5, we describe the experimental feasibility of the proposed scheme. Finally, section 6 presents a summary.

2. Atom–field interaction

We briefly review the atom–cavity field interaction and introduce our notation. The time evolution of the atom–field system is determined by

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi(t=0)\rangle = \mathrm{e}^{-\mathrm{i}gt(a^\dagger\sigma_- + a\sigma_+)}|\psi(t=0)\rangle. \tag{1}$$

In (1), g is the atom–field coupling constant assumed to be real, σ_- and σ_+ are atomic ladder operators defined as

$$\begin{aligned} \sigma_- &= |g\rangle\langle e|, \\ \sigma_+ &= |e\rangle\langle g|, \end{aligned} \tag{2}$$

with $|g\rangle$ and $|e\rangle$ denoting the lower and upper states of the atom, and the photon annihilation and creation operators a and $a^\dagger$ are given by

$$\begin{aligned} a &= \sum_{n=1}^{\infty} f_n |n-1\rangle\langle n|, \\ a^\dagger &= \sum_{n=1}^{\infty} f_n^* |n\rangle\langle n-1| = \sum_{n=0}^{\infty} f_{n+1}^* |n+1\rangle\langle n|. \end{aligned} \tag{3}$$

Note that we leave f_n of (3) unspecified, because our purpose is to propose an experimental scheme that proves the commutation relation $[a, a^\dagger] = 1$.

If the initial state of the atom–cavity system is given by $|g\rangle \sum_n c_n|n\rangle$, corresponding to the situation where initially the atom is prepared in $|g\rangle$ and the cavity field is in an arbitrary pure state $\sum_n c_n|n\rangle$, (1) yields

$$|\psi(t)\rangle = |g\rangle C_g(t) \sum_n c_n|n\rangle - \mathrm{i}|e\rangle S_g(t) \sum_n c_n|n\rangle. \tag{4}$$

If the initial state is given by $|e\rangle \sum_n c_n|n\rangle$, then (1) yields

$$|\psi(t)\rangle = |e\rangle C_e(t) \sum_n c_n|n\rangle - \mathrm{i}|g\rangle S_e(t) \sum_n c_n|n\rangle. \tag{5}$$

In (4) and (5), $C_g(t)$, $S_g(t)$, $C_e(t)$ and $S_e(t)$ are field operators defined as

$$C_g(t) = \cos(gt\sqrt{a^\dagger a}) = 1 - \frac{(gt)^2}{2!}a^\dagger a + \frac{(gt)^4}{4!}a^\dagger a a^\dagger a - \cdots, \tag{6a}$$

$$S_g(t) = a\frac{\sin(gt\sqrt{a^\dagger a})}{\sqrt{a^\dagger a}} = gta - \frac{(gt)^3}{3!}aa^\dagger a + \frac{(gt)^5}{5!}aa^\dagger a a^\dagger a - \cdots, \tag{6b}$$

$$C_e(t) = \cos(gt\sqrt{aa^\dagger}) = 1 - \frac{(gt)^2}{2!}aa^\dagger + \frac{(gt)^4}{4!}aa^\dagger aa^\dagger - \cdots, \tag{6c}$$

$$S_e(t) = a^\dagger\frac{\sin(gt\sqrt{aa^\dagger})}{\sqrt{aa^\dagger}} = gta^\dagger - \frac{(gt)^3}{3!}a^\dagger aa^\dagger + \frac{(gt)^5}{5!}a^\dagger aa^\dagger aa^\dagger - \cdots, \tag{6d}$$

Equations (4) and (5) can, respectively, be expressed, with the help of (3), as

$$|\psi(t)\rangle = |g\rangle \sum_n c_n \cos(|f_n|gt)|n\rangle - \mathrm{i}|e\rangle \sum_n c_n \frac{\sin(|f_n|gt)}{|f_n|}(a|n\rangle) \tag{7}$$

and

$$|\psi(t)\rangle = |e\rangle \sum_n c_n \cos(|f_{n+1}|gt)|n\rangle - \mathrm{i}|g\rangle \sum_n c_n \frac{\sin(|f_{n+1}|gt)}{|f_{n+1}|}(a^\dagger|n\rangle). \tag{8}$$

3. The scheme

The scheme we propose to prove the commutation relation consists of three two-level atoms and a single-mode cavity. The three atoms enter the cavity and interact with the cavity field one after another. We label them atoms 1, 2 and 3 in the order they enter the cavity. Atoms 1 and 3 are prepared initially in a singlet Bell state $\frac{1}{\sqrt{2}}(|e\rangle_1|g\rangle_3 - |g\rangle_1|e\rangle_3)$, while atom 2 is prepared in its upper state $|e\rangle_2$. The cavity field, assumed to be resonant with the atomic transition, can be in any arbitrary pure state, $|\psi_c(t=0)\rangle = \sum_n c_n|n\rangle$. The experiment proceeds as atoms 1, 2 and 3, one after another, enter the cavity and interact with the cavity field for times t_1, t_2 and t_3, respectively. Measurement is made on the three atoms after the interactions are over, and the experiment is accepted only if the atoms are measured to be in the state $|e\rangle_1|g\rangle_2|e\rangle_3$.

Straightforward algebra yields that the final state of the cavity field, contingent upon post-selection of the three atoms in $|e\rangle_1|g\rangle_2|e\rangle_3$, is given by

$$|\psi_c(t_1,t_2,t_3)\rangle = N\left\{\left[S_g(t_3)S_e(t_2)C_e(t_1) - C_e(t_3)S_e(t_2)S_g(t_1)\right]\sum_n c_n|n\rangle\right\}, \tag{9}$$

where $N\{\}$ signifies that the unnormalized state inside the curly bracket should be normalized. The physical interpretation of (9) is straightforward. The first term inside the curly bracket on the right-hand side results from a series of interactions with atom 1 prepared in $|e\rangle_1$ remaining in $|e\rangle_1$ $[C_e(t_1)]$, atom 2 making a transition $|e\rangle_2 \to |g\rangle_2[S_e(t_2)]$, and atom 3 prepared in $|g\rangle_3$ making a transition $|g\rangle_3 \to |e\rangle_3[S_g(t_3)]$, whereas the second term results from a series of interactions with atom 1 prepared in $|g\rangle_1$ making a transition $|g\rangle_1 \to |e\rangle_1[S_g(t_1)]$, atom 2 making a transition $|e\rangle_2 \to |g\rangle_2[S_e(t_2)]$, and atom 3 prepared in $|e\rangle_3$ remaining in $|e\rangle_3[C_e(t_3)]$. The $-$ sign between the two terms arises from the phase difference of π in the initial singlet Bell state of atoms 1 and 3. Substituting (6*a*)–(6*d*) into (9) and using (3), we can express (9) as

$$\begin{aligned}|\psi_c(t_1,t_2,t_3)\rangle = N\Bigg\{&\sum_n c_n \frac{\sin(|f_{n+1}|gt_3)}{|f_{n+1}|}\frac{\sin(|f_{n+1}|gt_2)}{|f_{n+1}|}\cos(|f_{n+1}|gt_1)(aa^\dagger|n\rangle)\\ &- \sum_n c_n \cos(|f_{n+1}|gt_3)\frac{\sin(|f_n|gt_2)}{|f_n|}\frac{\sin(|f_n|gt_1)}{|f_n|}(a^\dagger a|n\rangle)\Bigg\}.\end{aligned} \tag{10}$$

We desire this state to be equal to the state $|\psi_{\text{desired}}\rangle = N\{(aa^\dagger - a^\dagger a)\sum_n c_n|n\rangle\}$. The commutation relation can then be proven by showing experimentally that this state $|\psi_c(t_1,t_2,t_3)\rangle$ coincides with the initial cavity state $\sum_n c_n|n\rangle$. It can be easily seen that, if one takes short interaction times such that

$$\begin{aligned}&|f_{\bar{n}}|gt_1 = |f_{\bar{n}}|gt_3 \equiv |f_{\bar{n}}|gt \ll 1,\\ &|f_{\bar{n}}|gt_2 \ll 1,\end{aligned} \tag{11}$$

where $\bar{n}$ denotes the average number of photons in the initial cavity field, (10) becomes

$$|\psi_c(t_1,t_2,t_3)\rangle \simeq N\left\{(aa^\dagger - a^\dagger a)\sum_n c_n|n\rangle\right\}, \tag{12}$$

and our mission is accomplished. (Strictly speaking, the limit of short interaction times represented by (11) should be given in terms of $n_{\max}$, the maximum number of photons in the initial radiation field for which $|c_n|$ is non-negligible, instead of in terms of $\bar{n}$. We assume, however, that the radiation field has a smooth, continuous distribution of photon numbers, in which case the condition $|f_{n_{\max}}|gt \ll 1$ implies $|f_{\bar{n}}|gt \ll 1$.) We can understand this result by noting that, in the limit of short interaction times, we have $C_g(t) \simeq 1$, $C_e(t) \simeq 1$, $S_g(t) \simeq (gt)a$ and $S_e(t) \simeq (gt)a^\dagger$, and (9) becomes (12). Note that, due to the normalization in (12), the commutator $[a,a^\dagger] = aa^\dagger - a^\dagger a$ is realized by our scheme only up to a constant. By showing that the cavity field state after the interaction with the three atoms, with the interaction times t_1, t_2 and t_3 chosen to be sufficiently short to satisfy (11), is identical to the initial cavity state, we can thus prove $[a,a^\dagger] = K$, where K is an arbitrary constant. To complete the proof, one needs to show that the constant K is indeed one.

One way to prove $K = 1$ is to measure the success probability, i.e. the probability of observing the atoms in the desired atomic state $|e\rangle_1|g\rangle_2|e\rangle_3$ after the interaction. The success probability of our scheme can be easily calculated to be given by

$$\begin{aligned} P = \frac{1}{2}\sum_n |c_n|^2[&\sin(|f_{n+1}|gt_3)\sin(|f_{n+1}|gt_2)\cos(|f_{n+1}|gt_1) \\ &- \cos(|f_{n+1}|gt_3)\sin(|f_n|gt_2)\cos(|f_n|gt_1)]^2. \end{aligned} \tag{13}$$

In the limit (11) of short interaction times, we obtain

$$P \simeq \frac{1}{2}(gt)^2(gt_2)^2\sum_n |c_n|^2(|f_{n+1}|^2 - |f_n|^2)^2. \tag{14}$$

Noting that $|f_{n+1}|^2 = \langle n|aa^\dagger|n\rangle$ and $|f_n|^2 = \langle n|a^\dagger a|n\rangle$, (14) reduces to

$$P \simeq \frac{1}{2}(gt)^2(gt_2)^2\sum_n |c_n|^2(\langle n|aa^\dagger - a^\dagger a|n\rangle)^2 = \frac{1}{2}K^2(gt)^2(gt_2)^2. \tag{15}$$

Measurement of the success probability thus determines the value of the constant K.

As described above, in order to prove the commutation relation with our scheme, one needs to go to the limit (11) of short interaction times. Since the success probability $P \propto (gt)^2(gt_2)^2$, our scheme suffers from a very low success probability. The problem of low success probability also exists in the optical scheme proposed by Kim *et al* [8]. This problem of low success probability may not be a serious threat to the optical scheme, because the traveling optical field can be provided at a high pulse rate. For the cavity-QED-based scheme, however, atoms are provided at a relatively low rate, and therefore it is desirable to find a way of enhancing the success probability. In the next section, we demonstrate that if the cavity field is prepared initially in a superposition of a small number (two) of Fock states, the success probability can be significantly enhanced.

4. The case when the cavity field is prepared in a superposition of two Fock states

In this section, we consider the simple case in which the cavity field is initially prepared in a superposition of two Fock states, $\alpha|N_1\rangle + \beta|N_2\rangle$. Substituting $c_n = \alpha\delta_{nN_1} + \beta\delta_{nN_2}$ (δ denotes the Kronecker delta function) into (10), we obtain

$$\begin{aligned}|\psi_c(t_1, t_2, t_3)\rangle = N\Bigg\{&\alpha\frac{\sin(|f_{N_1+1}|gt_3)}{|f_{N_1+1}|}\frac{\sin(|f_{N_1+1}|gt_2)}{|f_{N_1+1}|}\cos(|f_{N_1+1}|gt_1)(aa^\dagger|N_1\rangle)\\ &+\beta\frac{\sin(|f_{N_2+1}|gt_3)}{|f_{N_2+1}|}\frac{\sin(|f_{N_2+1}|gt_2)}{|f_{N_2+1}|}\cos(|f_{N_2+1}|gt_1)(aa^\dagger|N_2\rangle)\\ &-\alpha\cos(|f_{N_1+1}|gt_3)\frac{\sin(|f_{N_1}|gt_2)}{|f_{N_1}|}\frac{\sin(|f_{N_1}|gt_1)}{|f_{N_1}|}(a^\dagger a|N_1\rangle)\\ &-\beta\cos(|f_{N_2+1}|gt_3)\frac{\sin(|f_{N_2}|gt_2)}{|f_{N_2}|}\frac{\sin(|f_{N_2}|gt_1)}{|f_{N_2}|}(a^\dagger a|N_2\rangle)\Bigg\}.\end{aligned} \tag{16}$$

In order for our scheme to realize the commutator $[a, a^\dagger]$, this state needs to be the same as the desired state $|\psi_{\text{desired}}\rangle = N\{(aa^\dagger - a^\dagger a)(\alpha|N_1\rangle + \beta|N_2\rangle)\}$, which in turn requires

$$\begin{aligned}\frac{\sin(|f_{N_1+1}|gt_3)}{|f_{N_1+1}|}\frac{\sin(|f_{N_1+1}|gt_2)}{|f_{N_1+1}|}\cos(|f_{N_1+1}|gt_1) &= \frac{\sin(|f_{N_2+1}|gt_3)}{|f_{N_2+1}|}\frac{\sin(|f_{N_2+1}|gt_2)}{|f_{N_2+1}|}\cos(|f_{N_2+1}|gt_1)\\ &= \cos(|f_{N_1+1}|gt_3)\frac{\sin(|f_{N_1}|gt_2)}{|f_{N_1}|}\frac{\sin(|f_{N_1}|gt_1)}{|f_{N_1}|}\\ &= \cos(|f_{N_2+1}|gt_3)\frac{\sin(|f_{N_2}|gt_2)}{|f_{N_2}|}\frac{\sin(|f_{N_2}|gt_1)}{|f_{N_2}|} \equiv Q.\end{aligned} \tag{17}$$

The success probability in this case is given, according to (13), by

$$\begin{aligned}P = \frac{1}{2}\{&|\alpha|^2[\sin(|f_{N_1+1}|gt_3)\sin(|f_{N_1+1}|gt_2)\cos(|f_{N_1+1}|gt_1)\\ &-\cos(|f_{N_1+1}|gt_3)\sin(|f_{N_1}|gt_2)\cos(|f_{N_1}|gt_1)]^2\\ &+|\beta|^2[\sin(|f_{N_2+1}|gt_3)\sin(|f_{N_2+1}|gt_2)\cos(|f_{N_2+1}|gt_1)\\ &-\cos(|f_{N_2+1}|gt_3)\sin(|f_{N_2}|gt_2)\cos(|f_{N_2}|gt_1)]^2\}.\end{aligned} \tag{18}$$

If (17) is satisfied, the success probability reduces to

$$P = \frac{1}{2}Q^2\Big[|\alpha|^2(|f_{N_1+1}|^2 - |f_{N_1}|^2)^2 + |\beta|^2(|f_{N_2+1}|^2 - |f_{N_2}|^2)^2\Big]. \tag{19}$$

The problem reduces to finding the interaction times t_1, t_2 and t_3 that satisfy (17). One obvious choice is the limit (11) of short interaction times. With such short interaction times, one sees that (17) is satisfied with $Q \simeq (gt)(gt_2)$ and (19) yields $P \simeq \frac{1}{2}(gt)^2(gt_2)^2[|\alpha|^2(|f_{N_1+1}|^2 - |f_{N_1}|^2)^2 + |\beta|^2(|f_{N_2+1}|^2 - |f_{N_2}|^2)^2]$, which is consistent with (14). As pointed out earlier, however, this success probability is very low. It should be noted from (19) that P is of the order of Q^2. It can be seen from (17) that if one can find non-small values of t_1, t_2 and/or t_3 that satisfy (17), then Q and consequently P may not necessarily be small. In order to pursue this possibility, we consider the simplest case of $N_1 = 0$ and $N_2 = 1$. In this case, condition (17) becomes

$$\frac{\sin(|f_1|gt_3)}{|f_1|}\frac{\sin(|f_1|gt_2)}{|f_1|}\cos(|f_1|gt_1) = \frac{\sin(|f_2|gt_3)}{|f_2|}\frac{\sin(|f_2|gt_2)}{|f_2|}\cos(|f_2|gt_1)$$
$$= \cos(|f_2|gt_3)\frac{\sin(|f_1|gt_2)}{|f_1|}\frac{\sin(|f_1|gt_1)}{|f_1|} \equiv Q. \tag{20}$$

Let us take the limit (11) of short interaction times for t_1 and t_3 ($|f_{\bar{n}}|gt_1 = |f_{\bar{n}}|gt_3 \equiv |f_{\bar{n}}|gt \ll 1$) but leave t_2 arbitrary. Equation (20) then becomes

$$(gt)\frac{\sin(|f_1|gt_2)}{|f_1|} = (gt)\frac{\sin(|f_2|gt_2)}{|f_2|} \equiv Q. \tag{21}$$

Although f_1 and f_2 are left unspecified, it can be seen that it is not too difficult to find non-small values of gt_2 that satisfy (21). Q is then of the order of (gt) and P is of the order of $(gt)^2$. Although this success probability is still small, it represents a significant enhancement over $P \propto (gt)^2(gt_2)^2$, the success probability as given by (15) for the case when all three interaction times t_1, t_2 and t_3 are chosen small. With substitution of the actual values $|f_1| = 1$ and $|f_2| = \sqrt{2}$, equation (21) is satisfied, for example, by $gt_2 = 3.7613$. The corresponding success probability is given by $P = \frac{1}{2}(gt)^2 \sin^2 3.7613 \simeq (0.17)(gt)^2$. Further enhancement in the success probability is possible if one can find non-small values of all three interaction times t_1, t_2 and t_3 that satisfy (20). There is no reason why one cannot find such non-small values that satisfy (20) at least approximately. For example, with substitution of the actual values $|f_1| = 1$ and $|f_2| = \sqrt{2}$, (20) is satisfied with a 15% error by taking $gt_1 = gt_2 = gt_3 = 2.70$. The corresponding success probability is 0.014.

In summary, our scheme to prove the commutation relation $[a, a^\dagger] = K$ with the success probability significantly higher than the scheme of section 3 goes as follows. As in the scheme of section 3, we prepare atoms 1 and 3 in the entangled state $\frac{1}{\sqrt{2}}(|e\rangle_1|g\rangle_3 - |g\rangle_1|e\rangle_3)$ and atom 2 in $|e\rangle_2$. The cavity, however, is prepared in a superposition of two Fock states, for example, in a superposition of the vacuum and the one-photon state. We then let the three atoms interact with the cavity field one after another for interaction times t_1, t_2 and t_3, respectively. We post-select the case when the atoms after their interaction with the cavity field are observed to be in the state $|e\rangle_1|g\rangle_2|e\rangle_3$. We choose the interaction times in such a way that (17) is satisfied. For the case when the cavity is prepared initially in a superposition of the vacuum and the one-photon state, we can choose $gt_2 = 3.7613$, while $gt_1 = gt_3 \equiv gt$ is chosen sufficiently small to satisfy (11). The success probability of the scheme will then be of the order of $(gt)^2$. Or we can choose $gt_1 = gt_2 = gt_3 = 2.70$ if an error of $\sim 15\%$ can be tolerated. The success probability of the scheme will then be ~ 0.014. By showing that the final cavity state after interaction with the atoms is identical to the initial cavity state, we prove experimentally that $[a, a^\dagger] = K$.

To prove $K = 1$, we can exploit interaction between a single atom and a cavity field. We prepare the atom in $|e\rangle$ and the cavity field in vacuum, $|\psi(t = 0)\rangle = |e\rangle|0\rangle$. We let the atom interact with the cavity field for time t and post-select the case when the atom is observed to remain in $|e\rangle$ after the interaction. The probability of finding the atom in $|e\rangle$ is given by

$$P = \cos^2(|f_1|gt). \tag{22}$$

Noting that $|f_1|^2 - |f_0|^2 = \langle 0|aa^\dagger - a^\dagger a|0\rangle = K$ and that $a|0\rangle = 0$ and therefore $f_0 = 0$, equation (22) can be written as

$$P = \cos^2(\sqrt{K}gt). \tag{23}$$

One can therefore determine the value of K by monitoring the periodic variation of the success probability as the interaction time is varied and determining the period of the variation.

5. Experimental feasibility

In this section, we discuss the experimental feasibility of our scheme by giving realistic considerations to some experimental issues. In order to prove the commutation relation with the proposed scheme, it is necessary to verify that the final cavity state produced by the scheme is identical to the initial cavity state. Since the scheme operates in the strong coupling regime, direct measurement of the field state inside the cavity is not allowed. One thus needs to rely on probe atoms to obtain information on the field state. One such method that allows direct measurement of the Wigner distribution function of the cavity field is proposed [13] and demonstrated [14] experimentally for the vacuum and the one-photon state.

The theory given in the previous sections assumes a lossless cavity and neglects atomic decay. In this regard, we note that our scheme needs to operate in the strong coupling regime in which the atom–field coupling constant (or the vacuum Rabi frequency) g is large compared with the cavity decay rate κ and the atomic decay rate γ. For the scheme described in section 3, the interaction time t needs to be chosen sufficiently small that $gt \ll 1$. Since in the strong coupling regime g is large compared with κ and γ, the inequalities $\kappa t \ll 1$ and $\gamma t \ll 1$ are well satisfied. For the scheme of section 4, the parameter gt may be of order unity, $gt \simeq 1$, but neglecting the cavity decay and atomic decay can still be justified. Let us take as an example the rubidium atom in a high-Q microwave cavity operating on hyperfine levels. We have $g \simeq 10$–100 kHz, whereas $\kappa \simeq 0.01$–1 kHz (corresponding to $Q \simeq 3 \times 10^{10}$–$3 \times 10^8$) and $\gamma \simeq 0.1$–10 kHz [15]–[17], and thus the conditions $\kappa t \ll 1$ and $\gamma t \ll 1$ can easily be satisfied. If we consider cesium atoms operating in the optical regime, we have, for example, $g \simeq 34$ MHz, $\kappa \simeq 4.1$ MHz and $\gamma \simeq 2.5$ MHz [18]. Although both cavity decay and atomic decay are more important in the optical regime, it is still possible to operate under the condition where they play relatively insignificant roles.

Perhaps the most demanding part of the scheme is the initial preparation of the atomic states and the cavity field states. Let us first consider the initial atomic states. Two atoms are required to be prepared in a maximally entangled singlet Bell state. We recall that it is exactly the entangled atoms in a singlet Bell state that were generated in the experiment in which generation of entanglement in massive particles was first demonstrated [19]. Subsequently, various elaborate cavity-QED methods of generating entangled atoms have been proposed and shown to be experimentally feasible [20]–[25]. Recently, entangled atoms or ions have been generated to teleport unknown atomic states [26, 27]. One can therefore say that it is within the reach of current experimental technology to prepare atoms in the entangled state required by our scheme. There, however, remains a problem because, due to experimental errors, the entangled state actually prepared may not exactly coincide with the desired entangled state. One possible source of error is the uncertainty in the interaction time arising from the finite resolution of the atomic velocity. Let us consider, for example, the scheme of [19] to prepare two atoms in the singlet Bell state. Two atoms, atoms A and B, prepared in the excited state and in the ground state, respectively, interact one after the other with a cavity prepared initially in vacuum. The interaction times, t_A and t_B, for atoms A and B, respectively, are chosen such that $gt_A = \frac{\pi}{4}$ and $gt_B = \frac{\pi}{2}$. Due to experimental error, however, one may actually have $gt_A = \frac{\pi}{4}(1+\epsilon_A)$ and $gt_B = \frac{\pi}{2}(1+\epsilon_B)$, where $\epsilon_A \ll 1$ and $\epsilon_B \ll 1$. The state of the system, atom A + atom B + cavity field, after the interactions then becomes $\frac{1}{\sqrt{2}}(1-\frac{\pi}{4}\epsilon_A)|e_A, g_B, 0\rangle - \frac{1}{\sqrt{2}}(1+\frac{\pi}{4}\epsilon_A)|g_A, e_B, 0\rangle + \mathrm{i}\frac{\pi}{2\sqrt{2}}\epsilon_B|g_A, g_B, 1\rangle$. Post-selection of the cavity field in vacuum results in the two atoms in the entangled state $\frac{1}{\sqrt{2}}(1-\frac{\pi}{4}\epsilon_A)|e_A, g_B\rangle - \frac{1}{\sqrt{2}}(1+\frac{\pi}{4}\epsilon_A)|g_A, e_B\rangle$. Atoms A and B now play the role of atoms 1 and 3 in our proposed scheme. The initial state of the three atoms for our scheme is then given by

$$\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\epsilon\right)|e_1, e_2, g_3\rangle - \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{4}\epsilon\right)|g_1, e_2, e_3\rangle, \tag{24}$$

where $\epsilon \equiv \epsilon_A$ represents the error arising from a finite velocity resolution of the $\frac{\pi}{4}$ atomic pulse. Through straightforward algebra, we obtain that the final state of the cavity field produced by our scheme of section 3 is given, under the limit (11) of short interaction times, by

$$|\psi_c(t_1, t_2, t_3)\rangle \simeq N\left\{\sum_n c_n|n\rangle - \frac{\pi}{4}\epsilon\sum_n c_n(2n+1)|n\rangle\right\}. \tag{25}$$

The fidelity of this state with respect to the desired state $|\psi_{\mathrm{desired}}\rangle = (aa^\dagger - a^\dagger a)\sum_n c_n|n\rangle = \sum_n c_n|n\rangle$, defined as $F = |\langle\psi_{\mathrm{desired}}|\psi_c(t_1, t_2, t_3)\rangle|^2$, is then given by $F \simeq 1 - \frac{\pi^2}{8}\epsilon^2(2\bar{n}+1)^2$. This indicates that if our scheme is to work accurately against the error ϵ, the inequality $\epsilon^2(2\bar{n}+1)^2 \ll 1$, i.e. $\bar{n} \ll 1/2\epsilon$, must be satisfied. Taking $\epsilon \simeq 0.005$ corresponding to a typical velocity resolution of 0.5% [28], we must have $\bar{n} \ll 100$. The experimental error in the initial preparation of atomic entanglement puts restrictions on the average photon number of the initial field state. We can also carry out a similar analysis for the scheme of section 4. Taking the initial atomic state to be given by (24) and taking the initial cavity field state as $\alpha|0\rangle + \beta|1\rangle$, we obtain for the final cavity state, under the condition of short interaction times t_1 and t_3,

$$|\psi_c(t_1, t_2, t_3)\rangle \simeq N\left\{\left(1-\frac{\pi}{4}\epsilon\right)\alpha|0\rangle + \left(1-\frac{3\pi}{4}\epsilon\right)\beta|1\rangle\right\}. \tag{26}$$

The fidelity of this state with respect to the desired state $|\psi_{\mathrm{desired}}\rangle = \alpha|0\rangle + \beta|1\rangle$ is given by

$$F \simeq 1 - \left(\frac{\pi}{2}|\alpha|^2 + \frac{3\pi}{2}|\beta|^2\right)^2\epsilon^2.$$

The fidelity remains very close to 1 for $\epsilon \simeq 0.005$. We can conclude that our scheme is not very sensitive to the error in the initial preparation of atomic entanglement, as long as the average photon number of the initial cavity state is not too large. In this respect, our scheme favors small $\bar{n}$.

As described in section 4, if our scheme is to perform with a reasonably high success probability, the cavity field needs to be prepared in a superposition of two Fock states. Various schemes to generate an arbitrary finite superposition of Fock states have been proposed [29]–[36] and demonstrated experimentally [37]–[39] in the past. In particular, an arbitrary superposition of the vacuum and the one-photon state can be generated using the optical quantum scissors device or other methods, as already demonstrated experimentally [37]–[39]. It appears therefore that the requirement to prepare the cavity field in a superposition of two Fock states is also within the reach of current experimental technology, although it is certainly not an easy requirement to be fulfilled.

Finally, we consider the effect due to the error in atomic state detection. Atomic detectors have miscounts of 2–5% [16, 40]. This is potentially a serious problem for our scheme, because the miscount probability may be significantly higher than the success probability of the scheme and, as a consequence, it may be difficult to discern the success of the scheme from misfiring of the detector. We note, however, that the miscount rate can be made arbitrarily low at the expense of the detector efficiency and higher miscount probabilities of complementary detectors [40]. In order to quantitatively estimate the effect of the error in atomic state detection, we consider the case in which atoms 1 and 2 are post-selected correctly but atom 3 is post-selected incorrectly, i.e. the case when atom 3 in $|g\rangle_3$ is incorrectly identified as in $|e\rangle_3$. The incorrect post-selection of atom 3 yields, for the final state of the cavity field,

$$|\psi_c(t_1,t_2,t_3)\rangle = N\left\{\left[C_g(t_3)S_e(t_2)C_e(t_1)+S_e(t_3)S_e(t_2)S_g(t_1)\right]\sum_n c_n|n\rangle\right\} \tag{27}$$

as opposed to (9), which results when the correct post-selection is made. Under the limit (11) of short interaction times, (27) reduces to $|\psi_c(t_1,t_2,t_3)\rangle \simeq N\{a^\dagger \sum_n c_n|n\rangle\}$. The fidelity of this state with respect to state $(aa^\dagger - a^\dagger a)\sum_n c_n|n\rangle = \sum_n c_n|n\rangle$ is given by

$$F' = \frac{\left|\sum_{n=1} c_{n-1}^* c_n \sqrt{n}\right|^2}{\bar{n}+1},$$

which reduces, for the initial coherent state, to $F' = \bar{n}/(\bar{n}+1)$. One thus sees that if $\bar{n}$ is large, the final cavity state produced by our scheme is insensitive to the error in atomic state detection. This insensitivity is rather undesirable, because erroneous data resulting from the detection error need to be discarded. The insensitivity makes it difficult to judge whether the coincidence of the final state produced by the scheme with the desired state actually proves the commutation relation or results simply from the insensitivity. One therefore sees that, in this respect, too, our scheme favors small $\bar{n}$. If $\bar{n}$ is large, it is difficult to detect the detection error simply by looking at photon statistics. One may have to rely on other more sensitive means, such as looking for the negativity of the Wigner distribution function. Another problem in atomic state detection in addition to misreading of atomic states is the possibility of atoms not being detected. In order to avoid errors arising from having more than one atom in a detection window, one needs to have atoms well separated in space from one another. When atoms are not detected, the experiment can be aborted. The detection efficiency of atomic detectors typically ranges from 50 to 90% [28], [40]–[44]. Taking, for example, the detection efficiency to be 0.8, the success probability of the scheme will be smaller by a factor of $(0.8)^3 \simeq 0.5$ (the detection efficiency of one detector should be cubed because our scheme requires detection of three atoms) than the success probability with perfect detectors. The finite detection efficiency only lowers the success probability and it does not cause any error, as long as there is one or less than one atom in each detection window. Considering, however, that the proposed scheme generally has a low success probability, further lowering of the success probability is undesirable. It appears therefore that, for successful execution of the proposed scheme, it is important to minimize the error arising from misreading of atomic states and at the same time to maximize the detection efficiency of the atomic detectors.

6. Summary

We have proposed a cavity-QED-based scheme to prove the photon commutation relation $[a, a^\dagger] = 1$. The scheme exploits interaction between a cavity field and three atoms, two of which are prepared in an entangled state. The proof is carried out in two steps. Firstly, the relation $[a, a^\dagger] = K$, where K is an arbitrary constant, is proven by showing that the field state after the interaction, contingent upon post-selection of atoms in the desired state, is identical to the initial cavity state. Secondly, the constant K is proven to be one by measuring the success probability, i.e. the probability to find atoms in the desired state. If the scheme is to work for any arbitrary initial state of the cavity field, one needs to choose sufficiently short interaction times t_1, t_2 and t_3, as described in section 3. The choice of the short interaction times, however, leads inevitably to the problem of low success probability of the scheme. If we restrict our consideration to the case when the initial cavity state is prepared in a superposition of only two Fock states, it is then generally possible to choose non-small values of the interaction times that yield the final cavity state identical to the state that results when the commutation operation is applied to the initial cavity state, as described in section 4. The corresponding success probability is significantly higher than that of section 3. This higher success probability, however, comes at the expense of having to prepare the cavity field in a superposition of two Fock states.

Acknowledgments

HWL thanks Professor M S Kim and Dr J Cho of Queen's University, Belfast for fruitful discussions at the time of his visit to Queen's University. SYL is supported by an NPRP grant 1-7-7-6 from the Qatar National Research Fund. This research was supported by a grant from the Korea Research Institute for Standards and Science (KRISS).

References

[1] Kim M S 2008 *J. Phys. B: At. Mol. Opt. Phys.* **41** 133001
[2] Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492
[3] Agarwal G S and Tara K 1992 *Phys. Rev.* A **46** 485
[4] Zavatta A, Viciani S and Bellini M 2004 *Science* **306** 660
[5] Zavatta A, Viciani S and Bellini M 2005 *Phys. Rev.* A **72** 023820
[6] Zavatta A, Parigi V and Bellini M 2007 *Phys. Rev.* A **75** 052106
[7] Parigi V, Zavatta A, Kim M S and Bellini M 2007 *Science* **317** 1890
[8] Kim M S, Jeong H, Zavatta A, Parigi V and Bellini M 2008 *Phys. Rev. Lett.* **101** 260401
[9] Zavatta A, Parigi V, Kim M S, Jeong H and Bellini M 2009 *Phys. Rev. Lett.* **103** 140406
[10] Kim H J, Park J and Lee H W 2010 *J. Opt. Soc. Am.* B **27** 464
[11] Sun Q, Al-Amri M and Zubairy M S 2008 *Phys. Rev.* A **78** 043801
[12] Lee S Y, Park J, Ji S W, Ooi C H R and Lee H W 2009 *J. Opt. Soc. Am.* B **26** 1532
[13] Lutterbach L G and Davidivich L 1997 *Phys. Rev. Lett.* **78** 2547

[14] Bertet P, Auffeves A, Maioli P, Osnaghi S, Meunier T, Brune M, Raimond J M and Haroche S 2002 *Phys. Rev. Lett.* **89** 200402
[15] Walther H, Varcoe B T H, Englert B G and Becker T 2006 *Rep. Prog. Phys.* **69** 1325
[16] Weidinger M, Varcoe B T H, Heerlein R and Walther H 1999 *Phys. Rev. Lett.* **82** 3795
[17] Raimond J M, Brune M and Haroche S 2001 *Rev. Mod. Phys.* **73** 565
[18] Miller R, Northup T E, Birnbaum K M, Boca A, Boozer A D and Kimble H J 2005 *J. Phys. B: At. Mol. Opt. Phys.* **38** S551
[19] Hagley E, Maitre X, Nogues G, Wunderlich C, Brune M, Raimond J M and Haroche S 1997 *Phys. Rev. Lett.* **79** 1
[20] Plenio M B, Huelga S F, Beige A and Knight P L 1999 *Phys. Rev.* A **59** 2468
[21] Hong J and Lee H W 2002 *Phys. Rev. Lett.* **89** 237901
[22] Sorensen A S and Molmer K 2003 *Phys. Rev. Lett.* **90** 127903
[23] Sorensen A S and Molmer K 2003 *Phys. Rev. Lett.* **91** 097905
[24] Duan L M and Kimble H J 2003 *Phys. Rev. Lett.* **90** 253601
[25] Lee J, Park J, Lee S M, Lee H W and Khosa A H 2008 *Phys. Rev.* A **77** 032327
[26] Riebe M *et al* 2004 *Nature* **429** 734
[27] Barrett M D *et al* 2004 *Nature* **429** 737
[28] Jones M L, Wilkes G J and Varcoe B T H 2009 *J. Phys. B: At. Mol. Opt. Phys.* **42** 145501
[29] Pegg D T, Phillips L S and Barnett S M 1998 *Phys. Rev. Lett.* **81** 1604
[30] Barnett S M and Pegg D T 1999 *Phys. Rev.* A **60** 4965
[31] Dakna M, Clausen J, Knoll L and Welsch D G 1999 *Phys. Rev.* A **59** 1658
[32] D'Ariano G M, Maccone L, Paris M G A and Sacchi M F 2000 *Phys. Rev.* A **61** 053817
[33] Koniorczyk M, Kurucz Z, Gabris A and Janszky J 2000 *Phys. Rev.* A **62** 013802
[34] Lund A P and Ralph T C 2002 *Phys. Rev.* A **66** 032307
[35] Miranowicz A 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** 142
[36] Miranowicz A, Ozdemir S K, Bajer J, Koashi M and Imoto N 2007 *J. Opt. Soc. Am.* B **24** 379
[37] Resch K J, Lundeen J S and Steinberg A M 2002 *Phys. Rev. Lett.* **88** 113601
[38] Lvovsky A I and Mlynek J 2002 *Phys. Rev. Lett.* **88** 250401
[39] Babichev S A, Ries J and Lvovsky A I 2003 *Europhys. Lett.* **64** 1
[40] Varcoe B T H, Brattke S and Walther H 2004 *New J. Phys.* **6** 97
[41] Stibor A, Kraft S, Campey T, Komma D, Gunther A, Fortagh J, Vale C J, Rubinsztein-Dunlop H and Zimmermann C 2007 *Phys. Rev.* A **76** 033614
[42] Gunther A, Bender H, Stibor A, Fortagh J and Zimmermann C 2009 *Phys. Rev.* A **80** 011604
[43] Teper I, Lin Y J and Vuletic V 2006 *Phys. Rev. Lett.* **97** 023002
[44] Kuhr S, Alt W, Schrader D, Muller M, Gomer V and Meschede D 2001 *Science* **293** 278