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i-CONVEXITY OF MANIFOLDS WITH REAL PROJECTIVE STRUCTURES

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ABSTRACT. We compare the notion of higher-dimensional convexity, as defined by Carrière, for real projective manifolds with the existence of hemispheres. We show that if an *i*-convex real projective manifold M of dimension n for an integer i with $0 < i < n$ has an *i*-dimensional hemisphere, then M is projectively homeomorphic to S^n/Γ where Γ is a finite subgroup of $O(n+1, \mathbf{R})$ acting freely on S^n .

A real projective structure (\mathbf{RP}^n -structure) on a smooth manifold of dimension n , $n > 0$, is given by an atlas of charts to the sphere S^n where transition functions of the charts are restrictions of projective automorphisms of S^n . An \mathbf{RP}^n -manifold is a manifold with an \mathbf{RP}^n -structure, and a *projective map* is an immersion-preserving real projective structures locally. The Klein model of hyperbolic geometry implies that n -dimensional hyperbolic manifolds provide examples of \mathbf{RP}^n -manifolds. (See [2] and [3].)

Throughout this paper, let M be an \mathbf{RP}^n -manifold; let \tilde{M} denote the universal cover of M with the induced \mathbf{RP}^n -structure. M has a developing map $\text{dev}: \tilde{M} \rightarrow S^n$, a projective map. The sphere S^n has a standard Riemannian metric μ of curvature 1 and the associated distance metric d . Note that dev induces a Riemannian metric μ on \tilde{M} from S^n . Associated with μ is the induced distance metric d on \tilde{M} . The completion \check{M} of \tilde{M} is obtained by completing d . Let σ be the frontier set $\check{M} - \tilde{M}$. The sets \check{M} and σ are topologically independent of the choice of dev , and dev extends uniquely to a distance decreasing map on \check{M} . The extended map is also called a developing map and is denoted by the same symbol dev .

Let i be an integer such that $0 < i \leq n$ holds. A *great i -sphere* is a totally geodesic i -dimensional sphere imbedded in S^n ; a subset of \check{M} for which the restriction of dev is an imbedding onto a great i -sphere in S^n is also called a *great i -sphere*. A *great i -ball* is a hemisphere of a great i -sphere S^i in S^n ; a

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subset of \tilde{M} for which the restriction of dev is an imbedding onto a great i -ball is also called a *great i -ball*. An n -dimensional open hemisphere of S^n has a natural affine structure. An *i -simplex* is the convex hull of $i+1$ independent points in the hemisphere (under affine geometry). An *i -simplex* in \tilde{M} is a subset for which the restriction of dev is an imbedding onto an i -simplex in an n -dimensional open hemisphere of S^n .

We introduce the definition given by Carrière [1]. We say that M is *i -convex* for an integer i with $0 < i < n$ if the following holds: Given an $(i+1)$ -simplex T in \tilde{M} , if F_1 is a face of T such that $T \cap \sigma = F_1 \cap \sigma$, then $T \subset \tilde{M}$. (Note that the i -convexity of M implies the j -convexity of M where $i \leq j < n$.)

A subset A of \tilde{M} is called *convex* if every two points of A are connected by an imbedded arc α such that $\text{dev}|_\alpha$ is an imbedding onto a segment of d -length $\leq \pi$ (see [2, §1] for more detail). It is easy to see that given a convex open subset A of \tilde{M} , the map $\text{dev}|_A$ is isometric with respect to the metrics on \tilde{M} and S^n . Thus, $\text{dev}|_{\text{Cl}(A)}$ is an imbedding onto $\text{Cl}(\text{dev}(A))$ where $\text{Cl}(A)$ is the closure of A in \tilde{M} .

We prove the following theorem in this paper.

Theorem. *Suppose that M is i -convex where $0 < i < n$ holds. Suppose that \tilde{M} includes a great j -ball or a great j -sphere for $i \leq j < n$. Then \tilde{M} is projectively homeomorphic to S^n .*

The conclusion implies that M is projectively homeomorphic to S^n/Γ' where Γ' is a subgroup of the projective automorphism group $\text{Aut}(S^n)$ acting freely and properly discontinuously on S^n . It is a standard fact that Γ' is conjugate to a finite subgroup of $O(n+1, \mathbf{R})$, the group of isometries of S^n . Thus, M is projectively homeomorphic to S^n/Γ where Γ is a finite subgroup of $O(n+1, \mathbf{R})$ acting freely on S^n .

Proof of the Theorem. We give the proof of the theorem assuming that Lemma 1, which follows, holds. We can prove the theorem by using induction on j . Suppose that $j = n - 1$. Then by Lemma 1 \tilde{M} includes a great sphere of dimension n . Therefore, \tilde{M} is projectively homeomorphic to S^n .

Suppose that the conclusion is true for the case where $j = k > i$ holds. We verify the conclusion for the case where $j = k - 1 \geq i$ holds. By Lemma 1, \tilde{M} includes a great k -sphere. By the induction hypothesis, \tilde{M} is projectively homeomorphic to S^n . This completes the proof.

Let us discuss Lemma 1. For an integer j with $0 < j < n$, a $(j+1)$ -*bihedron* in S^n is a closed domain in a great $(j+1)$ -sphere S^{j+1} in S^n bounded by two great j -balls with common boundary equal to a great $(j-1)$ -sphere or the set of two points antipodal to each other; a $(j+1)$ -*bihedron* in \tilde{M} is a subset for which the restriction of dev is an imbedding onto a $(j+1)$ -bihedron in S^n . The bounding great j -balls of a bihedron are called *faces*. A bihedron in S^n or \tilde{M} is *convex* if and only if the interior angle between two faces of the bihedron is less than or equal to π .

Lemma 1. *Assume that M is i -convex for an integer i with $0 < i < n$. Suppose that \tilde{M} includes a great j -ball B_0 where $i \leq j < n$. Then \tilde{M} includes a great $(j+1)$ -sphere.*

Proof. We choose a convex $(j+1)$ -bihedron T_0 including B_0 in a neighborhood of B_0 with faces B_a and B_b such that $\delta B_a = \delta B_b = \delta B_0$ holds. (Assume that B_0 is not a face of T_0 .) Let us agree that bihedra in this proof are $(j+1)$ -dimensional always. Let A^+ be the set of convex bihedra with a face B_0 and including B_a , and let A^- be the set of convex bihedra with a face B_0 and including B_b . Then there is a unique great $(j+1)$ -sphere S^{j+1} in S^n including the images of elements of A^+ and A^- under dev .

We may parameterize A^+ and A^- by positive intervals. Given an element T of A^+ or A^- , let $\theta(T)$ be the interior angle between B_0 and the other face of T . This defines a function θ from the set of elements of A^+ and A^- to \mathbf{R} . Let T_a be the bihedron in T_0 bounded by B_0 and B_a ; let T_b be the bihedron in T_0 bounded by B_0 and B_b . Suppose that T' and T'' are two bihedra in A^+ with $\theta(T') = \theta(T'')$. Then Lemma 2 implies that $T' = T''$. Thus $\theta|_{A^+}$ is an injective map into $[\theta(T_a), \pi]$. Similarly, $\theta|_{A^-}$ is an injective map into $[\theta(T_b), \pi]$.

If we have $t < t'$ where $t' \in \theta(A^+)$ and $t \in [\theta(T_a), \pi]$ hold, then t is realized as the angle of a bihedron in A^+ which is included in the bihedron corresponding to t' . It follows that $\theta(A^+)$ is connected. Similarly, $\theta(A^-)$ is connected.

Let $T \in A^+$. Then $\text{dev}|T$ is an imbedding onto a convex bihedron $\text{dev}(T)$. Choose an open neighborhood N of T in \widetilde{M} such that $\text{dev}|N$ is an imbedding onto an open subset of S^n . Then for every convex bihedron T' in $\text{dev}(N)$ the open subset N includes a convex bihedron T'' such that $\text{dev}(T'') = T'$. This implies that $[\theta(T_a), \pi]$ includes an open neighborhood of $\theta(T)$ whose elements are realized by bihedra in A^+ . Hence, $\theta(A^+)$ is an open subset of $[\theta(T_a), \pi]$. Similarly, $\theta(A^-)$ is an open subset of $[\theta(T_b), \pi]$.

We claim that $\theta(A^+)$ is closed. (We use i -convexity now.) Suppose that it is not closed. Then $\theta(A^+)$ is the half-open interval $[\theta(T_a), t^+)$ for a real number t^+ less than or equal to π . Let $T_1 = \bigcup_{T \in A^+} T^\circ$. Since by Lemma 2, $\text{dev}|T_1$ is injective, $\text{dev}|T_1$ is an imbedding onto the interior of a convex bihedron of angle t^+ . For the closure $\text{Cl}(T_1)$ of T_1 in \widetilde{M} , the map $\text{dev}| \text{Cl}(T_1)$ is an imbedding onto $\text{Cl}(\text{dev}(T_1))$. Since $\text{Cl}(\text{dev}(T_1))$ is a convex bihedron, so is $\text{Cl}(T_1)$. The bihedron $\text{Cl}(T_1)$ has two faces B_0 and $B_{\text{Cl}(T_1)}$ where $\sigma \cap \text{Cl}(T_1) \subset B_{\text{Cl}(T_1)}^\circ$. Since $\sigma \cap B_{\text{Cl}(T_1)}$ is compact, $\sigma \cap B_{\text{Cl}(T_1)}^\circ$ is a bounded subset of the open great ball $B_{\text{Cl}(T_1)}^\circ$. Thus, the bihedron $\text{Cl}(T_1)$ includes a $(j+1)$ -simplex K such that $\sigma \cap \text{Cl}(T_1) \subset K_1^\circ$ where K_1 is a j -dimensional face of K and is included in $B_{\text{Cl}(T_1)}$. The definition of i -convexity implies that $K_1 \cap \sigma = \emptyset$ and $\text{Cl}(T_1) \subset \widetilde{M}$ hold. Since $\text{Cl}(T_1) \supset T_a$, we have $\text{Cl}(T_1) \in A^+$ and hence $t^+ \in \theta(A^+)$. This is absurd. Therefore, $\theta(A^+)$ and, similarly, $\theta(A^-)$ are closed.

We therefore have $\theta(A^+) = [\theta(T_a), \pi]$ and $\theta(A^-) = [\theta(T_b), \pi]$. Let T^+ be the convex bihedron corresponding to π belonging to A^+ , and let T^- be that belonging to A^- . We have $T^+, T^- \subset \widetilde{M}$. The maps $\text{dev}|T^+$ and $\text{dev}|T^-$ are imbeddings onto great $(j+1)$ -balls in the great $(j+1)$ -sphere S^{j+1} . Since the intersection $\text{dev}(T^+) \cap \text{dev}(T^-)$ is a great j -sphere and, hence, is path-connected, Lemma 2 implies that $\text{dev}|T^+ \cup T^-$ is an imbedding onto S^{j+1} . Therefore, $T^+ \cup T^-$ is a great $(j+1)$ -sphere in \widetilde{M} . This completes the proof of Lemma 1.

Lemma 2. *Suppose that A and B are path-connected compact subsets of \widetilde{M} such that $\text{dev}|_A$ and $\text{dev}|_B$ are imbeddings. Suppose that $A \cap B \neq \emptyset$ and that $\text{dev}(A) \cap \text{dev}(B)$ is a path-connected subset of S^n .*

Then $\text{dev}|_{A \cup B}$ is an imbedding onto $\text{dev}(A) \cup \text{dev}(B)$.

Proof. We only need to deduce the injectivity of $\text{dev}|_{A \cup B}$ from the fact that given a path in S^n and an initial point in \widetilde{M} there is at most one lift of the path to \widetilde{M} (see [1, Proposition 1.3.1]).

Let us end this paper with the following remark: Carrière conjectured in 1988 that the homotopy groups in dimensions greater than or equal to i , $i > 1$, for an i -convex affine manifold are trivial. This conjecture is still open. What we did here may aid us in understanding similar questions for projective manifolds.

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