Valuing Derivatives on the Stock Index Volatility in a General Equilibrium Framework*

Byung Soo Kim, In Joon Kim and Tong Suk Kim

Financial Engineering Research Center, Graduate School of Management at Korea Advanced Institute of Science and Technology (KAIST), 207-43 Chongyang-ni, Dongdaemoon-gu, Seoul, KOREA

April 1998

Abstract

This paper presents a general equilibrium model of derivatives on the stock index volatility in which the prices of financial assets, the expected premium for volatility risk, and the riskless rate of interest are determined endogenously. It is found that the expected premium for the volatility risk is not proportional to the level of volatility and, hence the risk-neutral process of volatility does not follow a CIR square-root process. This makes the resulting model different from that of Grünbichler and Longstaff (1996) which is based on the assumption of a proportional risk premium and a CIR square-root process. The stochastic nature of interest rate allows us to investigate the difference between forward and futures contracts. We also correct some inconsistencies of Grünbichler and Longstaff (1996).

Keywords: Volatility, General equilibrium, Forward and Futures, Options, CIR square-root process

* Corresponding author : Tel.: 82-2-958-3968; fax : 82-2-958-3604; e-mail: s_trihemp@cais.kaist.ac.kr

1 Introduction

Empirical evidence on underlying asset prices and their derivatives strongly suggests that asset price volatility is stochastic. For traders whose portfolios contain options or securities with option-like features, hedging changes in volatility is, therefore, an important problem. Brenner and Galai (1989) and Whaley (1993) show that futures and options contracts on the volatility index can be used to manage the volatility risk exposure of portfolios containing options or securities with option-like features. The VIX index based on the implied volatility of S&P 100 index calls and puts is quoted since 1993. In Europe, the German Futures and Options Exchange (DTB) provides a volatility of DAX index calls and puts. Volatility futures and options on volatility indexes are currently being developed by a number of investment banking firms in the U.S. and Europe.

The work of Grünbichler and Longstaff (1996) is the first that addresses this interesting issue. In the paper, they provides excellent discussions of the implications for the properties of volatility derivatives. Since volatility is not a traded asset, its expected growth rate and investor attitudes toward risk become important to the pricing of derivatives on volatility. Grünbichler and Longstaff (1996), therefore, present an equilibrium model under the assumption that volatility follows a CIR square-root process and the expected premium for the volatility risk is proportional to the level of the volatility.

We present a general equilibrium model of the volatility derivatives in which the prices of financial assets, the expected premium for volatility risk, and the riskless rate of interest are endogenously determined. Unlike Grünbichler and Longstaff (1996), the riskless rate

of interest is allowed to vary stochastically. We find that the volatility risk premium is not proportional to the level of it. The general equilibrium model developed in this paper has many interesting implications for the price behaviors of futures, forward and option on the volatility. The futures price in a general equilibrium is different from the price of the Grünbichler and Longstaff (1996)'s model. This is because the process of the risk-neutral volatility is not a CIR square-root process unlike their assumptions. The change of the risk-neutral process is caused by non-proportionality of risk premium. The effect of a stochastic interest rate on the relationship between futures and forward prices is well known. In our result, the forward price on volatility is lower than the futures price. The call option prices in a general equilibrium are lower than the price of G-L model for the low level of a current volatility and higher for the sufficiently high level.

We organize the paper as follows. In Section 2, we develop our model using the Cox, Ingersoll, and Ross (1985) general equilibrium framework. In Section 3, we provide numerical solutions for the model of section 2. In Section 4 and 5, we investigate the price difference properties of futures, forward and call between a general equilibrium model and the Grünbichler and Longstaff model. Section 6 summarizes the results.

2 A General Equilibrium Valuation Model of Derivatives on the Stock Index Volatility

In this section, we use the general equilibrium framework of CIR (1985a) to develop a pricing model of derivatives on volatility. In doing this, we make the following assumptions. Economic activity takes place in the time interval $[0, T]$[1] . Times mentioned later will always be assumed to belong to this interval, $t \in [0, T]$. Only one good is produced. The technology to produce that good is exogenously given and exhibits constant stochastic returns to scale and the output is continually reinvested. If q_t is invested in production at date t, its returns are governed by the stochastic differential equation,

$$dq_t = q_t \mu dt + q_t V_t \, dw_{1t}, \tag{1}$$

where w_{1t} is a standard Wiener process. The constant, μ, is the mean of the rate of return on the production process and V_t is the instantaneous volatility. The state of technology can be represented by the stock index volatility, V_t. The state variable V_t is governed by the stochastic differential equations,

$$dV_t = (\alpha - \kappa V_t)dt + \sigma\sqrt{V_t}dw_{2t} \tag{2}$$

where α, κ, and σ are constant, w_{2t} is a standard Wiener process, and ρ is the instantaneous correlation coefficient, $\rho dt = (dw_{1t})(dw_{2t})$.

[1] We adopt a finite-horizon formulation to consider the effects of horizon length on contingent claim values. A bequest function assigning utility to terminal wealth could easily be added.

Without loss of generality, we assume that firms finance production only through equity claims. Markets are perfect; there are no transaction costs, taxes, or short-sales restrictions. Investors can buy and sell riskless bonds among themselves, so that riskless borrowing and lending can take place at a rate of interest, r. There are perfectly competitive continuous markets for a variety of contingent claims, $F(W, V, t)$, including derivatives on the stock index volatility. These have continuous payment flows, $\delta(W, V, t)$. We can in general write the stochastic differential equation governing the movement of the value of the claim, F, as

$$dF = (F\beta - \delta)dt + F(h_1 dw_{1t} + h_2 dw_{2t})$$

where β is the instantaneous expected rate of return and h_1 and h_2 are the instantaneous standard deviations of the return.

There are a fixed number of investors, identical in their endowments and preferences of the form,

$$E_0\left[\int_0^T e^{-\psi t}\ln(C_t)dt\right], \tag{3}$$

where C_t is the consumption at time t, E_0 is the conditional expectation operator conditional on the current endowment and the state of the economy. As in CIR (1985a), the representative investor's decision-problem is equivalent to maximizing Equation (3), subject to the budget constraint,

$$\begin{aligned} dW_t &= (aW_t(\mu - r) + bW_t(\beta - r) + rW_t - C)\,dt \\ &\quad + aW_t V_t dw_{1t} + bW_t(h_1 dw_{1t} + h_2 dw_{2t}), \end{aligned}$$

where W_t denotes wealth, aW_t is the amount of wealth invested in the production process,

and bW_t is the amount of wealth invested in the contingent claim.

From theorem 1 of CIR (1985a), the equilibrium risk-free rate of interest is,

$$r(V,t) = \mu - V_t^2.$$

This result is reasonable because the riskless rate of interest is lower than the expected return on risky investment. Since the riskless rate is the difference between the expected return on the market and the variance of the market return, it must be stochastic if the market volatility randomly changes. From CIR (1985a), Theorem 2, the equilibrium expected return on contingent claims is given by,

$$\beta F - rF = V_t^2 W F_W + \sigma\rho\sqrt{V_t}V_t F_V \tag{4}$$

Thus, the risk premium of a single state variable is the volatility of market return times the volatility of the state variable. Grünbichler and Longstaff (1996) assume that the dynamics of the volatility, V_t, is given by a CIR square-root process like Equation (2). They further assume that "the expected premium for volatility risk is proportional to the level of volatility, ζV_t."[2] Their assumptions of proportionality can not be maintained in our equilibrium, since the proportionality risk premium is inconsistent with the assumed volatility process, Equation (2). If the volatility follows a CIR square-root precess, the expected premium for the volatility risk could be $\sigma\rho\sqrt{V_t}V_t$. Then the risk-neutral process for V_t is

[2] To satisfy the proportionality of the risk premium, we can assume that V_t is the current variance value of returns for the stock index price instead of the volatility. That is, equation (1) is rewriten by $dq_t = q_t\mu dt + q_t\sqrt{V_t}\ dw_{1t}$. Then the risk premium is proportional to the variance, and the risk-neutral process is $dV_t = (\alpha - (\kappa + \sigma\rho)\, V_t)\, dt + \sigma\sqrt{V_t}dw_{2t}$. Or, we can use another process for the volatility, i.e. an Ornstein-Uhlenbeck process. Then the volatility dynamics is $dV_t = (\alpha - \kappa V_t)dt + \sigma dw_{2t}$ and the risk-neutral process is $dV_t = (\alpha - (\kappa + \sigma\rho)\, V_t)\, dt + \sigma dw_{2t}$.

$$dV_t = (\alpha - \kappa V_t - \sigma\rho\sqrt{V_t}V_t)dt + \sigma\sqrt{V_t}dw_{2t}. \tag{5}$$

Although the actual volatility is a CIR square-root process, the process when the drift is adjusted by the risk premium is not the CIR model. As a result, the risk-neutral process is not distributed as a non-central chi-squared variate.

We now turn to the fundamental valuation equation for any contingent claim on the stock index volatility[3] . Since we value securities whose contractual terms do not dependent explicitly on wealth but the volatility, the partial derivatives F_W, F_{WW}, and F_{WV} are equal to zero for such securities and the current value of any contingent claims to the stock index volatility, $F(V_0, T)$, satisfies the following fundamental valuation equation:

$$\frac{1}{2}\sigma^2 V F_{VV} + \left(\alpha - \kappa V - \sigma\rho V\sqrt{V}\right) F_V + F_t - r(V,t)F + \delta = 0, \tag{6}$$

where $r(V,t) = \mu - V_t^2$, subject to the boundary condition $F(V_T, 0) = P(V_T)$.

3 Numerical Solutions

We could not derive a closed-form solution to Equation (6). The risk-neutral process, Equation (5), is not distributed as a non-central chi-squared variate and the distribution of $\int_0^T r(V,t)dt$ is quite complicated because the integral value is the sum of the sequences of the correlated second moments of V_t, which may be correlated with V_T. We use a finite-difference approximation in this paper. It is important to recognize that "futures price is

[3] Theorem 3 of CIR (1985) state that the price of any contingent claim satisfies the partial differential equation, $F_t + (rW - C)F_W + \frac{1}{2}V^2W^2F_{WW} + \sigma\rho\sqrt{V}VF_{WV} + (\mu - \sigma\rho\sqrt{V}V)F_V + \frac{1}{2}\sigma^2VF_{VV} - rF + \delta = 0$,where $r(V,t) = \mu - V_t^2$.

equal to the value of an asset which receives a continual payout flow of $r(V,t)F$ and the amount V_T at time T."[4] In the present application, the futures price on the stock index volatility must satisfy the partial differential equation,

$$\frac{1}{2}\sigma^2 V F_{VV} + \left(\alpha - \kappa V - \sigma\rho V\sqrt{V}\right) F_V + F_t = 0, \tag{7}$$

subject to $F(V,T) = V_T$. Forward and option prices on the volatility have no payout; $\delta = 0$. These must satisfy the partial differential equation,

$$\frac{1}{2}\sigma^2 V F_{VV} + \left(\alpha - \kappa V - \sigma\rho V\sqrt{V}\right) F_V + F_t - \left(\mu - V^2\right) F = 0, \tag{8}$$

subject to $F(V,T) = V_T$ and $F(V,T) = \max(V_T - K, 0)$. The resulting numerical values of the forward and futures prices are presented in Table 1. Numerical values are computed using the 'Crank-Nicholson' method. The boundary values at the extreme mesh points are obtained by a quadratic extrapolation. Table 1 also reports the futures prices of the G-L model with the same parameter values. Since the risk-neutral process, Equation (5), is identical to Equation (5) of Grünbichler and Longstaff (1996) when $\rho = 0$, the futures price in a general equilibrium is equal to that of the G-L model. However, the forward prices are lower than those predicted by the G-L model. Table 2 compares the call values given by solving Equation (8) numerically with the values of Grünbichler and Longstaff (1996) under conditions identical to the general equilibrium model except for the constant rate of interest, $r = 0.1$. We compute the option values of G-L model using the same type of the finite-difference method used to calculate our model. Grünbichler and Longstaff (1996) propose the

[4] As discussed by CIR (1981) proposition 2 and 7, futures price would correspond to $\delta = r(V,t)F$ and $F(V,T) = V_T$.

normal approximation on the complementary non-central chi-squared distribution suggested by Sankaran (1963). However, the proposed approximation, Equation (13), has some errors. As shown in Table 2, the general equilibrium prices are higher than those of the G-L model for the sufficiently high level of a current volatility and lower than those for the low level.

4 Properties of Volatility Futures and Forward Prices in a General Equilibrium

In this section, we examine the properties of volatility futures and forward prices in the general equilibrium model. We also compare the results with the G-L model under the identical conditions.

4.1 Volatility Futures Price

In this subsection, the properties of the volatility futures price numerically computed by solving Equation (7) are examined. The correlation between the volatility with the rate of return on the stock index plays an important role for valuing the futures contracts in a general equilibrium. When the volatility is uncorrelated with the rate of return on the stock index, the futures price in a general equilibrium is identical to the price of the G-L model. Since the risk-neutral volatility process, Equation (5), is not a CIR square-root process when the volatility is correlated with the rate of return, the futures price is different from the price of the G-L model. The futures price when $\rho = 0$ is lower (higher) than the price for a negative (positive) correlation between the volatility process and the rate of return on the stock index. The sign of correlation coefficient, ρ, is also important. The price

difference between the case of negative correlation and of zero correlation increases with σ in the general equilibrium model . When ρ is positive, the opposite is true; an increase in σ decreases the price difference.

Figure 1 shows the results for different levels of correlation between the volatility and the rate of return on the stock index price. The curves are the price difference between the futures contract in the general equilibrium model and in the G-L model. As ρ increases, the price difference decreases. Figure 2 illustrates the price difference for different σ when the volatility is positively correlated with the rate of return on the stock index price. For the case of $\rho = 0.5$ ($\rho = -0.5$), the result of an increase in σ is to decrease (increase) the price difference of futures. When the volatility is higher, the price difference is increased. The effect of the sign of ρ on the futures price difference is symmetric.

From Equation (4), we can derive the equilibrium expected rate of return on volatility derivatives as

$$\beta - r = \sigma\rho\sqrt{V_t}V_t\frac{F_V}{F}. \tag{9}$$

The equilibrium expected rate of return, β, depends upon the correlation between the volatility and the rate of return on the stock index; the higher the correlation, the higher its equilibrium expected rate of return. Consequently, the price of any derivatives on volatility when $\rho = 0$, *ceteris paribus*, is higher than the price for a positive correlation and lower than for a negative correlation. When using the risk-neutral valuation, the equilibrium price of derivatives on volatility is the conditional expected value of the terminal payoff with the risk-neutral volatility discounted by the riskless rate of interest. The risk adjustment is

accomplished by reducing the drift of volatility process by the expected premium of the volatility risk, $\sigma\rho\sqrt{V_t}V_t$. The properties of volatility futures price in a general equilibrium equivalently can be explained by the impact that the correlation, ρ, has on the risk-neutral process, as a result, the terminal distribution of the volatility. Let $H(V_0, T)$ denote the futures price for a futures contract on V_0 with maturity T. Following Equation (46) of Cox, Ingersoll and Ross (1981), the futures price can be expressed as the expected value of V_T, $H(V_0, T) = \hat{E}_0(V_T)$, where $\hat{E}_0$ indicates the conditional expectation on current volatility V_0 taken with respect to a risk-neutral process for V_t. So, we understand that the futures price difference between the general equilibrium model and the G-L model is the expected value difference of the risk-neutral volatilities between when $\rho \neq 0$ and $\rho = 0$. However, the drift of the risk-neutral process, Equation (5), is a nonlinear equation unlike a CIR square-root process. We could not derive a closed-form solution for any order moment of the process. Figure 3 shows the conditional expectation of V_T given that $V_0 = 0.05$ or $V_0 = 0.25$. When $\rho = 0$, the process is a continuous time first-order autoregressive process. The properties are well known. The expected value when $\rho = -0.5$ ($\rho = 0.5$) is higher (lower) than the case of zero correlation, and approaches 0.153145 (0.147044) as $T \to \infty$. When $\rho \neq 0$, the process is similar to a first-order autoregressive process where the randomly moving volatility is pulled toward a long-term value. To understand the properties of the risk-neutral process when $\rho \neq 0$, we can rewrite Equation (5) as

$$dV_t = \left(\kappa + \sigma\rho\sqrt{V_t}\right)\left(\frac{\alpha}{\kappa + \sigma\rho\sqrt{V_t}} - V_t\right)dt + \sigma\sqrt{V_t}dw_{2t}. \tag{10}$$

In Equation (10), $\kappa + \sigma\rho\sqrt{V_t}$ would determine the speed of adjustment. Similarly, we would

interpret $\frac{\alpha}{\kappa+\sigma\rho\sqrt{V_t}}$ as a long-term value. When $\rho = 0.5$ ($\rho = -0.5$), the long-term value $\frac{\alpha}{\kappa-\sigma\rho\sqrt{V_t}}$ is a decreasing (increasing) function for $\sigma > -\frac{2\kappa}{\sqrt{V}}$ ($0 < \sigma < \frac{2\kappa}{\sqrt{V}}$) and the speed of adjustment, $\kappa + \sigma\rho\sqrt{V_t}$, an increasing (decreasing) function for σ. This means that when $\rho = 0.5$ ($\rho = -0.5$), the high terminal volatility become less (more) probable than the case of zero correlation as σ increases, therefore, the terminal distribution of the risk-neutral volatility is more positively (negatively) skewed than the distribution when $\rho = 0$. The sign of ρ may have an effect on the skewed direction of terminal distribution and the absolute value of the expected risk premium, $\left|\sigma\rho\sqrt{V_t}V_t\right|$, on the skewed magnitude.

Figure 4 illustrates the price difference for different time to maturity when volatility is negatively correlated with the rate of return on the stock price. When $\rho = -0.5$ ($\rho = 0.5$), an increase in time to maturity increases(decreases) the price difference for low volatility, but the price difference when $T = 0.5$ is less (larger) than the difference when $T = 0.3$ for sufficiently high volatility. This property may be caused by the mean reversion property of Equation (5). In Equation (10), the long-term value and the speed adjustment are affected by the mean reversion property of V_t. The sensitivity of the futures price difference to the current value V_0 would decreases as $T \to \infty$.

4.2 Volatility Forward Price

In this subsection, we investigate the properties of the volatility forward price. The forward price in a general equilibrium is affected by the stochastic nature of the riskless interest rate and the non-proportionality of the volatility risk premium. As a result, the forward price is different from the futures price. Several studies noted the critical role of stochastic rate of

interests. Jarrow and Oldfield (1981) provide a discussion of the contractual differences and use an arbitrage argument to show the importance of stochastic rate of interests. Richard and Sundaresan (1981) derive a continuous time equilibrium model and use it to analyze forward and futures contracts. Using an arbitrage argument, Cox, Ingersoll and Ross (1981) show the relationship between futures and forward prices.

Figure 5 illustrates the effect of the different correlation on the forward price difference between the general equilibrium model and the G-L model. When $\rho = -0.5$ ($\rho = 0.5$), the forward price difference is larger (smaller) than the case of zero correlation. This is similar to Figure 1.

Figure 7 shows the results for different σ when the volatility is uncorrelated with the rate of return on the stock index. When $\rho = 0$, we can investigate the effect of the stochastic rate of interest on the forward price except for the process change of the risk-neutral volatility when the volatility is correlated.. The result is that the forward price in a general equilibrium is lower than the price predicted by the G-L model. If $B(V_t, T-t)$ is the price at time t of a riskless discount bond paying one dollar at time T, from the proposition 6 of CIR (1981), the difference between forward price and futures price is $\frac{\int_0^T H(V_t,T-t)\mathrm{cov}[H(V_t,T-t),B(V_t,T-t)]dt}{B(V_0,T)}$, where $\mathrm{cov}[H(V_t, T-t), B(V_t, T-t)]$ represents the local covariance of the percentage change in the futures price H and the percentage change in the bond price B. The proposition shows that $\mathrm{cov}[H(V_t, T-t), B(V_t, T-t)] > 0$ for all $T-t$ implies that a forward price is higher than a futures price. In a general equilibrium model, the volatility forward price is lower than the futures price. Figure 6 shows the relationship between the bond price $B(V_0, T)$

in a general equilibrium and the bond price for the constant interest rate, $r = 0.1$. The bond price may be a complicated function of V_0. The value first increases as V_0 increases. Although not evident from Figure 6, it is true that, for sufficiently high volatility, the bond price decreases again, but the futures price is always an increasing function of V_0. In the general equilibrium model, the numerator may be negative. In Figure 7, an increase in σ is to decrease the magnitude of the price difference. This may be caused by the bond price which increases as σ increases. When the volatility is correlated with the rate of return, the results of an increase in σ is similar to the case of the futures price. When $\rho = -0.5$ ($\rho = 0.5$), the price difference is larger (smaller) than the difference for zero correlation.

Figure 8 illustrates the forward price difference for the different time to maturity. The forward price difference decreases as T increases. This may be because of the bond price. The bond price decreases as T increases. This causes an increase in the price difference, $\frac{\int_0^T H(V_t,T-t)\mathrm{cov}[H(V_t,T-t),B(V_t,T-t)]dt}{B(V_0,T)}$. Of course, the numerator is affected by increasing T. At the least, the denominator change may dominate the numerator change. The net effect is that the curve of the price difference shifts downward. Figure 5 and 8 provides a clue to the forward price difference when the volatility is correlated with the rate of return on the stock index. We note that the price difference when $T = 0.1$ is larger than the difference when $T = 0.3$, which is in turn larger than when $T = 0.5$ regardless of the sign of ρ. This property is different from the result of Figure 4. This may be because the effect of the time to maturity dominates the effect of the distribution change. When $\rho = -0.5$, $\int_0^T H(V_t, T-t)\mathrm{cov}[H(V_t, T-t), B(V_t, T-t)]\, dt$ can be positive. For instance, when $T = 0.3$

or $T = 0.5$, the forward price difference first is negative but increases as V_0 increases, and then is positive for sufficiently high volatility.

4.3 Relationship between Forward Price and Futures Price in a General Equilibrium

In the previous subsections, we examined the price difference between a general equilibrium model and a G-L model for futures price and forward price, respectively. In this subsection, we examine the relationship between a forward price and futures price in the general equilibrium. The price difference between a forward contract and a futures contract for different levels of σ is similar to the relationship shown in Figure 7 regardless ρ although the magnitude of the difference is slightly different. The difference when T changes is similar to the relationship shown in Figure 8. Figure 7 and 8 illustrate the relationship of the forward prices caused by the stochastic interest rate except for the non-proportionality of risk premium. Similarly, the relationship between a forward and futures price in a general equilibrium is caused by the stochastic rate of interest. The forward and futures price in a general equilibrium, both are affected by the change ofthe risk-neutral process caused by the non-proportionality when $\rho \neq 0$. However, the effect of the change on the forward price is offset by that of the futures price.

5 Properties of Volatility Call Price in a General Equilibrium

In this section, we examine the properties of the volatility option price. We also compare the results with the G-L model under the identical condition with the general equilibrium

model except for the constant rate of interest $r = 0.1$. In Figure 9, the call price differences are shown for changing ρ. For the case of zero correlation, the call price differences are negative for low volatilities and positive for the high volatilities. When $\rho = -0.5$ ($\rho = 0.5$), the price difference is larger (smaller) than the difference when $\rho = 0$ because of the effect of correlation on the equilibrium expected return, equivalently, on the risk-neutral volatility process. This result is consistent with Figure 1 and Figure 5.

Figure 10 illustrates the price difference for the different level σ when $\rho = 0$. The result is that the difference of the call prices is negative for the low level of a current volatility but positive for the sufficiently high level. Since the risk-neutral process when $\rho = 0$, Equation (5), is a CIR square-root process, the difference of call prices is only caused by the stochastic nature of the riskless interest rate. Figure 6 shows the bond price in the general equilibrium model and the price for the constant interest rate. The bond price in a general equilibrium is lower than the price with the constant interest rate for low volatility, but, higher than the price for sufficiently high V_0. Since the current value of a call option is the conditional expected value of the terminal payoff discounted by the riskless interest rate, the terminal payoff in the general equilibrium model is more discounted than the case of the constant interest rate for low V_0, but less discounted for sufficiently high V_0. Consequently, the call option price in a general equilibrium is lower (higher) than the price in the G-L model for low (high) volatility. Another interesting result is that, when a current volatility is low, the price difference when $\sigma = 0.2$ larger than the difference when $\sigma = 0.4$, which is in turn larger than the difference when $\sigma = 0.6$. When a current volatility is sufficiently high,

the reverse is true: the price difference increases as σ increases. This may be explained by the impact that a volatility coefficient of volatility, σ, has on the terminal distribution of volatility, V_T. As T sufficiently increases, the distribution of V_T will become similar to a gamma distribution with $\varpi \equiv \frac{2\kappa}{\sigma^2}$ and $\nu \equiv \frac{2\alpha}{\sigma^2}$[5] . We note that, as ν decreases, low terminal volatilityies become more probable in comparison with sufficiently high volatilities[6] . If V_0 decreases, the expected value of V_T and the variance of volatility decreases. The result is that the distribution of V_T is positively skewed, and the probability of low terminal volatility increases. If σ increases, ν decreases and the variance of volatility increases. High variance of volatility increases the chances for extrem volatility movement, but a decrease of ν increases more the probability of low volatility than the probability of sufficiently high volatility. As σ increases, the option value increases. However, when V_0 is low, the increament of option value in a general equilibrium will be less than the case of a constant interest rate. As a result, the price difference decreases as σ increases. If V_0 is sufficiently high, the distribution is negatively skewed and the variance of volatility is very high. In addition, an increase of σ increases the variance of volatility even though ν decreases. The net effect may be that an increament of option value is more then an increament of the G-L model. Therefore, the price difference may increases as σ increases.

Figure 11 shows the price difference for different time to maturity, T, when the volatility

[5] The expected value and variance of V_T are given by; $E(V_T) = \frac{\alpha}{2\kappa^2}\sigma^2(1-\exp(-\kappa T))^2 + \exp(-\kappa T)V_0$ and $Var(V_T) = \frac{\sigma^2}{\kappa}V_0(\exp(-\kappa T) - \exp(-2\kappa T)) + \frac{\alpha}{2\kappa^2}\sigma^2(1-\exp(-\kappa T))^2$. For sufficiently large T, the distribution of V_T will approach a gamma distribution whose pdf is $p(V_\infty) = \frac{\varpi^\nu}{\Gamma(\nu)}V^{\nu-1}e^{-\varpi V}$ where $\varpi \equiv \frac{2\kappa}{\sigma^2}$ and $\nu \equiv \frac{2\alpha}{\sigma^2}$.
See CIR(1985b).

[6] See Norman L. Hohnson et al (1994).

is uncorrelated with the return on the stock index. The price difference decreases as T increases. This is consistent with Figure 8. The sensitivity of the option price difference to the current volatility value V_0 also decreases as T increases. This may be because of the mean reversion property of a first-order autoregressive process. The bond price in a general equilibrium is a complicated function of V_0. However, the dependency of the bond price on the current volatility value would decrease as T increases

6 Conclusion

In this paper, we use the general equilibrium framework of CIR (1985a) to develop the pricing model of derivatives on volatility. In this model, the riskless rate of interest is endogenously determined with a stochastic process. The derivatives on the stock index volatility is examined using numerical methods when the default free rate of interest is stochastic. In our general equilibrium, we find that the expected premium for the volatility risk is not proportional to the level of volatility. The risk-neutral volatility process is not a CIR square-root process even though the actual volatility is the CIR model.

The correlation between the volatility with the rateof return on the stock index plays an important role for valuing the volatility derivatives in a general equilibrium. The equilibrium expected rate of return on volatility derivatives depends upon the correlation; the higher the correlation, the higher its equilibrium expected return. Consequently, the price of any derivatives on volatility when $\rho = 0$, *ceteris paribus*, is higher than the price for a positive correlation and lower than for a negative correlation. In the case of a futures price, when

the volatility is uncorrelated with the rate of return on the stock index, the futures price of G-L model is identical to the general equilibrium model, however, underprices the volatility futures contract when negatively correlated and overprices it when positively correlated. The forward prices and option values in a general equilibrium are affected by the stochastic nature of the riskless interest rate. When $\rho = 0$, the forward price is lower than the price predicted by the G-L model. This is because of the effect of the stochastic rate of interest on the forward price. The call option prices in a general equilibrium is lower than the G-L model price for the low level of a current volatility and higher than the price for the sufficiently high level because of the stochastic nature of the riskless interest rate.

References

Bailiy, W., and R. M. Stulz, March 1989, The pricing of stock index options in a general gquilibrium model. Journal of Financial and Quantitative Analysis 41, No. 1, 1-12.

Black, F., 1976, The pricing of commodity contracts, Journal of Financial Economics 3, 167-179.

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637-654.

Brenner, M. and D. Galai, 1989, New financial instruments for hedging changes in volatility, Financial Analysts Journal, July/August, 61-65.

Cox, J., J.E. Ingersoll, and A. Ross, 1981, The relation between forward prices and futures prices, Journal of Financial Economics 9, 321-346.

Cox, J., J.E. Ingersoll, and A. Ross, 1985a, An intertemporal general equilibrium model of asset prices, Econometrica 53, 363-384.

Cox, J., J.E. Ingersoll, and A. Ross, 1985b, A theory of the term structure of interest rate, Econometrica 53, 385-407.

French, Kenneth R., 1981, The pricing of futures contracts, Unpublished working paper (Graduate School of Management, University Rochester, Rochester, NY).

French, K.R., G.W. Schwert and R.F. Stambaugh, 1987, Expected stock returns and volatility, Journal of Financial Economics 19, 3-30.

Friedman, A., 1975, Stochastic differential equations and applications, vol. 1 (Academic Press, New York).

Grünbichler, A. and F.A. Longstaff, 1996, Valuing futures and options on volatility, Journal of Banking & Finance 20, No.6, 985-1001.

Hemler, M.L. and F.A. Longstaff, 1991, General equilibrium stock index futures prices: Theory and empirical evidence, Journal of Financial and Quantitative Analysis 26, 287-308.

Heston, S.A., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, The Review of Financial Studies 6, 327-343.

Hull, J., and A. White, 1987, The pricing of options on assets with stochastic volatility, Journal of Finance 42, 281-301.

Jarrow, R.A. and G.S. Oldfield, 1981, Forward contracts and futures contracts, Journal of Financial Economics 9, 373-382.

Johnson, H., and D. Shanno, June 1987, Option pricing when the variance is changing, Journal of Financial and Quantitative Analysis 22, 143-153.

Johnson, N.L, S. Kotz, and N. Balakrishnan, 1994, Continuous Univariate Distributions, vol. 1 (JOHN WILEY & SONS, INC.).

Margrabe, William, 1976, A theory of forward and futures prices. Unpublished working paper (The Wharton School, University of Pennsylvania, Philadelphia, PA).

Merton, Robert C., 1971, Optimum consumption and portfolio rules in a continuous-time model, Journal of Economic Theory 3, 373-413.

Merton, Robert C., Sep. 1973, An intertemporal capital asset pricing model, Econometrica 41, No. 5, 867-887.

Press,W.H., S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, 1992, Numerical Recipes in C (Cambridge University Press).

Rabinovitch, Ramon, Dec. 1989, Pricing stock and bond options when the default-free rate is stochastic, Journal of Financial and Quantitative Analysis 24, No. 4, 447-457.

Richard, S.F., and M. Sundresan, 1981, A continuous time equilibrium model of forward prices and futures prices in a multigood economy, Journal of Financial Economics 9, 347-371.

Sankaran, M., 1963, Approximations to the non-central chi-square distribution, Biometrica 50, 199-204.

Sheikh, A.M., 1993, The behavior of volatility expectations and their effects on expected returns, The Journal of Business 66, 93-116.

Scott, L., Dec. 1987, Option pricing when the variance changes randomly: Theory, estimation and application, Journal of Financial and Quantitative Analysis 22, 419-438.

Stein, A.M. and J.C. Stein, 1991, Stock price distributions with stochastic volatility: An analytic approach, The Review of Financial Studies 4, 727-752.

Sundaresan, M., 1980, A study of commodity futures prices, Unpublished doctoral dissertation (Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, PA).

Vasicek, O.A., 1977, An equilibrium characterization of the term structure, Journal of Financial Economics 5, 177-188.

Whaley, R.E., 1993, Derivatives on market volatility: Hedging tools long overdue, Journal of Derivatives (Fall), 71-84.

Wiggins, J., Dec. 1987, Stochastic volatility option valuation: Theory and empirical estimates, Journal of Financial Economics 19, 351-372.

Table 1: Comparison of futures prices and forward prices of General Equilibrium Model and G-L Model: Parameters:$\alpha = 0.6$, $\kappa = 4$, $\rho = 0$, $\sigma = 0.6$, $T = 0.1$, $\mu = 0.2$

	General Equilibirum		G-L	Difference	
Volatility	Futures	Forward	Futures	Futures	Forward
0.00	0.049452	0.048481	0.049452	0	-0.000971
0.02	0.062858	0.061635	0.062858	0	-0.001223
0.04	0.076265	0.074798	0.076265	0	-0.001467
0.06	0.089671	0.087972	0.089671	0	-0.001699
0.08	0.103078	0.101160	0.103078	0	-0.001918
0.10	0.116484	0.114362	0.116484	0	-0.002122
0.12	0.129890	0.127583	0.129890	0	-0.002307
0.14	0.143297	0.140823	0.143297	0	-0.002474
0.16	0.156703	0.154086	0.156703	0	-0.002617
0.18	0.170110	0.167372	0.170110	0	-0.002738
0.20	0.183516	0.180685	0.183516	0	-0.002831
0.22	0.196922	0.194027	0.196922	0	-0.002895
0.24	0.210329	0.207400	0.210329	0	-0.002929
0.26	0.223735	0.220806	0.223735	0	-0.002929
0.28	0.237142	0.234247	0.237142	0	-0.002895
0.30	0.250548	0.247726	0.250548	0	-0.002822
0.32	0.263954	0.261245	0.263954	0	-0.002709
0.34	0.277361	0.274806	0.277361	0	-0.002555
0.36	0.290767	0.288412	0.290767	0	-0.002355
0.38	0.304174	0.302064	0.304174	0	-0.002110
0.40	0.317580	0.315766	0.317580	0	-0.001814
0.42	0.330986	0.329519	0.330986	0	-0.001467
0.44	0.344393	0.343325	0.344393	0	-0.001068
0.46	0.357799	0.357188	0.357799	0	-0.000611
0.48	0.371206	0.371109	0.371206	0	-0.000097
0.50	0.384612	0.385091	0.384612	0	0.000479

Table 2: Comparison of Call Prices of General Equilibrium Model and G-L Model; Parameters:$\alpha = 0.6$, $\kappa = 4$, $\rho = 0$, $\sigma = 0.6$, $T = 0.1$, $\mu = 0.2$, $K = 0.15$

Volatility	General Equilibirum	G-L Model	Difference
0.00	0.000073	0.000074	-0.000001
0.02	0.000431	0.000434	-0.000003
0.04	0.001300	0.001311	-0.000011
0.06	0.002923	0.002947	-0.000023
0.08	0.005513	0.005556	-0.000043
0.10	0.009221	0.009288	-0.000067
0.12	0.014131	0.014226	-0.000095
0.14	0.020255	0.020382	-0.000127
0.16	0.027549	0.027706	-0.000157
0.18	0.035925	0.036108	-0.000183
0.20	0.045267	0.045467	-0.000200
0.22	0.055445	0.055651	-0.000206
0.24	0.066332	0.066527	-0.000195
0.26	0.077803	0.077968	-0.000165
0.28	0.089751	0.089863	-0.000112
0.30	0.102079	0.102112	-0.000033
0.32	0.114711	0.114635	0.000076
0.34	0.127581	0.127364	0.000271
0.36	0.140641	0.140249	0.000392
0.38	0.153852	0.153246	0.000606
0.40	0.167186	0.166326	0.000860
0.42	0.180621	0.179466	0.001155
0.44	0.194144	0.192647	0.001497
0.46	0.207743	0.205857	0.001886
0.48	0.221412	0.219087	0.002325
0.50	0.235148	0.232332	0.002816

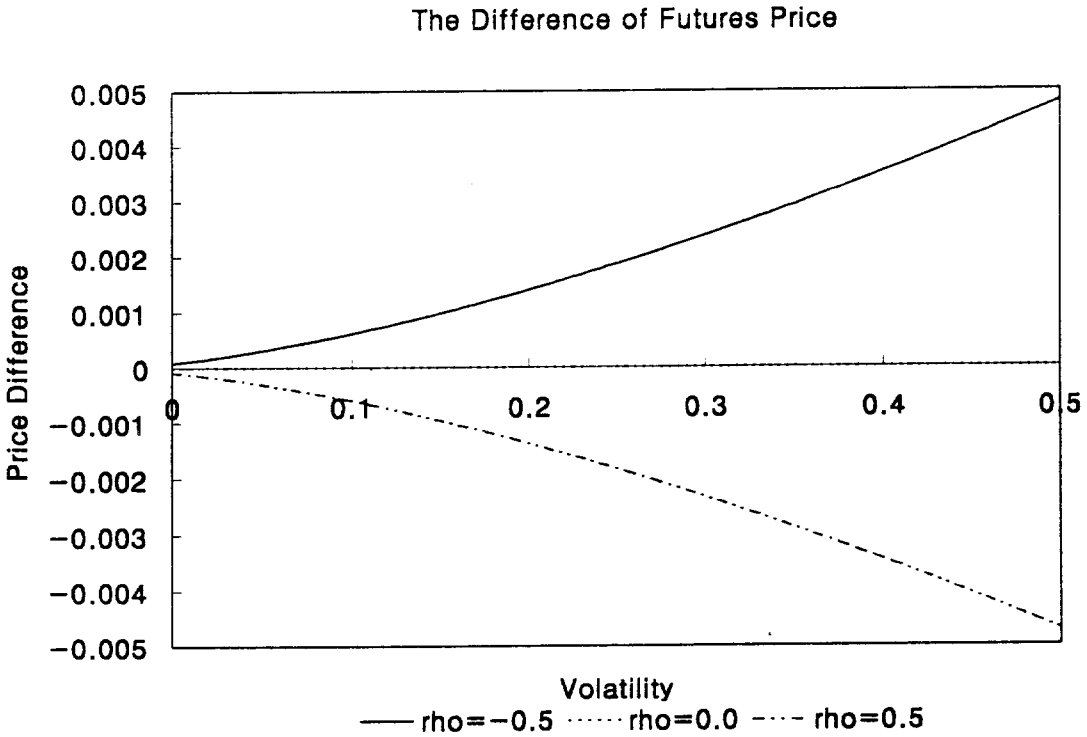

Figure 1: Effect of Varying ρ When $\alpha = 0.6, \kappa = 4, \sigma = 0.4, T = 0.1$.

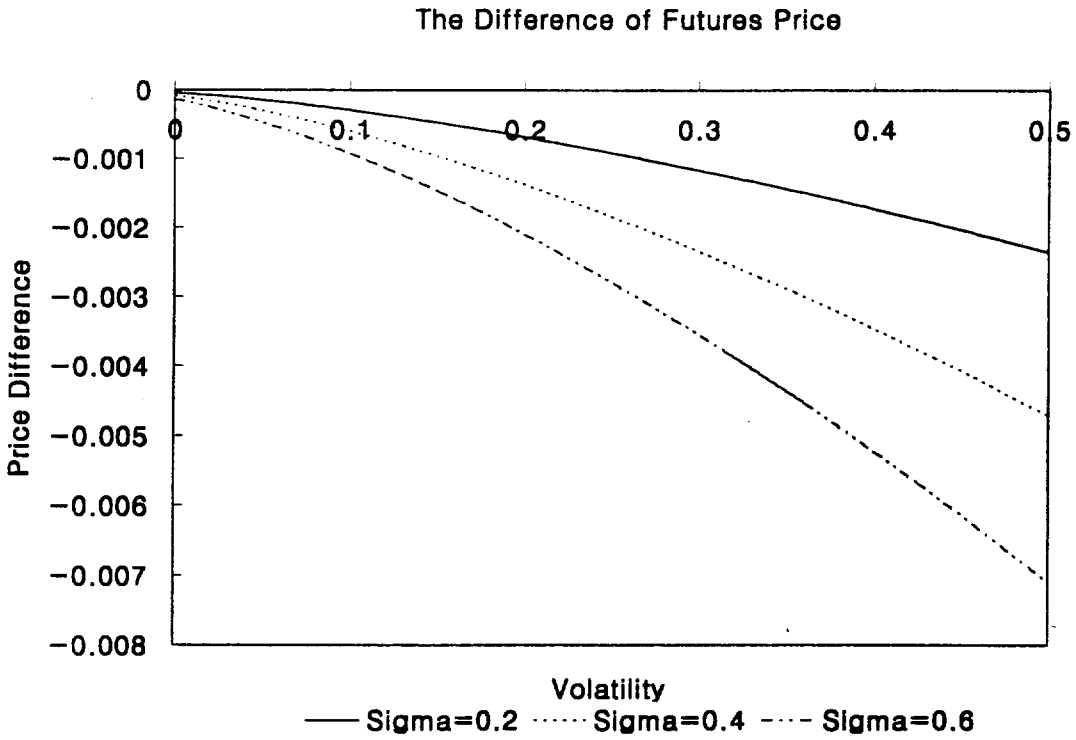

Figure 2: Effect of Varying σ When $\alpha = 0.6, \kappa = 4, T = 0.1, \rho = 0.5$.

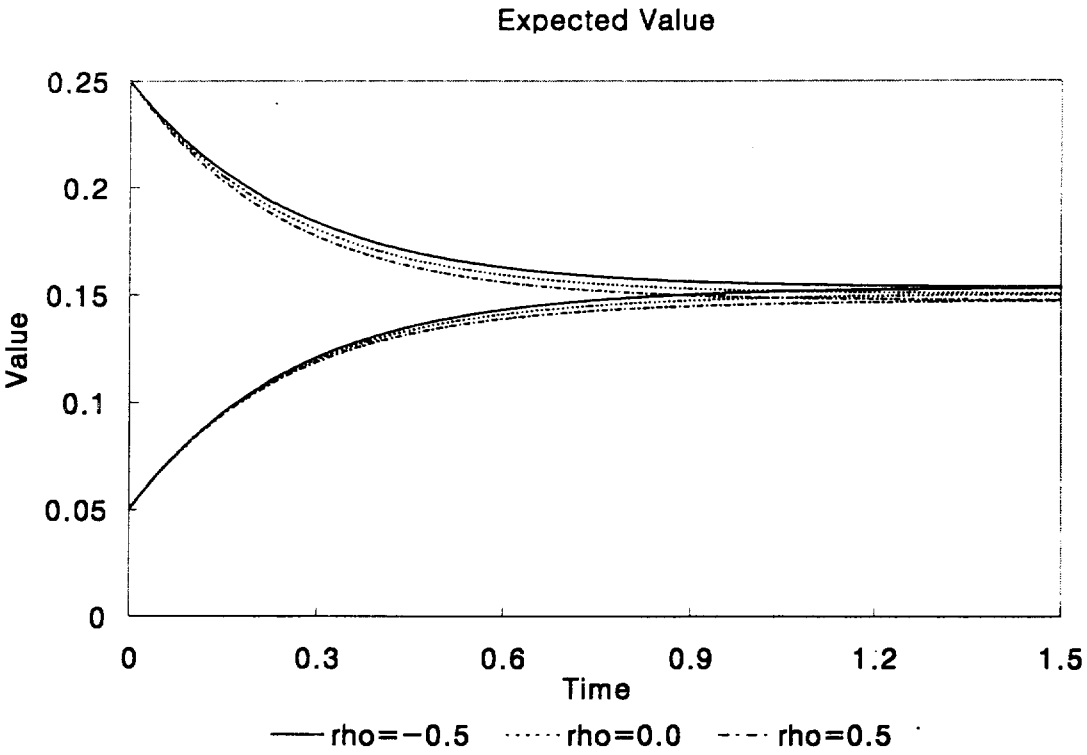

Figure 3: The Expected Value of V_T When $\alpha = 0.6, \kappa = 4, \sigma = 0.4$.

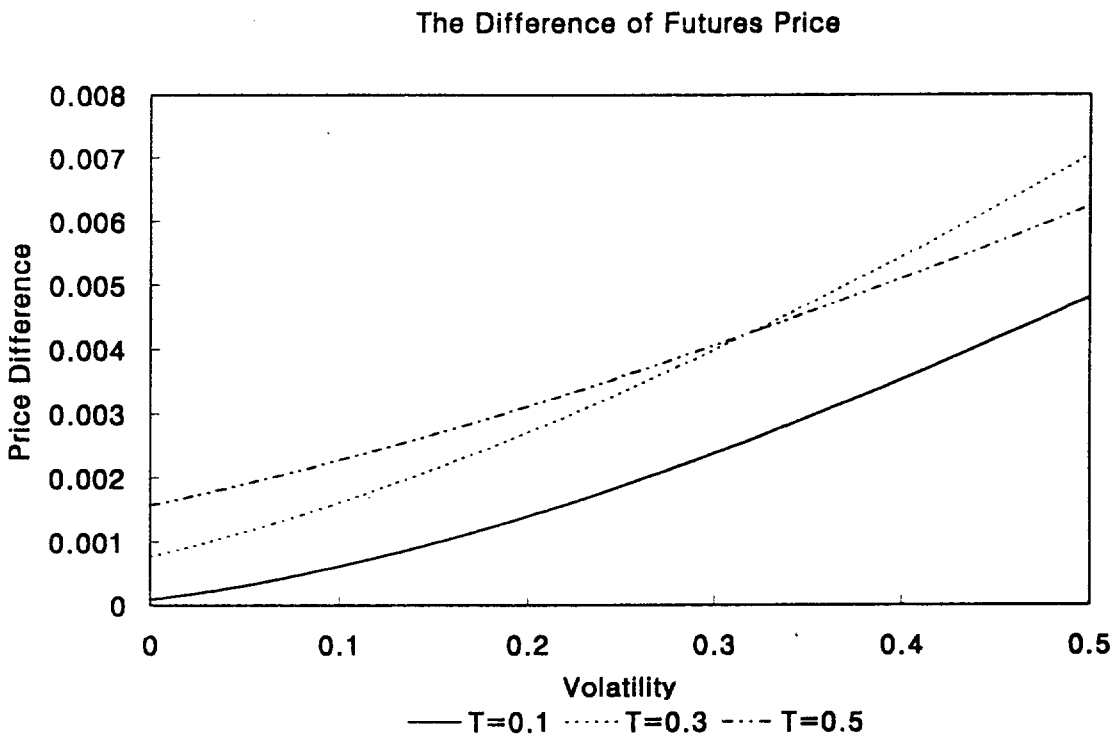

Figure 4: Effect of Varying T When $\alpha = 0.6$, $\kappa = 4$, $\sigma = 0.4$, $\rho = -0.5$.

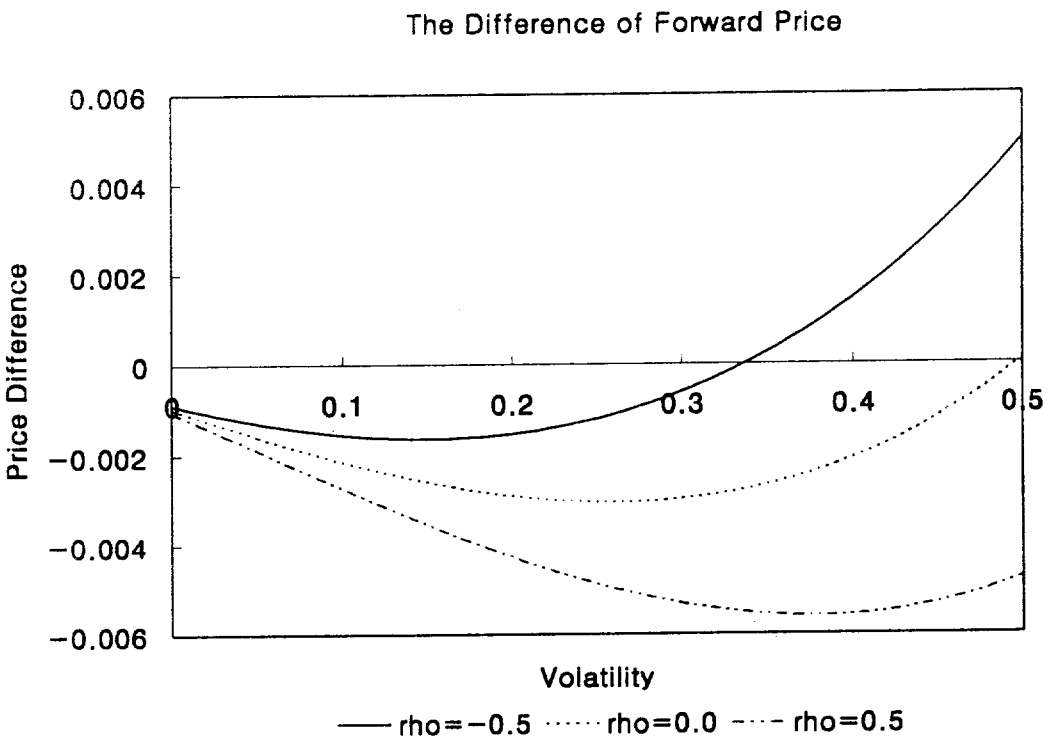

Figure 5: Effect of Varying ρ When $\alpha = 0.6$, $\kappa = 4$, $\sigma = 0.4$, $\mu = 0.2$, $T = 0.1$.

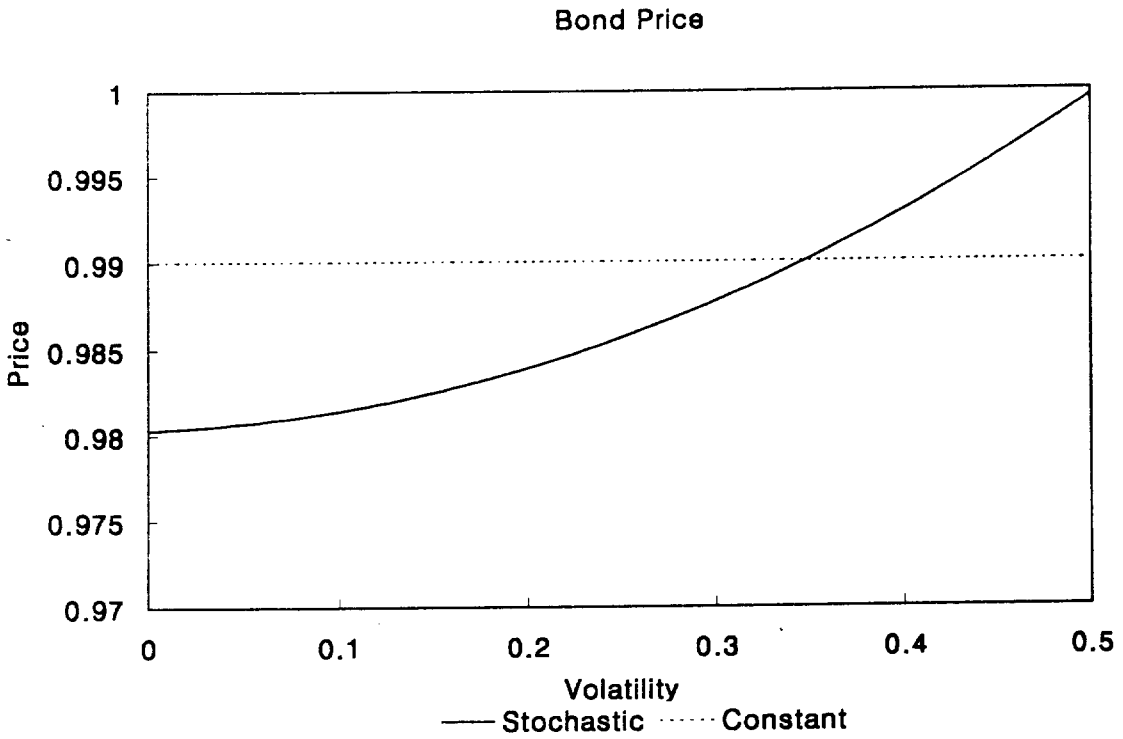

Figure 6: Bond Price When $\alpha = 0.6$, $\kappa = 4, \sigma = 0.4$, $\mu = 0.2$, $T = 0.1$, $\rho = 0$.

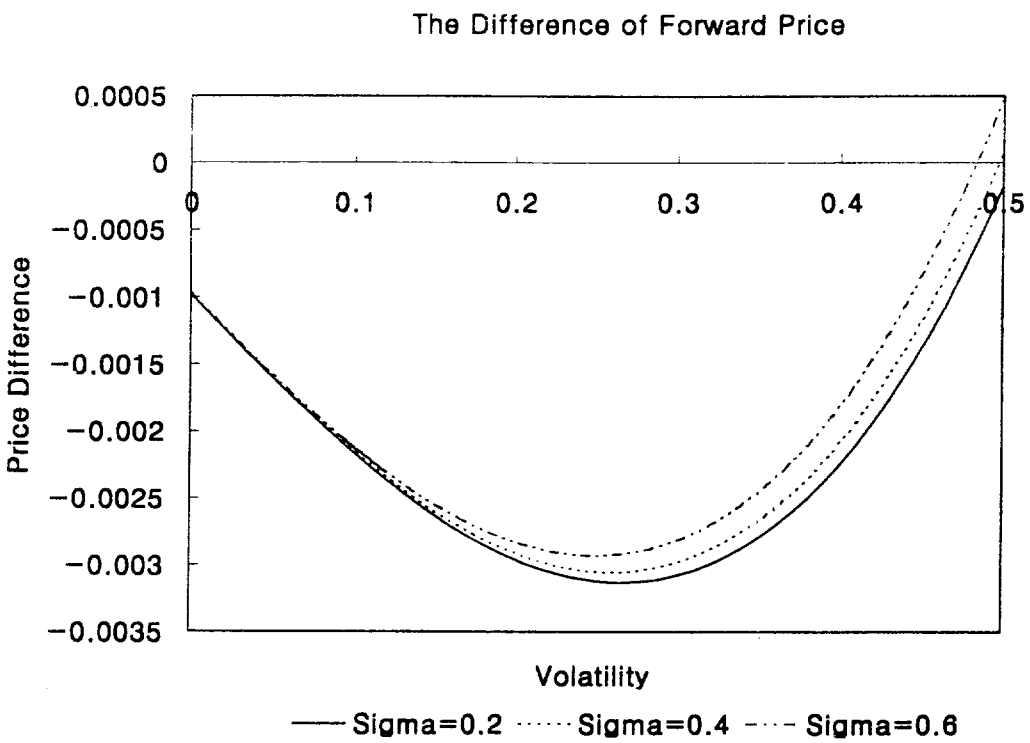

Figure 7: Effect of Varying σ When $\alpha = 0.6$, $\kappa = 4$, $\rho = 0$, $\mu = 0.2$, $T = 0.1$.

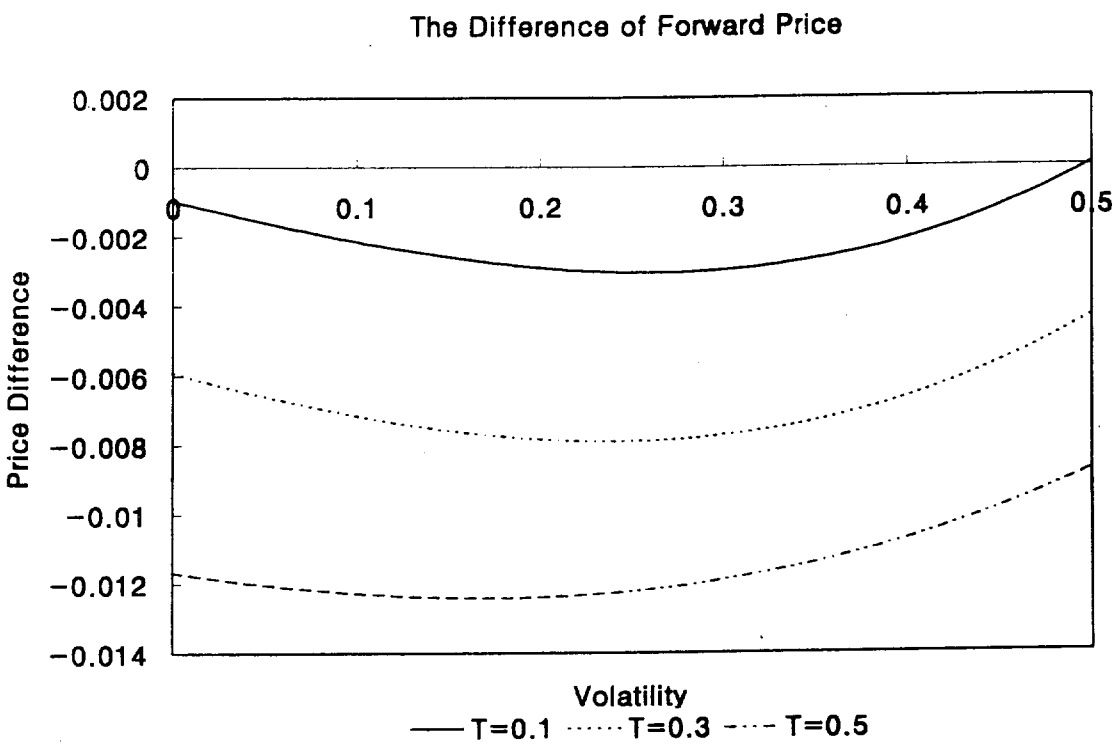

Figure 8: Effect of Varying T When $\alpha = 0.6$, $\kappa = 4, \sigma = 0.4$, $\mu = 0.2$, $\rho = 0$.

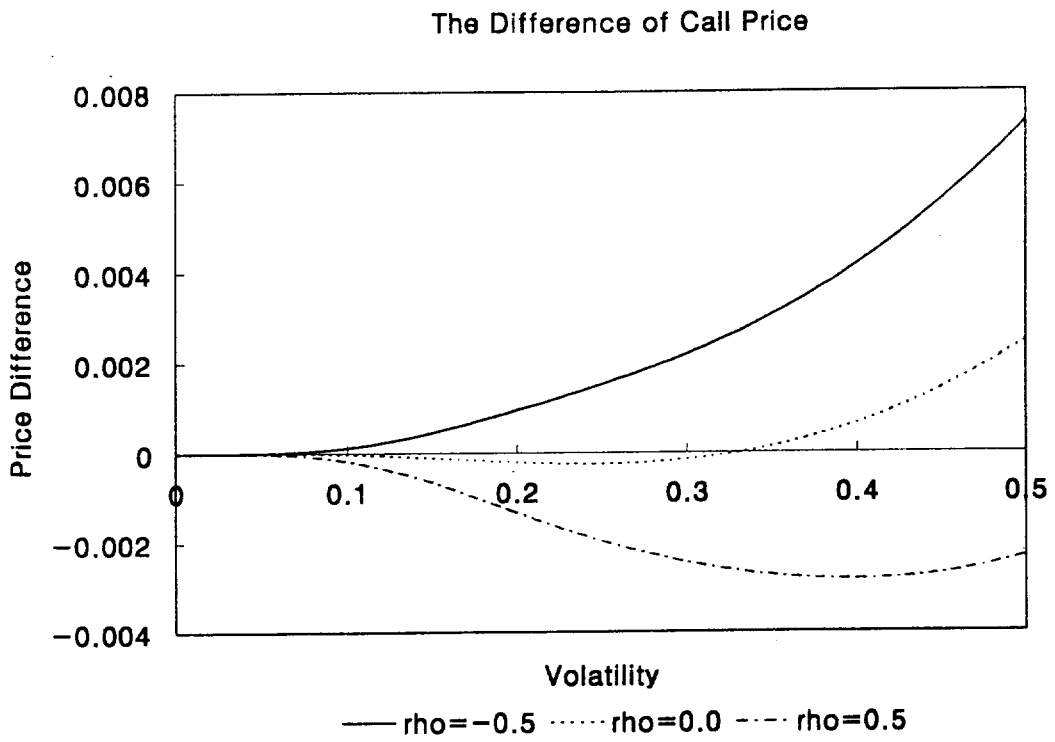

Figure 9: Effect of Varying ρ When $\alpha = 0.6$, $\kappa = 4$, $\sigma = 0.4$, $\mu = 0.2$, $T = 0.1$, $K = 0.15$.

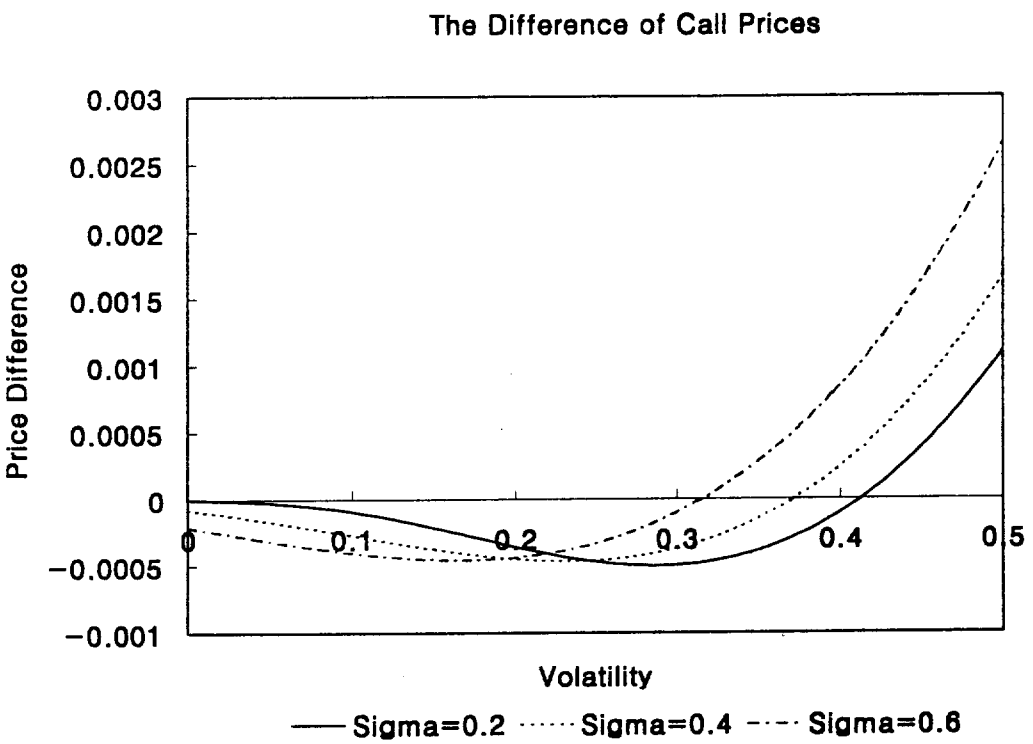

Figure 10: Effect of Varying σ When $\alpha = 0.6$, $\kappa = 4$, $\mu = 0.2$, $T = 0.3$, $K = 0.15$, $\rho = 0$.

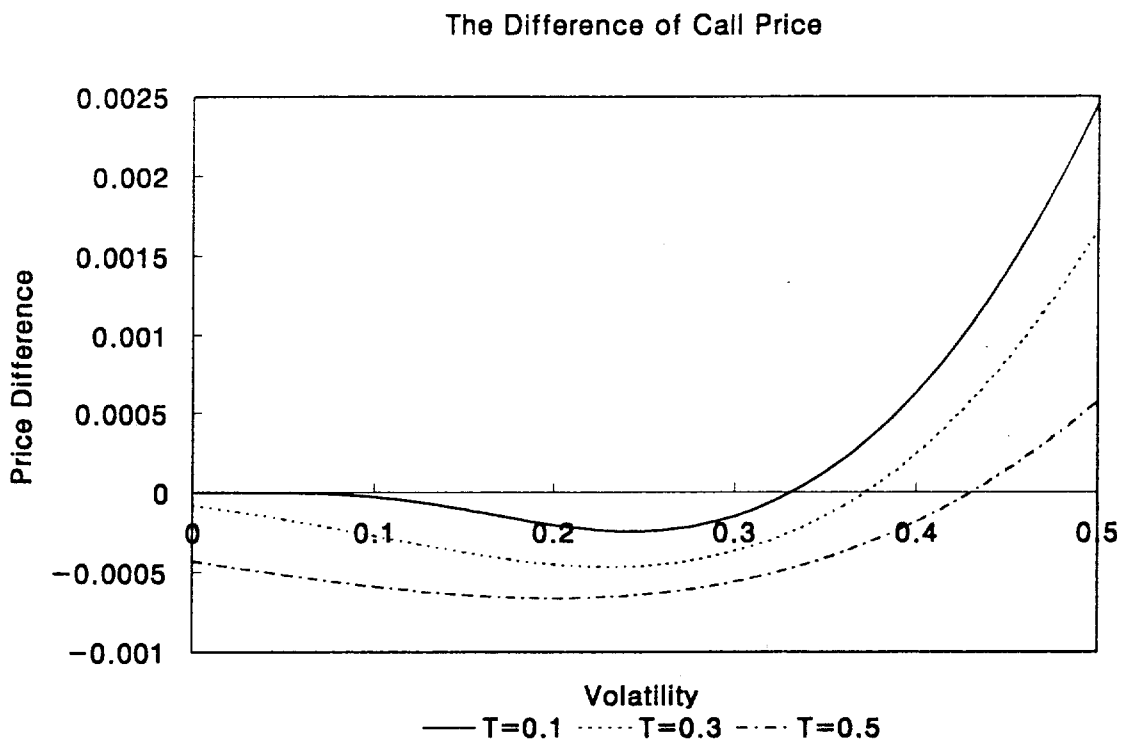

Figure 11: Effect of Varying T When $\alpha = 0.6$, $\kappa = 4$, $\mu = 0.2$, $\sigma = 0.4$, $K = 0.15$, $\rho = 0$.

제9분과 파생상품Ⅲ (4 : 30 – 6 : 30)

좌 장 : 강종만(한국증권연구원)

발 표 자	토 론 자	논 문 제 목
곽노성(동국대) 임준환(금융연구원)	변종국 (영남대)	파생금융상품의 활용과 통화정책
오규택(중앙대) 신성환(홍익대)	강장구 (서울시립대)	다이아몬드펀드의 파생상품 거래손실 사례분석
강태훈(계명대)	김명직 (한양대)	An Asymmetric EGARCH Option Pricing Model
김진호(이화여대)	홍정훈 (국민대)	ARFIMA with GARCH-M 모형을 통한 주가변동 위험의 추정