Comparative economic analysis between direct and indirect wiring in the copper-based local loop

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In a copper-based local loop, a circuit pair is indirectly connected to each subscriber usually via a primary cross connection point (PCP) which acts as a buffer to absorb circuit demand fluctuations among subareas. But some telephone operators adhere to the classic practice of direct wiring based on technological preferences without taking advantage of cost-efficient flexibility points. We analyse the extra cost of maintaining the old practice of direct wiring over the popular one using flexibility points in a single PCP area. For that, the expectation of circuit shortages in subareas during a single replenishment period for the direct wiring is first obtained. Exploiting the convexity of the expectation, we then present a procedure for optimally allocating circuits among subareas, which not only serves its own purpose of circuit provisioning for the direct wiring but also precisely calculates the extra cost over the indirect case.

Keywords: telecommunications; inventory; allocation;

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Introduction

The traditional copper loop has a ‘tree and branch’ structure as shown in Figure 1. One pair per telephone line is allocated to each customer, but the flexibility points, called Primary Cross-Connection Points (PCPs), are introduced to allow bundles of pairs to be split and diverted to meet circuit demands. The final drop to customer premises is usually via a Distribution Point (DP), but some premises are allowed to be directly fed from the exchange. The pairs within cables between the exchange (or the central office) and a PCP are designated as the Main side (or Feeder side) and the remaining part of the loop as the Distribution side.1,2

Generally, an exchange area is partitioned into areas, called PCP catchment areas (or PCP areas in short), which are further sub-divided into smaller areas called DP catchment areas (or DP areas). The PCP equipment makes a connection to a customer by simple adjustment of a jumper in its cabinet, thereby serving as a buffer to resolve discrepancies between the forecasted and actual demands of DP areas. However, some countries such as Korea and Japan do not take advantage of the cost-efficient flexibility of PCPs. That is, customer premises are directly wired from the exchange, the loop structure of which is still of the tree and branch type, but without using PCPs. The advantages of this deployment method are two-fold: costs of setting up and maintaining PCPs are not incurred, and the noise level is kept lower from the direct connection. The absence of PCPs, and therefore the resulting lack of flexibility, forces telecommunications operators (telcos) to resort to the cost-incurring wire rearrangement (or reallocation) to cope with demand fluctuations among DP areas. In other words, wire shortage in a DP area has to be resolved by rewiring to the DP some wire-pairs oversupplied to other DP areas in the same PCP area.3

This study, focusing on the economic aspect, compares two deployment methods, with and without PCPs. To facilitate the comparative analysis, we assume the same tree and branch loop configuration from the exchange, which allows us to concentrate our attention on a single branch, that is a single PCP area (or simply the area). Rearranging wires to and from DP areas (or subareas) are assumed to be conducted only within the area, fitting the real-world practice. Since the area-level shortages are treated the same for both direct and indirect wiring cases, or for both non-PCP and PCP cases, attention is naturally

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Figure 1. Local loop structure.
centered on examining the relative effectiveness of indirect wiring over the direct one in handling the fluctuations among the subarea-level demands.

It is interesting to note that there is no study in the literature addressing the cost difference between these two circuit provisioning practices. The objective of this study, the development of which is centered on the case of direct wiring, is two-fold: to find the optimal circuit allocation strategy under the direct wiring practice, and to exactly estimate the extra cost over the indirect case, thereby allowing telcos to figure out how much overhead they are paying for adhering to their classic wiring practices.

Model

Two inventory systems, one for each of the two cases, are defined with the following assumptions and parameters.

1. The area consists of $r$ subareas (or DP, $i = 1, \ldots, r$). Attention is confined to a certain reinvestment (or replenishment) period of duration $T$ with the supply amount to the area fixed at $n$.

2. The amount supplied to and the demand occurred at DP, are $n_i$ and $X_i$, respectively. The $X_i$'s are assumed to have independent Poisson distributions with parameters $\lambda_i$. The demand in the whole area, $X = \sum_{i=1}^{r} X_i$, therefore has a Poisson distribution with parameter $\lambda = \sum_{i=1}^{r} \lambda_i$. We also define the $r$-vector, $n = (n_1, \ldots, n_r)$, to represent the circuit allocation in the area and assume that the circuits supplied to the area are allocated exhaustively, $\sum_{i=1}^{r} n_i = n$.

3. Among the three cost elements in the usual inventory system, only the shortage cost is considered here. Note that the carrying and replenishment costs are exactly the same for both direct and indirect cases in our setting. Shortages are classified into two kinds: the P-level for the area and the D-level for subareas. The P-level shortages do not have to be differentiated between the two cases, rendering the common per circuit shortage cost $p_D$. On the other hand, shortages in the D-level are handled in a significantly different manner, simple jumper adjustment for the indirect case, and the cost-incurring wire rearrangement for the direct case. Since the cost for jumper adjustment is negligible, the per circuit D-level shortage cost for the indirect case is set at zero. For the direct case, the difficulty of wire rearrangements varies with DP locations. But neglecting the differences, we simply set the per circuit D-level shortage cost for the direct case at $p_D$.

Comparison of two deployment methods

Case 1—Indirect wiring

Primary level shortage. The P-level shortage occurs when the demand in the area is greater than the supply, $X > n$. $X$ is a nonnegative discrete random variable with probability distribution $F_X(\cdot)$ and cumulative distribution function $F_X(\cdot)$. Define the random variable $Z_p$ as the amount of P-level shortage,

$$Z_p = \begin{cases} X - n & X \geq n + 1 \\ 0 & \text{otherwise} \end{cases}$$

which will be simply expressed as

$$Z_p = (X - n)^+,$$

where $a^+$ means $|a|$ if $a > 0$ and zero otherwise. Denoting $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$, we have the expectation of $Z_p$ as

$$E[Z_p] = \sum_{i=n+1}^{\infty} (i-n)\bar{F}_X(i)$$

$$= \lambda \bar{F}_X(n) - n \bar{F}_X(n),$$

where the second equality comes from that the demand, $X$, has a Poisson distribution with parameter $\lambda$. Recall our assumption that resolving the D-level shortage in the indirect case costs nil, from which attention would not be paid to the level of such shortage.

Case 2—Direct wiring

Primary level shortage. The P-level shortage occurs when $X > n$, irrespective of the realised sizes of the $X_i$'s. Note the reality that supplying an additional circuit to the area from outside is much costlier than a wire rearrangement made within the area, implying $p_D \ll p_p$. Therefore the shortage in the P-level is considered only after all the possible rearrangements are made. The amount of P-level shortage and its expectation for the indirect case still hold here.

Distribution level shortage. The D-level shortages are first resolved by making as many rearrangements as possible within the set of circuits already supplied to the area. Dividing the supply level into two cases, $X \leq n$ and $X > n$, we have the number of possible rearrangements, represented by random variable $Z_D$, as

$$Z_D(n) = \begin{cases} \sum_{i=1}^{n} (X_i - n_i)^+ & X \leq n \\ \sum_{i=1}^{r} (n_i - X_i)^+ & X > n. \end{cases}$$

The expected level can then be written as

$$E[Z_D(n)] = E[E[Z_D(n)|X]]$$

$$= \sum_{X \leq n} \left( \sum_{i=1}^{n} (x_i - n_i)^+ \right) \cdot P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r)$$

$$+ \sum_{X > n} \left( \sum_{i=1}^{r} (n_i - x_i)^+ \right) \cdot P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r).$$
Considering the cardinality of the set, \( \{(x_1, x_2, \ldots, x_r) | \sum_{i=1}^r x_i = n, x_i \in \mathbb{N}, i = 1, \ldots, r\} \), it looks computationally prohibitive to calculate the above expectation. It is judicious to find the following fact, which makes its calculation within \( O(m) \) summations. When \( X \leq n \), the shortages, \( x_i - n_i \), at DP, can be resolved by reallocating as many circuits with the leftovers from \( n - n_i \), circuits after meeting all demands of the remaining DPs. Symmetrically for \( X > n \), the surpluses, \( n_i - x_i \), at a certain DP, can also be used to fill shortages in other DPs, bound to occur at least that many. Taking this dichotomous view on each of the summation terms, we now have the following simple form of expectation.

\[
E[Z_D(n)] = \sum_{x_i=n_i+1}^{n} \sum_{x_{i-1}=0}^{x_i-1} (x_i - n_i) \cdot f_X(x_i) \cdot F_{X-X}(n-x_i) \\
+ \sum_{x_i=0}^{n} (n_i - x_i) \cdot f_X(x_i) \cdot F_{X-X}(n-x_i).
\]

**Cost comparison**

It would be worthwhile to compare the total costs incurred during a single period \( T \) by the two methods, when the number of circuits supplied to the area is fixed at \( n \). Recall that the other two costs for carrying and replenishing inventories are incurred the same for both cases, and thus are not accounted for in this study. Recall that \( p_P \) and \( p_D \) are the unit P-level and D-level shortage costs. The expected shortage cost for the indirect case is

\[
p_P \cdot E[Z_P] = p_P \cdot (\lambda\tilde{F}_X(n-1) - n\tilde{F}_X(n)),
\]

while the one for the direct case becomes

\[
p_P \cdot E[Z_P] + p_D \cdot E[Z_D(n)] = p_P \cdot (\lambda\tilde{F}_X(n-1) - n\tilde{F}_X(n)) \\
+ p_D \cdot \sum_{i=1}^r \sum_{x_i=n_i+1}^{n} (x_i - n_i) \cdot f_X(x_i) \cdot F_{X-X}(n-x_i) \\
+ \sum_{x_i=0}^{n} (n_i - x_i) \cdot f_X(x_i) \cdot F_{X-X}(n-x_i).
\]

Since the extra cost for the direct wiring case is simply \( p_D \cdot E[Z_D(n)] \), our attention from here on will be confined to obtaining and characterising the expected shortage \( E[Z_D(n)] \).

**Cost-minimising wire allocation for the case of direct wiring**

Given the supplied number \( n \) of circuits for the area, it is interesting to find the optimal circuit allocation among subareas, which minimises the expected number of rearrangements. The problem is formulated as the following integer programming model:

\[
\text{(AP)} \quad \min \sum_{i=1}^r h_i(n_i) \\
\text{subject to} \\
\sum_{i=1}^r n_i = n \\
n_i \in \mathbb{N}, i = 1, \ldots, r
\]

where

\[
h_i(n_i) = \sum_{x_i=n_i+1}^{n} (x_i - n_i) \cdot f_X(x_i) \cdot \tilde{F}_{X-X}(n-x_i) \\
+ \sum_{x_i=0}^{n} (n_i - x_i) \cdot f_X(x_i) \cdot \tilde{F}_{X-X}(n-x_i).
\]

Note that this allocation problem (AP) is none other than the so-called cell problem (or the separable problem)\(^4\) with the following properties.

**Property 1** The expected number of rearrangements at each DP, \( h_i(n_i) \), is a convex function of \( n_i \) for \( i = 1, \ldots, r \).

**Proof** The convexity of function \( g() \) defined on the integer domain can be proved by showing that the inequality

\[
g(a-1) + g(a+1) \geq 2g(a)
\]

holds for any three integers \((a-1, a, a+1)\) in the domain.\(^5\) The convexity of \( h_i(n_i) \) is proven by noting that

\[
h_i(n_i-1) + h_i(n_i+1) - 2h_i(n_i) = h_i(n_i) + f_X(n_i) \quad \text{and} \quad f_X(n_i) \geq 0.
\]

**Property 2** If \( \mu \) exists such that \( h_i(n_i) - h_i(n_i-1) \leq \mu \leq h_i(n_i+1) - h_i(n_i) \) for \( i = 1, \ldots, r \) for some feasible allocation \( n, \) then \( n \) is optimal.

**Proof** Associating Lagrange multiplier \( \mu \) with the resource (or feasibility) constraint \( \sum_{i=1}^r n_i = n \), we have the Lagrangian subproblem:

\[
\text{(LAP)} \quad \min \sum_{i=1}^r h_i(n_i) - \mu \left( \sum_{i=1}^r n_i - n \right) \\
\text{subject to} \\
n_i \in \mathbb{N}, i = 1, \ldots, r.
\]

Rewriting the objective function as

\[
\min \sum_{i=1}^r (h_i(n_i) - \mu n_i) + \mu n,
\]

we see the separability that each minimising \( n_i \) obtained separately and independently would from the optimal \( n^* \) for (LAP).
From the convexity of $h_i(n_i)$, and thus that of $(h_i(n_i) - \mu n_i)$, the following inequalities hold at each minimizing $n_i^*$:

\[
\begin{align*}
    h_i(n_i^*) - \mu n_i^* &\leq h_i(n_i^* - 1) - \mu(n_i^* - 1), \\
    h_i(n_i^*) - \mu n_i^* &\leq h_i(n_i^* + 1) - \mu(n_i^* + 1).
\end{align*}
\]

If this $n^*$ is feasible to the original allocation problem (AP), that is, $\sum_{i=1}^r n_i = n$, then $n^*$ is optimal to AP.

**Property 3** A feasible allocation $n$ is an optimal solution of the allocation problem (AP) if $\min_{i=1,...,r}(F_X(n_i)) \geq \max_{i=1,...,r}(F_X(n_i - 1))$.

**Proof** We first note the following relation from the definition of $h_i(n_i)$:

\[h_i(n_i) - h_i(n_i - 1) = F_X(n_i - 1) - F_X(n_i).\]

Let $\mu = \min_{i=1,...,r}(F_X(n_i)) - F_X(n_i)$. This $\mu$ then satisfies the inequality condition in Property 2. The proof is completed by noting that the inequality can be rearranged as $\min_{i=1,...,r}(F_X(n_i) - n_i) \geq \max_{i=1,...,r}(F_X(n_i - 1))$.

This is similar to the result in either the multi-item newsvendor problem with a single constraint or the competitive newsvendor problem that the optimal amount of order (or allocation) depends only on the probability of shortage.

The methods to allocate inventories can be applied to our problem with some modification. The incremental allocation procedure developed by Fox can also be used, which requires exactly $n$ iterations to reach optimality. Taking all these previous studies into account, we now present a practically more efficient solution procedure.

**Procedure interchange allocation**

1. find an initial allocation $n$ such that $\sum_{i=1}^r n_i = n$ and $n_i \in \mathbb{N}$, $i = 1, \ldots, r$.
2. while not $\min_{i=1,...,r}(F_X(n_i)) \geq \max_{i=1,...,r}(F_X(n_i - 1))$ do begin
   - let $j = \arg \min_{i=1,...,r}(F_X(n_i))$ and $k = \arg \max_{i=1,...,r}(F_X(n_i - 1))$;
   - set $n_j = n_j + 1$ and $n_k = n_k - 1$; end; end;

Let $n^0$ and $n^*$ be an initial and an optimal solution respectively. Then the number of iterations required by the above algorithm is bounded by $\sum_{i=1}^r (n_i^* - n_i^0)^+ \leq n$. To expedite the convergence, we can simply set the initial allocation to $\text{DP}^i, n_i^0$, to be proportional to the average demand of the subarea, that is, $n_i^0 = n \times E[X]/E[X]$, which some telcos are actually using in practice.

**Example**

Consider a case where the PCP area consists of 20 DP areas, $r = 20$, and the demand of each DP $i$ for a given reinvestment period $T$ has an independent Poisson distribution with its own parameter, for example, $\lambda_i = i$ for $i = 1, \ldots, 20$. Both types of expected shortages are obtained by varying the ratio of supply to the expected demand from 0.9 to 1.9 as shown in Table 1. As expected, both shortages decrease as the supply increases. But the D-level shortage does not easily vanish, whereas the pace of fading in the P-level shortage is rapid, reaching almost zero even at the ratio of 1.2. This implies that a sufficiently large level of circuit provisioning is not cost-effective when introducing PCPs in the local loop.

Note that the difference in expected shortage is not so significant between the optimal allocation and the initial one based on the expected demand, $n_i = n \times E[X]/E[X]$ for $i = 1, \ldots, 20$. But the practical significance of this seemingly minor difference for a single PCP area cannot be emphasized enough when considering the enormous number of PCP areas a typical telco has to support.

Another point to note is that the Do-loop of Interchange Allocation Procedure was iterated 9.7 times on the average in this experiment. This is in good contrast with approximately 300 iterations required under the pure Incremental Allocation procedure.

**Conclusions and future studies**

We examined the circuit shortages occurring in the copper-based local loop system, and compared the expected shortages in a single PCP area under two circuit deployment methods, direct and indirect wiring. With focus naturally placed on the number of rearrangements for the case of direct wiring, we derived an $O(n^2)$ computational formula to calculate its expectation. Also derived was an

<table>
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<th>$n$</th>
<th>$E[Z_p] = \frac{n \times E[X]}{E[X]}$</th>
<th>$E[Z_0(n)] = \frac{n \times E[X]}{E[X]}$</th>
<th>Interchange allocation</th>
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<tr>
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optimality condition for allocating to the DP areas the
given number of circuits for the PCP area, based on
which a practically efficient allocation procedure was
proposed.

The allocation procedure can be directly applied to the
exchange level, to the process of distributing a bundle of
circuits assigned to an Exchange (or CO) to PCP areas.
Furthermore, this allocation strategy can be extended, with
some technology-specific modifications, to the construction
of fiber loops.\textsuperscript{1,2} Considering the immense amount of
investment in building the emerging dominant form of
local loops, the realistic importance of our optimisation-
based investment decision cannot be emphasised too much.
This type of analysis may be found useful in logistics where
a company with many sales agencies has to decide how to
distribute its goods among them while minimising the
associated costs.

We can also think of the interesting alternative problem
with a resource constraint replaced by a budget
constraint.\textsuperscript{5,8} Then the constraint coefficients should not
be uniform reflecting the distances from the PCP to DPs,
and the fixed number of circuits supplied to the PCP area
should be replaced by the fixed budget for the area. The
one-to-one rewireability of a circuit demand may have to
be relaxed to accommodate the partial rewireability or
substitutability\textsuperscript{7,9} as well.

A comprehensive extension of our study would be to find
the optimal number of circuits supplied to the PCP area at
each replenishment period which minimises all the costs
involved in this replenishment-inventory system over some
planning horizon. This problem, once solved effectively,
would contribute considerable cost savings in circuit
management operation for telcos employing the direct-
wiring policy. Despite this importance, this investigation
is reserved for a future study due to the problem's
complexity.

\textbf{References}

Telecommunications PLC.
2. Reed DP (1992). \textit{Residential Fiber Optic Networks: An Engi-
eering and Economic Analysis}. Artech House: Norwood,
USA.
expansion in local access telecommunications networks. \textit{Ann
for solving problems of optimum allocation of resources. \textit{Ops
Res} 11: 399–417.
7. Lippman SA and McCardle KF (1997). The competitive news-
multi-item newsboy problem with a single constraint. \textit{Naval Res
Logist} 31: 463–474.
simultaneously determining the optimal brand collection and

Received June 1998;
accepted December 1998 after one revision