Local Harmonic $B_z$ Algorithm With Domain Decomposition in MREIT: Computer Simulation Study

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Abstract—Magnetic resonance electrical impedance tomography (MREIT) attempts to provide conductivity images of an electrically conducting object with a high spatial resolution. When we inject current into the object, it produces internal distributions of current density $J$ and magnetic flux density $B = (B_x, B_y, B_z)$. By using a magnetic resonance imaging (MRI) scanner, we can measure $B_z$ data where $z$ is the direction of the main magnetic field of the scanner. Conductivity images are reconstructed based on the relation between the injection current and $B_z$ data. The harmonic $B_z$ algorithm was the first constructive MREIT imaging method and it has been quite successful in previous numerical and experimental studies. Its performance, however, degraded when the imaging object contains low-conductivity regions such as bones and lungs. To overcome this difficulty, we carefully analyzed the structure of a current density distribution near such problematic regions and proposed a new technique, called the local harmonic $B_z$ algorithm. We first reconstruct conductivity values in local regions with a low conductivity contrast, separated from those problematic regions. Then, the method of characteristics is employed to find conductivity values in the problematic regions. One of the most interesting observations of the new algorithm is that it can provide a scaled conductivity image in a local region without knowing conductivity values outside the region. We present the performance of the new algorithm by using computer simulation methods.

Index Terms—Conductivity image, domain decomposition, harmonic $B_z$, magnetic resonance electrical impedance tomography (MREIT).

I. INTRODUCTION

Magnetic resonance electrical impedance tomography (MREIT) was motivated to deal with the ill-posed nature of the image reconstruction problem in electrical impedance tomography (EIT) [1]. MREIT is expected to provide conductivity images of an electrically conducting object with a high spatial resolution [2]–[5]. We sequentially inject multiple currents through chosen pairs of surface electrodes to produce current density $J = (J_x, J_y, J_z)$ and also magnetic flux density $B = (B_x, B_y, B_z)$ distributions inside the object [6]. Using an magnetic resonance imaging (MRI) scanner, we can measure $B_z$ data where $z$ is the direction of the main magnetic field of the scanner [7]–[9].

The conductivity image reconstruction in MREIT is based on the relationship between the injection current and measured $B_z$ data [10]–[14]. We assume that $\Omega$ represents a 3-D domain to be imaged and $\sigma$ is an isotropic conductivity distribution in $\Omega$. The current density $J$ in $\Omega$ subject to an injection current is determined by $\sigma$, electrode configuration, and boundary shape of $\Omega$. In this paper, we assume that electrode configuration and boundary shape are given from MR magnitude images. Measured $B_z$ data in $\Omega$ conveys the information about any local change of $\sigma$ which intermediates the following relationship:

$$B_z(r) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{(x-x') J_y(r') - (y-y') J_z(r')}{|r-r'|^3} \, dx' + \text{effects of current outside } \Omega$$

with

$$\nabla \cdot J(r) = 0 \text{ and } \nabla \times \left( \frac{J(r)}{\sigma(r)} \right) = 0 \text{ for } r \in \Omega$$

where $r = (x, y, z)$ and $\mu_0$ is the magnetic permeability of the free space.

Seo et al. proposed the first constructive $B_z$-based MREIT algorithm called the harmonic $B_z$ algorithm [10]. Noting the noise amplification property of the algorithm, various other methods were also suggested to improve the image quality [13]–[17]. Previous studies of numerical simulations and phantom experiments showed that the harmonic $B_z$ algorithm is quite successful in producing high-resolution conductivity images [18], [19]. In those studies, however, imaging objects had homogeneous backgrounds. Lately, we have been trying animal experiments and found that their conductivity distributions are very inhomogeneous with a wide range of contrast [20], [21]. Applying currently available MREIT image reconstruction algorithms, the image quality turned out to be lower than that of...
phantom images especially when imaging slices contained low-conductivity regions.

Inside the vertebrate body, there exist local regions with very low conductivity values. Such internal regions may include bones, lungs, and air-filled stomach. Since the air is surrounding the body, the outermost boundary is also facing the insulator. Regardless of a chosen electrode configuration, the internal current density becomes tangential to the boundary of such an insulating or almost insulating region. On the other hand, the harmonic $B_z$ algorithm requires us to produce at least two internal current density distributions that are not parallel in the domain to be imaged [10]–[12]. Therefore, by using the harmonic $B_z$ algorithm, it is difficult to correctly reconstruct conductivity values near the boundary of an internal low-conductivity region and also the outer boundary of the imaging object itself. We realized that, in previous studies of numerical simulation and phantom imaging, we unconsciously took advantage of a homogeneous background which eliminated this difficulty. In the recent experimental study of animal imaging [21], we therefore restricted conductivity image reconstructions only inside the canine brain instead of the head itself and assumed that the periphery of the brain had a constant conductivity value.

In order to effectively handle low-conductivity regions inside animal and human bodies, we need to develop a new image reconstruction method that can handle the problem of parallel current densities near boundaries of those regions. Carefully analyzing the structure of internal current density, in this paper, we propose a new MREIT image reconstruction algorithm based on a divide-and-conquer strategy. It consists of three major steps: 1) separate problematic regions where we expect almost parallel current densities, 2) reconstruct conductivity values in regions away from problematic regions, and 3) recover conductivity values near problematic regions. For the first step, we adopt an image segmentation method utilizing the available MR magnitude image. For the second step, we use a modified version of the harmonic $B_z$ algorithm called the local harmonic $B_z$ algorithm. For the third step, we employ the method of characteristics. In this paper, we exclude the low-conductivity region itself in conductivity image reconstructions. A simple method is to set the conductivity value of such a problematic region to be a certain small value. Alternatively, we may use the method proposed by Lee et al. to recover conductivity values inside the low-conductivity region [20]. We will show the performance of the new algorithm using computer simulation methods.

II. HARMONIC Bz ALGORITHM AND ITS LIMITATION

A. Setup of $B_z$-Based MREIT

Let the subject to be imaged occupy a 3-D bounded domain $\Omega \subset \mathbb{R}^3$ and let $R$ be a subdomain of $\Omega$ occupying low-conductivity regions. The region $R$ may include bones, lungs, air-filled stomach, and others. We assume that the conductivity distribution in $\Omega$, denoted by $\sigma$, is isotropic and $\sigma \approx 0$ in $R$. Let $\Omega_0 := \Omega \setminus R \setminus \{z = 0\}$ denote the slice of $\Omega$ cut by the plane $\{z = 0\}$. We denote boundaries of $\Omega$ and $R$ by $\partial \Omega$ and $\partial R$, respectively. Let $S := \partial R \cup \partial \Omega$ and let $\Sigma := S \cap \{z = 0\}$ as shown in Fig. 1. We say that $\sigma$ has a conductivity contrast $\lambda$ in a region $T$ if $\lambda = (\sup_T \sigma / \inf_T \sigma)$ where $\sup_T \sigma$ and $\inf_T \sigma$ denote the supremum and infimum of $\sigma$ in $T$, respectively.

We inject two independent electrical currents $I_1$ and $I_2$ through two pairs of surface electrodes $\mathcal{E}^+ \cup \mathcal{E}^-$ and $\mathcal{E}_j^+ \cup \mathcal{E}_j^-$, respectively. Then, the injection current $I_j$ with $j = 1, 2$ gives rise to an electrical potential, denoted by $u_j[\sigma]$, which satisfies the following boundary value problem:

\[
\begin{align*}
\nabla \cdot (\sigma \nabla u_j[\sigma]) &= 0 \quad \text{in } \Omega \setminus R \\
I &= \int_{\mathcal{E}_j^+} \sigma \frac{\partial u_j[\sigma]}{\partial n} \, ds = -\int_{\mathcal{E}_j^-} \sigma \frac{\partial u_j[\sigma]}{\partial n} \, ds \\
\nabla u_j[\sigma] \times \mathbf{n}_{\mathcal{E}_j^+} &= 0, \quad \nabla u_j[\sigma] \times \mathbf{n}_{\mathcal{E}_j^-} = 0 \\
\sigma \frac{\partial u_j[\sigma]}{\partial n} &= 0 \quad \text{on } \partial R \cup (\partial \Omega \setminus \mathcal{E}_j^+ \cup \mathcal{E}_j^-)
\end{align*}
\]

where $\mathbf{n}$ is the outward unit normal vector, $\mathbf{r} = (x, y, z)$, $(\partial u_j / \partial n) = \nabla u \cdot \mathbf{n} \times \mathbf{e}$, and $ds$ the surface area element. Note that we excluded the region $R$ from the imaging domain. The injection current $I_j$ induces a magnetic flux density whose $z$-component $B_{z,j}$ conveys the information about any local change of $\sigma$ via the Biot-Savart law: for $j = 1$ and 2,

\[
B_{z,j}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathbf{r}} \sigma(\mathbf{r'}) \left( \frac{(x-x') \frac{\partial u_j[\sigma]}{\partial y} (\mathbf{r'}) - (y-y') \frac{\partial u_j[\sigma]}{\partial x} (\mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \right) d\mathbf{r'}
\]

for $\mathbf{r} = (x, y, z) \in \Omega$.

B. Harmonic $B_z$ Algorithm

The harmonic $B_z$ algorithm is based on the $z$-component of the curl of the Ampere's law $\nabla \times \mathbf{J} = (1 / \mu_0) \nabla \times \nabla \times \mathbf{B}$ which provides the following identity [10]:

\[
\begin{bmatrix}
\frac{\partial u_1[\sigma]}{\partial y}(\mathbf{r}) \\
\frac{\partial u_1[\sigma]}{\partial x}(\mathbf{r}) \\
\frac{\partial u_2[\sigma]}{\partial y}(\mathbf{r}) \\
\frac{\partial u_2[\sigma]}{\partial x}(\mathbf{r})
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \phi}{\partial x}(\mathbf{r}) \\
\frac{\partial \phi}{\partial y}(\mathbf{r}) \\
\frac{\partial \psi}{\partial x}(\mathbf{r}) \\
\frac{\partial \psi}{\partial y}(\mathbf{r})
\end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix}
\nabla^2 B_{z,1}(\mathbf{r}) \\
\nabla^2 B_{z,2}(\mathbf{r})
\end{bmatrix}
\]

(2)
for \( r \in \Omega \). The above identity leads to the following implicit representation formula for \( \sigma \) and the iterative process to reconstruct \( \sigma \) is the harmonic \( B_z \) algorithm:

\[
\sigma(x, y, z_0) = \Phi_{\Omega_{z_0}}[\sigma](x, y) + L_{S_{z_0}} \sigma(x, y)
\]

for \( (x, y, z_0) \in \Omega_{z_0} \), where

\[
\Phi_{\Omega_{z_0}}[\sigma](x) := \frac{1}{2\pi} \int_{\Omega_{z_0}} \frac{x - x'}{|x - x'|^2} \cdot \mathbf{A}[\sigma]^{-1} \left[ \nabla^2 B_{1,z}(x', z_0) \right] \, d\mathbf{x}'.
\]

for \( (x', z_0) = (x, y, z_0) \in \Omega_{z_0} \) and

\[
L_{S_{z_0}} \sigma(x) := -\frac{1}{2\pi} \int_{S_{z_0}} \frac{(x - x') \cdot \nu(x')}{|x - x'|^2} \sigma(x', z_0) \, dl',
\]

Here, \( \mathbf{A}[\sigma](r) := \mu_0 \left[ \frac{\partial u_1[\sigma]}{\partial y}[\sigma](r) - \frac{\partial u_1[\sigma]}{\partial x}[\sigma](r) \right] \), \( \nu \) is the 2-D outward unit normal vector to \( \partial \Omega_{z_0} \), and \( dl' \) is the length element.

C. Effects of Low-Conductivity Region

In the presence of the low-conductivity region \( \mathcal{R}_3 \), the harmonic \( B_z \) algorithm suffers from the singularity of \( \mathbf{A}[\sigma] \) near the boundary \( \mathcal{S} := \partial \mathcal{R} \cup \partial \Omega \) since the direction of current density becomes tangential to \( \mathcal{S} \). This comes from the fact that \( \mathbf{n} \cdot \nabla u_j[\sigma] = 0 \) on \( \mathcal{S} \setminus (\mathcal{E}_j^+ \cup \mathcal{E}_j^-) \). If \( r \in \mathcal{S} \setminus (\mathcal{E}_j^+ \cup \mathcal{E}_j^-) \) and if \( \mathbf{n}(r) \) does not have a \( z \)-component, then \( [\partial u_1[\sigma]/\partial y], [\partial u_1[\sigma]/\partial x] \) and \( [(\partial u_2[\sigma]/\partial y), (\partial u_2[\sigma]/\partial x)] \) are parallel at \( r \) and therefore \( \det \mathbf{A}[\sigma](r) = 0 \). This singularity produces spurious spikes near \( \mathcal{S} \) and may deteriorate the overall image quality since excessive errors in one region may affect other regions according to (3). This is demonstrated numerically in Section IV [see images in Fig. 2(b) and (e)].

When \( \sigma \) is homogeneous near the boundary \( \partial \Omega \), that is, \( \nabla \sigma = 0 \) near \( \partial \Omega \), we have \( (1/\mu_0)\nabla^2 B_j = \nabla \sigma \times \nabla u_j = 0 \) near \( \partial \Omega \). In such a case, we can eliminate the singularity since

\[
\mathbf{A}[\sigma]^{-1} \left[ \nabla^2 B_{1,z} \right] = \mathbf{A}[\sigma]^{-1} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]
\]

in homogeneous regions. This is the primary reason why the harmonic \( B_z \) algorithm could produce high-quality conductivity images in previous studies of numerical simulations and phantom experiments where we unconsciously took advantage of this background homogeneity providing \( \nabla^2 B_z \approx 0 \) [18], [19].

In animal or human experiments, we cannot expect a homogeneous background conductivity and the imaging domain may contain low-conductivity regions [20], [21]. In this case, we should take into account the singularity of \( \mathbf{A}[\sigma] \) not only at the outermost insulating boundary \( \partial \Omega \setminus (\mathcal{E}_j^+ \cup \mathcal{E}_j^-) \) but also at boundaries of internal low-conductivity regions.

III. LOCAL HARMONIC \( B_z \) ALGORITHM

To deal with the singularity of \( \mathbf{A}[\sigma] \) near \( \mathcal{S} \), it is important to understand a direct relation between \( \sigma \) and \( B_z \). Let \( L := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \). We can view the data \( \nabla^2 B_z \) as the \( L \nabla u \)-directional change of \( \sigma \) since

\[
\begin{bmatrix} \partial x u_1[\sigma] \\ \partial y u_1[\sigma] \\ \partial x u_2[\sigma] \\ \partial y u_2[\sigma] \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial x \sigma \\ \partial y \sigma \end{bmatrix} = \frac{1}{\mu_0} \nabla^2 B_{z,2}
\]

in \( \mathcal{O} \).

Observation 3.1: The change of the conductivity \( \sigma \) along the tangential direction of the level set of potential \( u_j[\sigma] \) at each 2-D slice \( \Omega_{z_0} \) is dictated by the distribution of \( \nabla^2 B_{z,j} \) in the following ways:

- If \( B_{z,j} \) is convex at \( r, \sigma \) is increasing at \( r \) in the direction \( L \nabla u_j[\sigma](r) \).

- If \( B_{z,j} \) is concave at \( r, \sigma \) is decreasing at \( r \) in the direction \( L \nabla u_j[\sigma](r) \).

- If \( B_{z,j} \) is harmonic at \( r, \sigma \) is not changing at \( r \) in the direction \( L \nabla u_j[\sigma](r) \).

We can, therefore, estimate spatial changes of \( \sigma \) from measured \( B_{z,j} \) data, if we could predict the direction of \( L \nabla u_j \).

A. Conductivity Reconstruction Away From Low-Conductivity Region

Let \( u_j[\sigma_0] \) be a potential in (1) subject to a homogeneous conductivity \( \sigma_0 \). When the conductivity \( \sigma \) has a low contrast in \( \Omega \setminus \mathcal{R}_j \), we have assumed that the direction of \( \nabla u_j[\sigma] \) is little influenced by \( \sigma \). Instead, it is mostly dictated by the geometry of the boundary \( \mathcal{S} \) and the injection current \( I_j \). Indeed, the direction of the vector field \( \nabla u_j[\sigma] \) is similar to that of the vector.
field $\nabla u_f[\sigma_0]$, and the data $\nabla^2 B_{z,1}$ and $\nabla^2 B_{z,2}$ hold the major information of the conductivity contrast according to the Observation 3.1.

To be precise, let $\sigma_0 = 1$. The formula (2) can be decomposed as

$$
\left[ \begin{array}{c}
\frac{\partial}{\partial x}(r) \\
\frac{\partial}{\partial y}(r)
\end{array} \right] = (\beta(r)I + \text{Err}(r)) \times \mathcal{A}_\sigma^{-1}(r) \left[ \begin{array}{c}
\nabla^2 B_{z,1}(r) \\
\nabla^2 B_{z,2}(r)
\end{array} \right],
$$

where

$$
\beta(r) = \frac{\text{det} A[\sigma_0](r)}{\text{det} A[\sigma](r)}, \quad I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

and

$$
\text{Err}(r) = -\beta(r)I + \mathcal{A}_\sigma^{-1}(r)\mathcal{A}_\sigma[\sigma_0](r).
$$

If $\sigma$ is close to a constant, then $\beta \mathcal{A}_\sigma \approx \mathcal{A}_\sigma[\sigma_0]$ and $\text{Err}(r) \approx \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}$. Hence, we have assumed that the term $\text{Err}(r)$ is smaller than $\beta$ in a region with a low conductivity contrast.

**Observation 3.2:** Assume that the conductivity contrast is low in $\Omega \setminus \bar{D}$ and $\sigma_0 = 1$. Let $G$ be a local region contained in $\Omega \setminus \bar{D}$. If $\mathcal{A}_\sigma$ is not singular in $G$, then we have the following approximation:

$$
\left[ \begin{array}{c}
\frac{\partial}{\partial x}(r) \\
\frac{\partial}{\partial y}(r)
\end{array} \right] \approx \beta(r)\mathcal{A}_\sigma^{-1}(r) \times \left[ \begin{array}{c}
\nabla^2 B_{z,1}(r) \\
\nabla^2 B_{z,2}(r)
\end{array} \right], \quad r \in G.
$$

Let us call $\hat{\sigma}$ a scaled conductivity of $\sigma$ in $G$ when $\hat{\sigma}$ satisfies

$$
\left[ \begin{array}{c}
\frac{\partial \hat{\sigma}}{\partial x}(r) \\
\frac{\partial \hat{\sigma}}{\partial y}(r)
\end{array} \right] = \mathcal{A}_\sigma^{-1}(r) \left[ \begin{array}{c}
\nabla^2 B_{z,1}(r) \\
\nabla^2 B_{z,2}(r)
\end{array} \right]
$$

inside $G$. It is quite interesting to observe that $\hat{\sigma}$ reflects spatial contrast of $\sigma$ in $G$ because the term $\mathcal{A}_\sigma^{-1}(r)$ reflects the direction of $\left[ \begin{array}{c}
\frac{\partial \sigma}{\partial x}(r) \\
\frac{\partial \sigma}{\partial y}(r)
\end{array} \right]$. As numerically demonstrated in Section IV, a reconstructed scaled conductivity $\hat{\sigma}$ in a local region $G$ can provide fine details of the true conductivity contrast without any knowledge of the scale factor $\beta$ and the conductivity distribution outside $G$ (see Figs. 2 and 3). The next observation explains how to reconstruct $\sigma$ in a local disk $D$ contained in the slice $\Omega_{z_0}$.

**Observation 3.3:** Let $D$ be an arbitrary disk lying on $\Omega_{z_0}$. Denote by $\Phi_D[\sigma]$ the function defined in (4) with $\Omega_{z_0}$ replaced by $D$. The following identity holds: for $(x, z_0) \in D$:

$$
\sigma(x, z_0) - \sigma_{OD} = \Phi_D[\sigma](x) - \frac{1}{r \pi} \int_{\partial D} \frac{\mathbf{x} - \mathbf{x}'}{\|\mathbf{x} - \mathbf{x}'\|^2} \cdot \Phi_D[\sigma](\mathbf{x}') d\mathbf{x}',
$$

where $r$ is the radius of $D$, $x_0 = (x_0, y_0)$ is the center of $D$, and $\sigma_{OD} = (1/2\pi r) \int_{\partial D} \sigma dl$ the average of $\sigma$ along the boundary $\partial D$.

![Fig. 3. Robustness of the one-step local harmonic $B_2$ algorithm against added noise. Three different target conductivity distributions in (a), (d), and (g) with five insulating regions in black and inhomogeneous backgrounds were tried. Denoting the target conductivity distribution of (a) as $\sigma_a$, its background conductivity is 1 S/m and it includes different numbers of three different ellipses whose conductivity values are 2, 0.6, and 0.2 S/m. Denoting $\sigma_d$ and $\sigma_g$ as the target conductivity distributions of (d) and (g), respectively, we set $\sigma_a + 0.5 \cdot (\sin 50x \cdot \sin 50y + 1)$ and $\sigma_g \approx 0.9 \cdot (\sin 100x \cdot \sin 100y + 1) + 0.2$, respectively. The scaled conductivity images inside two circles were reconstructed using the one-step local harmonic $B_2$ algorithm with the initial guess $\sigma^0 = 1$. We added 10% noise for the cases of (b), (e), and (h). For (c), (f), and (i), the added noise was 20%.

For the proof of (9), see Appendix. Unlike the harmonic $B_2$ algorithm, the formula (9) does not require inverting the operator $L_{OD}$ defined in (5) and hence the local computation of $\sigma$ using (9) is straightforward. In the numerical computation for (9), we use an iteration process in such a way that

$$
\sigma^{n+1}(x, z_0) = \sigma^n_{OD} + \Phi_D[\sigma^n](x) - \frac{1}{r \pi} \int_{\partial D} \frac{\mathbf{x} - \mathbf{x}'}{\|\mathbf{x} - \mathbf{x}'\|^2} \cdot \Phi_D[\sigma^n](\mathbf{x}') d\mathbf{x}',
$$

for $x \in D$. This local reconstruction of (9) is robust against noise in $B_2$ data provided that $\mathcal{A}_\sigma[\sigma^n]$ is not singular. Hence, we can apply it to a region away from the boundary $S$.

**B. Conductivity Reconstruction Near Low-Conductivity Region**

We can not apply the formula (9) for the conductivity reconstruction near the boundary $S$ since $\mathcal{A}_\sigma[\sigma^n]$ is singular there. Indeed, we can apply the formula (9) to the interior region $\Omega^+_{z_0}[\delta]$ with $\delta$-distance away from $S$

$$
\Omega^+_{z_0}[\delta] := \{r \in \Omega_{z_0} : \text{dist}(r, S) \geq \delta\}
$$

where $\delta$ is an appropriate positive value. For example, we may select $\delta$ so that the condition number of $\mathcal{A}_\sigma[\sigma_0]$ is smaller than 10 in $\Omega^+_{z_0}[\delta]$. Hence, it remains to deal with the region $\Omega^-_{z_0}[\delta]$ near $S$

$$
\Omega^-_{z_0}[\delta] := \{r \in \Omega_{z_0} : \text{dist}(r, S) < \delta\}.
$$
Observation 3.4: Assume that \( |\mathbf{n}(\mathbf{r}) \times \mathbf{e}_z| \neq 0 \) for \( \mathbf{r} \in \mathcal{S}_{20} \) where \( \mathbf{e}_z = (0,0,1) \). Then, \( \mathbf{L} \nabla u_j[\sigma](\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \neq 0 \) for \( \mathbf{r} \in \mathcal{S}_{20} \setminus \mathcal{E}_{j}^+ \cup \mathcal{E}_{j}^- \), and we can use the method of characteristics to determine \( \sigma \) in \( \Omega_{20}[\delta] \) by solving (6) from the knowledge of \( \sigma \) in \( \Omega_{20}^+[\delta] \) provided that \( \delta \) is sufficiently small.

This observation is based on the following orthogonality property:

\[
\mathbf{L} \nabla u_j[\sigma] \cdot \nabla u_j[\sigma] = 0 \text{ in } \Omega \setminus \bar{\mathcal{R}}
\]

and

\[
\mathbf{n} \cdot \nabla u_j[\sigma] = 0 \text{ on } \mathcal{S} \setminus \mathcal{E}_{j}^+ \cup \mathcal{E}_{j}^-.
\]

This means that both \( \mathbf{L} \nabla u_j[\sigma] \) and the normal vector \( \mathbf{n} \) are orthogonal to \( \nabla u_j[\sigma] \) along the surface \( \mathcal{S} \setminus \mathcal{E}_{j}^+ \cup \mathcal{E}_{j}^- \). Hence, if \( \mathbf{n} \cdot \mathbf{e}_z = 0 \) in particular, it is easy to see that the vector \( \mathbf{L} \nabla u_j[\sigma] \) along the slice curve is pointing to the normal direction to the slice curve \( \mathcal{S}_{20} \setminus \mathcal{E}_{j}^+ \cup \mathcal{E}_{j}^- \) itself. Since \( |\mathbf{n} \times \mathbf{e}_z| > 0 \) on \( \mathcal{S}_{20} \), we get \( \mathbf{L} \nabla u_j[\sigma] \cdot \mathbf{n} \neq 0 \) on \( \mathcal{S}_{20} \setminus \mathcal{E}_{j}^+ \cup \mathcal{E}_{j}^- \).

Viewing \( \mathbf{L} \nabla u_j[\sigma] \cdot \nabla \sigma \) as a directional derivative of \( \sigma \) in the direction of \( \mathbf{L} \nabla u_j[\sigma] \), the identity \( \mathbf{L} \nabla u_j[\sigma] \cdot \nabla \sigma = (1/\mu_0) \nabla^2 B_{z,j} \) provides a change of \( \sigma \) in the direction \( \mathbf{h} = \mathbf{L} \nabla u_j[\sigma_0]/|\mathbf{L} \nabla u_j[\sigma_0]| \) which we call the characteristic direction. From the knowledge of \( \sigma \) in \( \Omega_{20}^+[\delta] \), we compute \( \sigma \) in the region \( \Omega_{20}^-[\delta] \) from the following approximation for \( (\mathbf{x}^*, z_0) \in \partial \Omega_{20}^+[\delta] \):

\[
\sigma(\mathbf{x}^* \pm \tau \mathbf{h}, z_0) 
\approx \sigma(\mathbf{x}^*, z_0) \pm \tau \mathbf{h} \cdot \nabla \sigma(\mathbf{x}^*, z_0) 
= \sigma(\mathbf{x}^*, z_0) \pm \tau \mathbf{L} \nabla u_j \times [\sigma_0](\mathbf{x}^*, z_0) \sqrt{2} B_{z,j}(\mathbf{x}^*, z_0)
\]

where \( \tau \) is a step size. Numerical implementation of this method of characteristics requires a special meshing near the boundary \( \mathcal{S} \) [see Fig. 4(b)] to compute \( \sigma \) along the characteristic direction \( \pm \mathbf{h} \). In a 2-D model, the direction of \( \mathbf{L} \nabla u_j[\sigma] \) is orthogonal to the insulating boundary, so the method of (11) works well provided that \( \delta \) is sufficiently small. We should mention that this method can not be applied near \( \mathbf{r} \in \mathcal{S} \) when \( |\mathbf{n}(\mathbf{r}) \times \mathbf{e}_z| \) is close to zero.

C. One-Step Local Harmonic \( B_z \) Algorithm

Based on the key observations, we now explain the one-step local harmonic \( B_z \) algorithm which produces a scaled conductivity image in a local region. We may use this one-step algorithm for the cases where we are primarily concerned about the local conductivity contrast.

1) We sequentially inject electrical currents \( I_1 \) and \( I_2 \) through two pairs of surface electrodes \( \mathcal{E}_{j}^+ \) and \( \mathcal{E}_{j}^- \), respectively.
2) Using an MR scanner, we get an MR magnitude image \( \mathcal{M} \) and induced magnetic flux density images, \( B_{z,j} \) for \( j = 1 \) and 2.
3) Using the MR magnitude image \( \mathcal{M} \), we perform segmentation of \( \partial \mathcal{S}, \mathcal{E}_{j}^+ \), and internal region \( \mathcal{R} \) with possibly low-conductivity values.
4) Solve the direct problem (1) with \( \sigma \) replaced by \( \sigma^0 = 1 \) in \( \Omega \setminus \mathcal{R} \).

5) For each \( z_0 \), select a local disk \( D \) in \( \Omega_{20}^-[\delta] \) in which we reconstruct a scaled conductivity image.
6) The scaled conductivity \( \sigma_D \) in the local disk \( D \) is obtained by computing

\[
\sigma_D(\mathbf{x}, z_0) = \Phi_{D,z_0}[\sigma_0](\mathbf{x})
- \frac{1}{\tau \pi} \int_{\partial D} \frac{(\mathbf{x} - \mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}_s)}{|\mathbf{x} - \mathbf{x}'|^2} \times \Phi_D[\sigma_0](\mathbf{x}') \, d\mathbf{x}',
\]

where \( \mathbf{x}_s = (x_s, y_s) \) and \( \tau \) are the center and radius of \( D \), respectively.

D. Iterative Local Harmonic \( B_z \) Algorithm

An iterative process is required to provide conductivity images of the whole domain. In the \( r \)th iteration, we first apply the one-step local harmonic \( B_z \) algorithm for several disks in \( \Omega_{20}^-[\delta] \) and get the updated conductivity \( \sigma^{r+1} \) in the region \( \Omega_{20}^+[\delta] \) by selecting disks covering the region \( \Omega_{20}^+[\delta] \). Next, we apply the method of characteristics to compute the updated conductivity \( \sigma^{r+1} \) in the remaining region \( \Omega_{20}^-[\delta] \). To be precise, we use the following iteration steps.
1) For \( n = 0, 1, 2, \ldots \), solve the direct problem (1) with \( \sigma \) replaced by \( \sigma^n \).

2) For each \( z_0 \), select several disks \( D_1, D_2, \ldots, D_m \) covering \( \Omega_{z_0}^+ \).

3) For each disk \( D_k \), compute

\[
\sigma_{D_k}^{n+1/2}(x, z_0) = \Phi_{D_k, z_0}[\sigma^n](x)
\]

where \( \Phi_{D_k, z_0} \) is the operator that was obtained in the previous step. The conductivity image reconstruction was found to be robust against noise (Fig. 3).

4) Choose constants \( c_1, \ldots, c_m \) so that \( \sigma_{D_k}^{n+1/2} + c_k = \sigma_{D_k}^{n+1/2} + c_k \) in \( D_k \cap D_j \). Determine \( \sigma^{n+1} \) in each slice \( \Omega_{z_0}^+ \) by assembling \( \sigma_{D_k}^{n+1/2}, \ldots, \sigma_{D_m}^{n+1/2} \) in such a way that \( \sigma^{n+1} = \sigma_{D_k}^{n+1/2} + c_k \) in \( D_k \cap D_j \). This process should be done in the \( z \)-direction.

5) Compute \( \sigma^{n+1} \) in each region \( \Omega_{z_0}^+ \) by using the method of characteristics in the observation 3.4. Here we use the knowledge of \( \sigma^{n+1} \) on \( \Omega_{z_0}^+ \) that was obtained in the previous step.

6) For the updated \( \sigma^{n+1} \), repeat the steps (1)–(5) until \( \|\sigma^{n+1} - \sigma^n\| \leq \varepsilon \) where \( \varepsilon \) is a given tolerance.

IV. NUMERICAL SIMULATION RESULTS

We carried out numerical simulations to test the feasibility of the proposed local harmonic \( B_2 \) algorithm. We considered a 2-D model of

\[ \Omega = \left\{ (x, y) \in \mathbb{R}^2 : \left( \frac{x}{0.15} \right)^2 + \left( \frac{y}{0.1} \right)^2 \leq 1 \right\} \]

where numbers are in meter. We used two different target conductivity distributions shown in Fig. 2(a) and (d). They contained an insulating region \( R \) including five internal insulators marked in black. The first and second target conductivity distributions had an inhomogeneous and homogeneous backgrounds, respectively and the conductivity contrast in \( \Omega \setminus \bar{R} \) was 10. We placed four electrodes \( E_j^+ \) and \( E_j^- \) at \(( \pm 0.15, 0)\) and \(( 0, \pm 0.1)\) in meter on the boundary of the model. The size of each electrode was 15 mm. Two different currents \( I_1 \) and \( I_2 \) were sequentially injected between two pairs of opposite electrodes \( E_j^- \). The amount of each injection current was 20 mA.

We used the PDE toolbox supported by Matlab (The Mathworks Inc., Natick, MA) to discretize the model \( \Omega \setminus \bar{R} \) into a finite element mesh with triangular elements. The maximal edge size of the elements was less than 3 mm. Using the standard finite element method, we solved the forward problem (1) to compute \( I_j \). We generated simulated data of \( B_{2,j} \) from the Ampere law \( \left( 1/\mu_0 \right) \nabla \times B_j = -\sigma \nabla u_j \), calculated \( \nabla^2 B_{2,j} \), and applied (9) to reconstruct an image of the target conductivity \( \sigma \). All numerical computations were performed using a PC with a Pentium IV processor, 1 GB RAM, and Windows XP Professional operating system.

During conductivity image reconstructions, we always fixed the initial guess \( \sigma_0 \) as 1. Reconstructed images in Fig. 2(b) and (e) show that the conventional harmonic \( B_2 \) algorithm produced a satisfactory image only when the background was homogeneous. The one-step local harmonic \( B_2 \) algorithm successfully reconstructed scaled conductivity images inside a disk shown in Fig. 2(c) and (f) for both cases. These results support the observation 3.3. Various other numerical simulations also supported the observation that the conductivity contrast was reasonably low.

In order to test the noise tolerance, we added 10% and 20% random noise to \( \nabla^2 B_{2,j} \). We generated noisy data \( \nabla^2 B_{2,j}^N \) by

\[ \nabla^2 B_{2,j}^N = \nabla^2 B_{2,j} + \sqrt{\frac{2}{|\Omega|}} \|\nabla^2 B_{2,j}\| \times NL \times RN \]

where \( \|\nabla^2 B_{2,j}\| \) is the \( L^2 \) norm of \( \nabla^2 B_{2,j} \) on \( \Omega \), \( RN \) a random number normally distributed in the interval \((-1, 1)\), \( NL \) the area of \( \Omega \) and \( NL \) a noise level representing the relative \( L^2 \)-error between the noisy data \( \nabla^2 B_{2,j}^N \) and the true data \( \nabla^2 B_{2,j} \), that is, \( NL = \|\nabla^2 B_{2,j} - \nabla^2 B_{2,j}^N\| / \|\nabla^2 B_{2,j}\| \). Fig. 3 shows reconstructed scaled conductivity images of three different target conductivity distributions with inhomogeneous backgrounds. Carefully examining the reconstructed scaled conductivity images, we found that very fine details of conductivity contrast are well recovered even in the presence of added noise.

Fig. 4 shows the process to reconstruct conductivity values near an insulating region as explained in the observation 3.4. Fig. 4(a) is a model of \( \Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq (0.12 \text{ m})^2 \} \) with a target conductivity distribution including one internal insulating region \( R \) marked in black. We divided the imaging domain into \( \Omega_{z_0}^+ \) and \( \Omega_{z_0}^- \) as shown in Fig. 4(b). Using the same numerical method, we first applied the one-step local harmonic \( B_2 \) algorithm to reconstruct a scaled conductivity image of the region \( \Omega_{z_0}^+ \) shown in Fig. 4(c). In order to reconstruct conductivity values in the problematic region \( \Omega_{z_0}^- \), we created a fine mesh using triangular elements to apply the method of characteristics. Fig. 4(d) shows a reconstructed conductivity image of the region \( \Omega_{z_0}^- \).

Fig. 5 shows the performance of the image reconstruction in the full domain using both the local harmonic \( B_2 \) algorithm and the method of characteristics. We performed only one iteration of the iterative algorithm described in Section III-D. Reconstructed conductivity images in Fig. 5 show that the method of characteristics works very well to produce conductivity images in the entire domain. The conductivity image reconstruction was very robust against added noise through noise effects were relatively bigger near the outermost boundary.

V. DISCUSSION

Using numerical simulations, we showed that the one-step local harmonic \( B_2 \) algorithm can reconstruct a scaled conductivity image inside a selected internal region where the condition number of the matrix \( A_{z_0}[\sigma_0] \) is small, for example 10 (Fig. 2). We found that this new algorithm is robust against noise (Fig. 3). Although it reconstructs a scaled conductivity image without providing absolute values, it reproduces fine details of conductivity contrast in a specified local region without any knowledge of conductivity values outside the region. Since \( \beta(\mathbf{r}) \) in (8) is positive, the scaled conductivity image \( \sigma \) defined in the Observation 3.2 never reverses the true conductivity contrast.
ever, future work is needed to quantitatively analyze how much distortion of the true conductivity contrast can occur in a reconstructed scaled conductivity image.

Under the assumption of low conductivity contrast, we found that the performance of the local harmonic $B_2$ algorithm is satisfactory. We note that conductivity values away from a local region has relatively little influence on voltage, current density, and also magnetic flux density inside the region. This well-known property causes the insensitivity problem in EIT limiting the spatial resolution. In MREIT, we speculate that it helps us to reconstruct scaled conductivity images with a high spatial resolution using only $B_2$ data. We will not go into the detail of this complicated issue since it is beyond the scope of this paper.

We should mention that Observations 3.3 and 3.4 are not rigorously proven in this paper. Their quantitative analysis requires a study of an extremely difficult mathematical problem to estimate the condition number of the matrix $A[\sigma]$. Even for the homogeneous conductivity case, it is quite difficult to estimate the lower bound for the determinant of $A[\sigma]$, which has been an open problem in a 3-D case. However, in practice, we can roughly estimate the condition number of $A[\sigma]$ based on our experience of many numerical simulations. In the numerical implementation of the local harmonic $B_2$ algorithm, we found that this kind of rough estimates are good enough.

For applications where we are primarily interested in the conductivity contrast inside a local region, we may use the local harmonic $B_2$ algorithm. However, some applications request conductivity images of the entire imaging domain. As shown in Figs. 4 and 5, the method of characteristics must be employed for these cases. Once properly implemented, the method of characteristics is quite robust against added noise. However, the most difficult part here is the generation of a special mesh as illustrated in Fig. 4 since it requires a very laborious work for a general geometry. For this reason, we tested the method of characteristics using a simpler model in Figs. 4 and 5 with just one insulating region instead of using the more complicated models in Figs. 2 and 3.

When we need to use the iterative algorithm in Section III.D, we face the problem of the convergence issue. Liu et al. lately studied the convergence characteristics of the harmonic $B_2$ algorithm assuming that the imaging domain has a low conductivity contrast [12]. We need to investigate the convergence property of the iterative version of the local harmonic $B_2$ algorithm in our future study.

In order to apply the local harmonic $B_2$ algorithm, we should perform a segmentation of an MR magnitude image. Extraction of the outermost boundary is easy and can be almost fully automated. However, segmentation of internal problematic regions having low conductivity values is not a trivial problem. Apart from the numerous technical issues in the general biomedical image segmentation problem, we face the question about which regions have possibly low conductivity values. Since we should determine the regions before trying a conductivity image reconstruction, we have no choice but to rely on our a priori knowledge. From animal experiments [20], [21], we found that there exist three problematic regions including the outer layer of the bone, the lung, and any gas-filled internal organ. They usually appear as dark regions in an MR magnitude image due to weak MR signals from them. We are currently using a semi-automatic image segmentation method to extract those internal regions. By performing more animal imaging experiments, we should accumulate more of our a priori knowledge on other problematic regions.

Future work should include a development of a 3-D image reconstruction software with a graphical user interface that facilitates the interactive use of the local harmonic $B_2$ algorithm. We need to develop an interactive segmentation tool that can easily incorporate our a priori knowledge into the semi-automatic segmentation process. Mesh generation tool must be imbedded in the software to be able to implement the method of characteristics for more general cases. Once the segmentation and mesh generation are performed, we should be able to interactively choose local imaging regions. Then, the local harmonic $B_2$ algorithm is fast enough to show conductivity images in the local regions within an acceptable time delay. We are currently developing such a software and plan to investigate its performance. We also plan to apply the new method proposed in this paper to measured data from postmortem and in vivo animal experiments.

VI. CONCLUSION

MREIT has not yet reached the stage of clinical applications primarily due to two technical difficulties. One is the limited capability of its image reconstruction algorithm to correctly recover an inhomogeneous conductivity distribution in the presence of a high conductivity contrast region including bones, lungs, and any air-filled organs. The other is the required amount of injection current that is higher than a physiologically acceptable level. Addressing the first issue, the local harmonic $B_2$ algorithm proposed in this paper is very promising since it may
provide a scaled conductivity image inside a specified local region of low conductivity contrast without knowledge of an inhomogeneous conductivity distribution outside the local region. Applications of the local harmonic $B_2$ algorithm may include high-resolution conductivity imaging of the brain, breast, liver, and other parts of the body.

Reduction of injection currents must be pursued utilizing all the latest developments in experimental MREIT techniques including improvements in RF coil, pulse sequence optimization, signal averaging, and denosing. We plan to perform in vivo anesthetized animal experiments using 5 mA injection current. Our short term goal is to provide high-resolution conductivity imaging of a canine brain followed by other parts. Preliminary human experiments are also being planned using 2 or 3 mA injection currents. Limbs will be the initial imaging areas of the human experiment.

**APPENDIX I**

**DERIVATION OF THE IDENTITY (9)**

According to the trace formula of the double layer potential [22, 23], we have

$$
\lim_{\nu \to 0} L_{\partial D} \sigma(x - \nu) = \frac{1}{2} \int_{\partial D} \frac{\nu(x') \cdot \nu(x')}{|x - x'|^2} d\sigma(x') + \frac{1}{2} \sigma_{\partial D}, \quad x \in \partial D.
$$

(14)

Since $D$ is the disk with the radius $r$, a direct computation yields

$$
\frac{\nu(x') \cdot \nu(x')}{|x - x'|^2} = -\frac{1}{r^2} \text{ for all } x, x' \in \partial D.
$$

Hence, the trace formula (14) becomes

$$
L_{\partial D} \sigma(x) = \frac{1}{2} \sigma(x, z_0) + \frac{1}{2} \sigma_{\partial D}, \quad x \in \partial D.
$$

(15)

From (3) and (15), we have the expression of $\sigma$ along the boundary:

$$
\sigma(x, z_0) = 2 \Phi_D[\sigma](x) + \sigma_{\partial D}, \quad x \in \partial D.
$$

(16)

Now, we substitute (16) into the integral term in (5) and combine with (3) to get

$$
\Phi_D[\sigma](x) = \sigma - \sigma_{\partial D} - \frac{1}{2 \pi} \int_{\partial D} \frac{\nu(x') \cdot \nu(x')}{|x - x'|^2} d\sigma(x')
$$

(17)

for $x \in D$. Using the fact that $-1 = (1/2\pi) \int_{\partial D} (\nu(x') \cdot \nu(x'))/|x - x'|^2 d\sigma(x')$ for all $x \in D$, the identity (17) becomes

$$
\Phi_D[\sigma](x) = \sigma - \sigma_{\partial D} - \frac{1}{2 \pi} \int_{\partial D} \frac{\nu(x') \cdot \nu(x')}{|x - x'|^2} \Phi_D[\sigma](x') d\sigma(x')
$$

(18)

for $x \in D$. The formula (9) follows from (3) and the above identity.

**REFERENCES**


