ADVERTISING COST INTERACTIONS AND THE OPTIMALITY OF PULSING*

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Whether pulsing, other than chattering, can be optimal is an important concern to both advertising practitioners and marketing scientists. In this paper, we explicitly incorporate various types of costs to a one-state advertising model to analyze the effect of these costs on the optimal advertising policy. We prove that the interaction of fixed and pulsing costs does make pulsing optimal under a reasonable condition. This result not only identifies an important factor that leads to the optimality of pulsing, but also generalizes the finding obtained by Sasieni (1971).

(ADVERTISING; PULSING; ADVERTISING COSTS)

1. Introduction

Whether it is best to adopt the pulsing policy,1 alternating between zero and high levels of spending with finite frequency, or the even policy, scheduling the exposures evenly, has been a fundamental research question to marketing scholars. Pulsing is a widely adopted advertising policy (Little 1979) which, many believe, may be better than other policies in practice. In a number of empirical research, the results consistent with this belief have been reported. As shown in Table 1, the pulsing policy could be superior to the blitz (Zielske 1959, J. L. Simon 1979, Mahajan and Muller 1986a), concentrating all the firm’s efforts in some initial periods, and to both the blitz and the even policy (Strong 1977, Katz 1980). Also shown is that the pulsing/maintenance, alternating between two nonzero levels of advertising, could be better than the even policy (Ackoff and Emshoff 1975, Rao and Miller 1975).

However, theoretical modeling studies exploring reasons that may lead to the superiority of pulsing have not been a great success. The models analyzed in the literature tended to prescribe either the even or the chattering policy (Sasieni 1971, 1989; Sethi 1977; Mahajan and Muller 1986a). The chattering policy, alternating between zero and high levels of advertising infinitely during the finite planning horizon, may be interpreted to imply in practice the faster the switching the better; but the chattering itself cannot be implemented. Even more dissatisfying is that the closest policy that can be implemented is not discernible from the even policy, adding a serious confusion to practitioners.

Observing the current state of the art of theoretical modeling studies, Little (1986) questions “are there any response models for which pulsing (other than chattering) would be optimal?” In fact, studies of optimal control (Hartl 1987, Kamien and Schwartz 1981) suggest that pulsing cannot be optimal for the class of models with one state variable which includes advertising models proposed by Vidale and Wolfe (1957) and Nerlove and Arrow (1962). Therefore, to address the issue, extended advertising models not included in the class need to be investigated. Recently, Luhmer et al. (1988) and Feinberg...
TABLE I
Findings of Major Empirical Studies on Pulsing

<table>
<thead>
<tr>
<th>Study</th>
<th>Investigated Policies</th>
<th>Effect Measure</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zielske (1959)</td>
<td>Blitz: 1 exposure per week with 13 repetitions.</td>
<td>Peak recall</td>
<td>Blitz is better</td>
</tr>
<tr>
<td></td>
<td>Pulsing or Even: 1 exposure every 4 week with 13 repetitions.</td>
<td>Average recall over 52 weeks</td>
<td>Pulsing or Even is better than Blitz(^b)</td>
</tr>
<tr>
<td>J. Simon (1979)</td>
<td>Re-analysis of Zielske's data.</td>
<td>Recall-weeks</td>
<td>Pulsing or Even is better than Blitz.</td>
</tr>
<tr>
<td>Mahajan &amp; Muller</td>
<td>Re-analysis of Zielske's Blitz data using their awareness model.</td>
<td>Total awareness</td>
<td>Chattering is the best.</td>
</tr>
<tr>
<td>(1986a)</td>
<td>Compared Blitz, Even, Pulsing, and Chattering.</td>
<td></td>
<td>Pulsing is better than Blitz.</td>
</tr>
<tr>
<td>Strong (1977)</td>
<td>Optimal timing of exposures based on Zielske's and some additional data.</td>
<td>Average annual recall</td>
<td>Grouping ads in a few flights, i.e., pulsing, is optimal.</td>
</tr>
<tr>
<td>Katz (1980)</td>
<td>Compared different patterns of flight schedules based on Strong's data.</td>
<td>Average awareness</td>
<td>Flights should be gradually spaced out.(^c)</td>
</tr>
<tr>
<td>Ackoff &amp; Emshoff</td>
<td>Compared Even, Blitz, and Pulsing/maintenance policy.</td>
<td>Increase in sales</td>
<td>Pulsing/maintenance and Blitz are better than Even. Media pulsing is also effective.</td>
</tr>
<tr>
<td>(1975)</td>
<td></td>
<td></td>
<td>Pulsing/maintenance is better than Even.</td>
</tr>
<tr>
<td>Rao &amp; Miller</td>
<td>Compared Even and Pulsing/maintenance policy.</td>
<td>Increase in sales, etc.</td>
<td></td>
</tr>
<tr>
<td>(1975)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) According to H. Simon (1982), further evidence supporting the superiority of pulsing has been provided by Haley (1977) and Sethi (1971).

\(^b\) Zielske and Henry (1980) compared the blitz, pulsing and three intermediate policies using television advertising data. Virtually the same conclusion has been drawn.

\(^c\) It implies the optimality of pulsing. Katz calls this a sliding schedule. On the other hand, according to Katz, O’Herlihy (1976) cited evidence that flight schedules were less efficient than the maximally spread even policy.

(1988) analyzed models with two or more state variables incorporating a filter that exponentially smooths out the advertising input to show the possible optimality of pulsing.

In this paper, we focus on the effect of various advertising costs, typically neglected in the analysis of optimal policies, under the premise that one-state advertising models reflect the reality adequately. Mahajan and Muller (1986b) commented that pulsing may be optimal with some kind of transaction cost in each pulse. However, they have not rigorously analyzed this issue any further. We analytically show that indeed the interaction of fixed and pulsing advertising costs can make pulsing, other than chattering, optimal.

### 2. Literature

Previous literature has shown that chattering can be optimal. Two factors are identified that lead to the optimality of chattering. They are *asymmetry* and *convexity* in dynamic response functions. By asymmetry, we mean that there exists a certain referent level of advertising such that over the level the advertising response is different from that below. Ackoff and Emshoff (1975) empirically observed a response such that a reduction in advertising increased sales, as did an increase. Pulsing is better than the even policy, in such a case, and chattering will be the best. Haley (1978) found that the response to the decrease in advertising was relatively slow and gradual whereas the response to the increase
was relatively abrupt and immediate. Haley's observation has been incorporated in ADPULS model by H. Simon (1982). Under the model, the firm gains additional revenue without incurring more expenditures by pulsing rather than adopting the even policy. The gain grows monotonically by increasing the number of alternations so that it is easily conjectured that the chattering will be optimal under the continuous version of ADPULS model. Mesak (1985) incorporated asymmetric advertising responses to positive and to zero advertising in his model by assuming different sales decay rates. If the asymmetry in the model is such that the firm gains by alternating between a positive and zero levels of advertising, the gain will monotonically grow as the number of alternations increases so that chattering will be optimal.

Another factor leading to the optimality of chattering is the convexity in advertising response functions. If there is convexity like an S-shaped model, there exists an interval of advertising levels in which pulsing is superior to the even policy (H. Simon 1982, Rao 1970). Furthermore, in this interval, chattering has been shown to be even better than pulsing (Sasieni 1971, 1989; Sethi 1977).

Studies in optimal control suggest that pulsing, unlike chattering, cannot be optimal at least for continuous time autonomous control models with one state variable (Sasieni 1971; Kamien and Schwartz 1981, p. 159; Hartle 1987). We introduce the monotonicity theorem (Hartl 1987) claiming this fact.

**MONOTONICITY THEOREM.** Let state \( x: [0, \infty) \rightarrow \mathbb{R} \) and control \( u: [0, \infty) \rightarrow \mathbb{R}^n \). Assume that \( S: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \), \( g: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \), and \( f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) are continuous in \( u \) and continuously differentiable in \( x \). Consider the following problem (HP).

\[
\begin{align*}
\max_u & \int_0^\infty e^{-rt} S(u, x) dt, \\
\text{s.t.} & \quad dx/dt = g(u, x), \\
f(u, x) & \geq 0, \quad x(0) = x_0.
\end{align*}
\]

If \( u^*(t), x^*(t) \) are the optimal solution of the autonomous one state variable problem (HP), and if \( x^*(t) \) is unique for all \( t \geq 0 \), then \( x^*(t) \) is monotonic for all \( t \geq 0 \).

We interpret the model as follows: control variable \( u(t) \) represents the effective advertising levels such as pages in printed advertisements, advertising frequency, or gross rating points; state variable \( x(t) \) represents the intermediate advertising effect such as the level of awareness (Mahajan and Muller 1986a), the number of people who adopted the product (Horsky and L. S. Simon 1983), or the stock of goodwill (Nerlove and Arrow 1962, Horsky 1977); and \( S(u, x) \) represents an ultimate advertising effect such as sales or profit amount. The function \( g(u, x) \) is the dynamic response function which represents how the change rate of \( x(t) \) is affected by \( u(t) \) and \( x(t) \). The function \( f(u, x) \) represents a certain managerial or physical constraint. For example, the advertising levels cannot be smaller than zero and will generally have an upper limit due to the budget constraint. As usual, \( r \) and \( t \) represent the discount rate and time respectively.

According to the theorem, the optimal path of one-state advertising models is monotone, under very general conditions (Luhmer et al. 1988; Feichtinger and Sorger 1988; Feinberg 1988, p. 39). Most continuous advertising models, including those of Vidale and Wolfe (1957) and Nerlove and Arrow (1962) are in this class. For these models, the optimal path \( x^*(t) \) must be monotonic. Since the change in \( u(t) \) affects \( x^*(t) \) through the equation \( dx/dt = g(u, x) \), \( x^*(t) \) being monotonic implies that the optimal policy \( u^*(t) \) is either even or chattering, ruling out the possibility of pulsing.

If \( g(u, x) \) is concave in \( u \), the optimal is the even policy. When there is convexity in \( g(u, x) \) with respect to \( u \), the \( g(u, x) \) can be replaced with its convex hull by allowing the chattering control (Sasieni 1971). These results by Hartl (1987) and Sasieni (1971)
show that the models in the class of (HP) prescribe only chattering or the even policy as optimal. To study the possible optimality of pulsing, it is therefore necessary to investigate response models not in this class. There may be two approaches in this direction. One is to modify the advertising response models assuming that they do not reflect the reality adequately. The other is to consider exogenous costs that affect the optimal policy based on one-state advertising models assuming that they are adequate.

Recently, adopting the former approach, there appeared a few studies analyzing models with two or more state variables. Näslund (1979) studied a two-state model reflecting the Nicosia's consumer behavior model. However, Muller (1983) notes that the conditions necessary for pulsing to be optimal are the ones rarely met in reality.

Luhmer et al. (1988) investigated a continuous variation of ADPULS model which has three state variables, sales, advertising goodwill, and the adaptation level. Unlike ADPULS, in which the adaptation level is defined as the level of advertising spending at the previous period, it is defined as the long run exponentially weighted average of advertising goodwill. As noted, the continuous version of ADPULS, which is an asymmetric model, prescribes the chattering policy as optimum. When a filter which exponentially smooths out the effect of chattering is explicitly incorporated through the model of goodwill, pulsing is shown to have a chance of being optimal. Feinberg (1988) also shows that the same result can be obtained by incorporating the filter to S-shaped response functions. The studies by Luhmer et al. (1988) and Feinberg (1988) show that pulsing can dominate chattering, and the even policy, when inertia for the motion of the state variable is introduced by filtering.

The other direction to study the possible optimality of pulsing is to bring in exogenous costs to the models included in (HP). It is the approach we take in this paper. We study if pulsing can be optimal when there is the interaction between fixed and pulsing advertising costs, based on an unmodified one-state advertising model included in (HP). Technically speaking, it may be seen as an approach that extends the one-state advertising models to have another state variable of which value is either 0 or 1 (Sorger 1986).²

3. Model

Advertising costs, like most other costs, have a fixed part charged to advertisers whenever advertising is done, and the variable part which depends on the level of advertising, in general. **Fixed advertising costs**, denoted by \( F \), are those independent of the specific advertising level \( u \) as far as it is nonzero. The presence of fixed costs implies that a very low level of advertising may not be economical. The **variable costs** of advertising, denoted by \( C(u) \), varies directly with the level of advertising \( u \).

**Pulsing costs**, denoted by \( P \), may be opportunity costs incurred when the firm starts up advertising after stopping for some time. Unlike fixed costs which are continuously incurred whenever advertising is done, the pulsing costs are incurred discretely only at the time when advertising is done after being stopped for some time. Haley (1978) notes that the pulsing costs ("cut-out costs" in his terminology) can be exceedingly high if the advertisers are using mostly network advertising with spots to bolster areas of weak coverage, a frequent type of media plan. In such situations the opportunity costs may incur, for example, because the firm cannot take advantage of discount schedules or use one of the most effective vehicles. Pulsing costs may also include additional production and inventory costs incurred by the affected sales fluctuations due to pulsing (H. Simon 1982). Both "start-up costs" in the investment theory (Davidson and Harris 1981) and

² Since we can create a 0-1 state variable which indicates whether pulsing costs are incurred or not, the extended model can be viewed as having two state variables. However, under the approach, we do not modify the original advertising model.
"re-entry costs" in the renewable resource theory (Lewis and Schmalensee 1982) which incur when the level of control is changed from zero to some positive level play the same role as pulsing costs in our context. Among the three types of costs, pulsing costs are of the stock dimension variety and the others are of the flow dimension.

The presence of the fixed and pulsing costs serves to remove the convexity that characterizes the cost functions normally used in the past studies. Our model explicitly incorporates the nonconvexity in the cost structure. Our profit maximizing advertiser is supposed to be faced with the following problem (OP).

\[
\text{Max } u \int_0^{\infty} e^{-\tau} S(u, x)\,dt - \sum_{i \in I} e^{-\tau_i} P
\]

s.t. \quad dx/dt = g(u, x) = u(N - x) - qx,

\[
S(u, x) = B(x) - W(u),
\]

\[
W(u) = F_{\bar{u}}(u) + C(u),
\]

\[
0 \leq u \leq \bar{u}, \quad x \geq 0, \quad x(0) = x_0,
\]

where

- \(x(t)\) is the number of people who are aware of the product,
- \(u(t)\) is the effective level of advertising,
- \(B(x)\) which is continuously differentiable in \(x\) is the gross profit function net of all costs except advertising,
- \(W(u)\) is the sum of fixed and variable advertising costs,
- \(C(u)\) is the variable advertising costs with \(C(0) = 0,\)
- \(\delta(u)\) is 0 if \(u = 0\), and 1 if \(u > 0,\)
- \(F\) is the amount of fixed advertising costs,
- \(P\) is the amount of pulsing costs,
- \(N\) is the number of people in the market,
- \(q\) is the decay rate,
- \(r\) is the discount rate,
- \(I\) is the set of switch time \(t\) when transition from zero to a positive level of advertising takes place, and
- \(\bar{u}\) is the maximum advertising possible for the firm.

The parameters \(N, \bar{u}, q\) and \(r\) are strictly positive numbers and \(F\) and \(P\) are constants which are either zero or positive. The response function \(g(u, x)\) specified above is essentially the model suggested by Vidale and Wolfe (1957).3

A problem similar to (OP) has been studied in the context of the optimal use of renewable resources (Lewis and Schmalensee 1979, 1982). The main concern there is in determining the optimal harvesting policy of renewable resources when nonconvexities are present in benefit, cost and production functions. While the harvesting problem shares similarities in the objective function with (OP), its response function representing the use and renewal of resources as well as the approach toward deducing propositions are different from ours. Also note that, in (OP), it is not optimal to advertise at all if the fixed costs or pulsing costs are very large relative to the revenue the firm gets in response to advertising. Such trivial cases are excluded in the following analysis.

To explore the possible optimality of pulsing, we first analyze the necessary conditions. The following lemma, proved in the Appendix, provides a crucial fact leading to the necessary conditions that pulsing is the optimal solution of (OP).

3 Using simpler specifications of \(g(u, x)\), such as the model suggested by Nerlove and Arrow (1962), we get the same result with simpler derivations.
Lemma 1. In (OP), whether \( P = 0 \) or \( P > 0 \), if the cost function \( W(u) \) is convex in \( u \), then the optimal advertising starts up at most once.

Lemma 1 implies that if pulsing is optimal in (OP), there exists nonconvexity in the cost function \( W(u) \). It holds not only for \( P = 0 \) but also for \( P > 0 \). Because previous studies (Sasieni 1971, Hartl 1987) made it clear that only the even or the chattering policy can be optimal when \( P = 0 \), we focus on the case when \( P \) is strictly positive.

The nonconvexity in \( W(u) \) can occur either through the nonconvexity in the variable cost function, i.e., \( F = 0 \) and \( C(u) \) is nonconvex, or through the existence of fixed costs \( (F > 0) \); but these are the necessary, not sufficient conditions for pulsing to be optimal. In the former case, the condition is clearly not sufficient. The nonconvexity in the cost function plays the same role as the convexity in the advertising response functions so that, in the absence of pulsing costs, chattering can be optimal. Since there exist pulsing costs in this problem, the advertiser can use the positive chattering, switching infinitely fast between two nonzero advertising levels, with the low level very close to zero. This will give almost the same effect as that of chattering without stopping advertising. Thus, the advertiser can avoid incurring the pulsing costs by adopting the positive chattering policy.

In case \( F \) is positive, there are two possibilities. One is that, in order to avoid pulsing costs, the firm never stops advertising once it starts up advertising. In this case, in the long run, the nonconvexity which occurs at \( u = 0 \) does not affect the decision. Therefore, the even policy, in case \( C(u) \) is convex, or the positive chattering, in case \( C(u) \) is concave, is adopted. The other possibility is that pulsing or chattering is adopted once the advertising starts up. Note that the presence of fixed costs may make chattering optimal (Sasieni 1971, Mahajan and Muller 1986a). However, the presence of pulsing costs tends to continue advertising so that it may keep chattering from being optimal. Thus, depending on the relative ratio of the two types of costs, there is a possibility that pulsing could be optimal. We rigorously investigate this possibility.

In studying the sufficient condition, it is helpful to refer to the following result (Sorger 1986, Feichtinger and Sorger 1986): If there exists a unique optimal solution \((u^*, x^*)\) in (OP), then \( x^*(t) \) is either monotonic in the long run or periodic in the long run. If there exists \( \tau \geq 0 \) such that the state \( x(t) \) is monotonic for \( t > \tau \), the solution is called monotonic in the long run. If there is \( \tau \geq 0 \) and \( \theta > 0 \) such that, for all \( t \geq \tau \), \( x(t) = x(t + \theta) \) then it is called periodic in the long run (see Figure 1). Note that either the even or chattering policy may correspond to the solution that \( x(t) \) is monotonic in the long run. The solution which is periodic in the long run in state \( x(t) \) corresponds to the periodic pulsing policy. An advertising policy \( u(t) \) is periodic if \( u(t + \theta) = u(t) \) for some positive \( \theta \) and if \( \theta \) is the smallest number for which the definition holds, then it is said to be periodic with period \( \theta \) (Feinberg 1988 p. 14).

Sorger's result is very useful for two reasons in providing sufficiency for the optimality of pulsing. First, it claims that in (OP), if pulsing is optimal, it must be periodic. Second, it shows that once there exists a unique solution, the optimality of the periodic solution with respect to \( x(t) \) is established if we can eliminate the possibility that the solution is monotonic with respect to \( x(t) \) in the long run. Since the periodic solution in \( x(t) \) implies the periodic pulsing policy with respect to \( u(t) \), we can establish the optimality of pulsing once we establish the optimality of the periodic solution.

The sufficiency then is established by identifying the condition that the best periodic solution with respect to \( x(t) \) dominates the best monotonic solution. The best periodic solution corresponds to the best pulsing policy, alternating between the maximal possible

\[ \text{During a start-up period, the advertising policy may show a different pattern from that at the steady state. When we mention advertising policies, we normally refer to those at the steady state. In other words, unless noted otherwise, we are mentioning the policies in the long run.} \]
level $u$ and zero level of advertising. In the following analysis, our focus is on showing that at least for certain reasonable cases, the condition $F > 0$ and $P > 0$ is not only necessary but also sufficient.

Let us assume for the rest of the paper that the variable cost function $C(u)$ and the benefit function $B(x)$ are linear such that $C(u) = cu$, for all $u > 0$ and $B(x) = bx$, for all $x > 0$. Here, $c$ and $b$ are all strictly positive. In the literature, the benefit function has been usually assumed to be linear (Sethi 1973) or concave (Nerlove and Arrow 1962, Gould 1970). The linearity assumption of the benefit function here enables us to focus our attention on the effect of advertising cost interactions. The variable advertising cost, when considered, has been generally assumed to be linear (Nerlove and Arrow 1962, Sethi 1973) or convex (Gould 1970, Sethi 1977) with respect to the effective advertising level. This amounts to assuming that the effective advertising level, such as advertising pages, gross rating points, etc., is linear or concave with respect to advertising dollars.\(^5\)

\(^5\) Although linear or convex variable cost function is used almost without exception in past studies, whether the assumption can be empirically supported is still an open question.
In our case, a linear variable cost function is used. The major motivation is in maintaining mathematical simplicity. However, our main result holds regardless of the shape of the benefit or variable cost function.

Now we prove the sufficiency, comparing the best periodic with the best monotonic solution. The proof is in the Appendix. The following notations are used to present and prove the sufficiency. As shown in Figure 1, for the periodic case, the upper and lower limit of \( x^*(t) \) are denoted as \( x_h \) and \( x_l \), respectively. In this case, the period is assumed to be \( \theta \). Once the advertising starts up, it is continued for a period of \( a \) and halted for a period of \( (\theta - a) \). Let \( r_m \) and \( r_p \) be the end of the start-up period for the monotonic and periodic case, respectively. Define \( r \) as the time when the first start-up occurs, in periodic case, since \( \max \{ r_m, r_p \} \), and \( t_i \) as the time of the \( i \)th advertising start-up, counting from \( r \). The subscript \( s \) represents a singular solution \( (u_s, x_s) \).

We further define the functions \( D_1, D_2, D_3, D_4, R \) and the discrimination function \( D(F) \) and present the sufficiency as follows:

\[
D_1 = \int_{r+\alpha}^{r+\theta} Fe^{-rt} dt = (F/r)[e^{-(r+\theta)} - e^{-(r+\alpha)}],
\]

\[
D_2 = \int_{r}^{r+\alpha} \{ b[(x_l - M)e^{-(uN/M)} + M] - cu \} e^{-rt} dt, \quad \text{where} \quad M = \bar{u}N/(\bar{u} + q),
\]

\[
D_3 = \int_{r+\alpha}^{r+\theta} bx_0 e^{-(r+\alpha)t} dt,
\]

\[
D_4 = \int_{r}^{\infty} (bx_t - cu_t)e^{-nt} dt,
\]

\[
R = \sum_{i=1}^{\infty} e^{-nt_i} = e^{-rt}/[1 - e^{-\alpha}], \quad \text{where} \quad t_1 = r,
\]

\[
D(F) = R[D_1 + D_2 + D_3] - D_4.
\]

**Lemma 2 (Sufficiency). Consider the problem (OP). If the discrimination function \( D(F) \) satisfies the inequality \( D(F) > R \cdot P \), then the optimal advertising policy of (OP) is the periodic pulsing, in the long run.**

The sufficiency lemma implies that periodic pulsing can be optimal under a reasonable condition. The condition is made clear if we interpret the discrimination function \( D(F) \). In the discrimination function \( D(F) \), \( R \) is the operator to calculate the present value of the sum of periodically incurring profits. \( D_1 \) is the saving of fixed costs in each period when pulsing is adopted instead of the policy keeping \( x(t) \) monotonic. \( D_2 \) and \( D_3 \) are profits during an advertising and nonadvertising period respectively, except for fixed and pulsing costs. Thus, \( R[D_1 + D_2 + D_3] \) represents the present value of total benefit the firm gets by adopting the pulsing policy except for fixed and pulsing costs. \( D_4 \) is the present value of all the profits except for fixed costs when a policy is adopted which keeps \( x(t) \) monotonic.

Now the meaning of \( D(F) \) becomes clear. It is the net benefit of adopting pulsing instead of the policy keeping \( x(t) \) monotonic, except for pulsing costs. If the pulsing policy is optimal, it must be greater than the present value of total pulsing costs. Therefore, the amount of pulsing costs relative to that of fixed costs, i.e., the \( P/F \) ratio, plays an

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6 When the Hamiltonian is linear in control \( u \), the coefficient of \( u \) in the Hamiltonian may be equal to zero over some period of time. Since the control does not affect the Hamiltonian during the periods, the choice of \( u \) must be determined by manipulating other conditions. In these cases, the value of \( u \) and the corresponding \( x \), denoted as \( (u_s, x_s) \) here, is said to be singular (Kamien and Schwartz 1981, p. 193).
important role in determining the optimal advertising policy. If the $P/F$ ratio is sufficiently large, the policy keeping $x(t)$ monotonic is preferable. If the ratio is sufficiently small, the pulsing policy is optimal. Thus, Lemma 2 implies that, for given amount of fixed costs, there exists a critical amount of pulsing costs $P^*$ such that the optimal state trajectory $x^*(t)$ can be either monotonic or periodic in the long run.\footnote{In Lemma 2, the focus is on the steady-state. If we take the start-up period into consideration, the actual value of $P^*$ will become different. But, the result claiming the existence of $P^*$ does not change.}

When the actual amount of pulsing costs is lower than $P^*$, then $x^*(t)$ must be periodic, implying that the periodic pulsing policy is optimal. In this case, the advertiser should set $u(t) = 0$ or $\bar{u}$, depending on the initial state, during the start-up period until $x(t)$ enters the steady state. In the steady state, the periodic pulsing policy, switching between $u(t) = 0$ and $\bar{u}$ periodically, is adopted.

Aggregating the implications of Lemma 2 and the findings from past studies, we present the following theorem.\footnote{We are indebted to one of the anonymous referees for organizing this theorem.} This theorem can be seen as a generalization of the result obtained by Sasieni (1971).

**THEOREM.** Assume that nonconvexity exists in the cost function in the form of fixed advertising costs. Assume further that there exists pulsing costs $P$ associated with each pulse. Then, there exists $P^*$ such that

(i) if $P^* < P$ the even policy is optimal,\footnote{This is true for linear or convex variable costs. If the variable cost function is specified to be concave, the optimal policy in this case is positive chattering.}
(ii) if $0 < P < P^*$ the pulsing policy is optimal, and
(iii) if $P = 0$ the chattering policy is optimal.

While (iii) of the theorem was shown by Sasieni (1971, 1989) and Mahajan and Muller (1986a), (i) and (ii) are implied by Lemma 2. The theorem reveals that the interaction of fixed and pulsing costs can make the periodic pulsing policy optimal under a model included in the class of (HP).

4. Discussion

This paper shows that pulsing can be optimal under a reasonable condition. We explicitly incorporated fixed and pulsing costs to a one-state advertising model and analyzed their effect on the optimal advertising policy. Our main result is that the $P/F$ ratio, the relative amount of pulsing costs to that of fixed costs, is a key factor in determining the optimal policy. Especially when the $P/F$ ratio is sufficiently small, the pulsing policy is optimal.

This finding can be viewed as a generalization of the famous result proved by Sasieni (1971). Sasieni has analyzed the case when there are no pulsing costs. We have generalized the result by taking both pulsing and fixed costs into consideration and suggested conditions under which the even, pulsing, and chattering policies are optimal. Our result clearly implies that advertising managers should understand the importance of various types of advertising costs and directly take $P/F$ ratio into consideration in making policy decisions.

To investigate the possible optimality of pulsing, Hartl's monotonicity theorem implies that one-state advertising models must be modified to have two or more state variables. Technically, we have bypassed this issue by introducing a new 0-1 state variable, considering exogenous costs, without modifying the one-state advertising model itself. Because of the approach we have taken, additional implications can be obtained. First, the force driving toward chattering is the convexity introduced from the cost function $W(u)$.
Hence, even if the *sales* response to advertising\(^{10}\) of the original one-state model does not have convexity, pulsing can be still optimal depending on the \(P/F\) ratio. Second, our result implies that at least for one-state advertising models, the cost interaction should be one of the rare factors that lead to the optimality of pulsing. It seems to be a difficult task to find another factor that can lead to the optimality of pulsing for these models without extending them to have two or more state variables.

Since our problem has some common features with that analyzed in the context of optimal harvesting of renewable resources, we might infer some implications from results obtained in the studies. For example, Lewis and Schmalensee (1979, 1982) propose that the optimal period \(\theta\) of periodic pulsing should be longer as “re-entry” cost gets larger. It might be translated such that the time period of the optimal pulsing policy becomes longer as the amount of pulsing costs gets larger. Also, from their “abandonment strategy” we might infer that no advertising is optimal in case we have a huge amount of fixed costs. To confirm those, however, further analyses are necessary.

An important direction for future research is to develop a computational procedure of the optimal periodic pulsing. For the purpose, the problem (OP) may be reformulated as a periodic optimal control problem for identifying the optimal path \(x^*(t)\). We may modify (OP) such that the objective is to maximize average profits per period, and introduce an additional constraint representing the periodic boundary condition \(x(t) = x(2\theta)\). Then the numerical approach suggested by Evans et al. (1987) might enable us to solve this periodic control problem. Another interesting future direction is to study a problem in which the amount of pulsing costs varies depending on the advertising levels being alternated. The variability of pulsing costs might lead to the optimality of the pulsing/maintenance policy which is also one of widely employed policies in practice. Also valuable will be studies which investigate other advertising models with two or more state variables, following the approach adopted by Luhmer et al. (1988) and Feinberg (1988). Among the models with two or more state variables, extremely interesting will be the ones incorporating competition. In competitive situations, Wells and Chinsky (1965) suggest that messages are most effective when they are delivered in bursts, implying the possible optimality of pulsing.\(^{11}\)

\(^{10}\) We are not including the cost part when we use the term *sales* response to advertising. Note that it is the shape of the *steady-state sales* response to advertising (Little 1979, p. 638), not including advertising costs, on which there is controversy.

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### Appendix

1. **Proof of Lemma 1**

(1) Case 1 \((P = 0)\): Let \((u^*, x^*)\) be an optimal path when \(P = 0\). Then, the optimal path is a *most rapid approach path* (MRAP) (Sethi 1973, Kamien and Schwartz 1981, p. 199). It approaches to some stationary level as fast as possible, and then remains there. Let \(x^*_t > 0\) be the stationary level. Being MRAP, it is optimal not to start up at all if \(u(0) > 0\) and \(x(0) \leq x^*\), and to start up only once, for all \(t\), otherwise.

(2) Case 2 \((P > 0)\): Let \((u^{**}, x^{**})\) be the optimal path when \(P > 0\).

(i) \(u(0) > 0\) and \(x(0) \leq x^*_t\): Compared to the maximum profit attainable with no start-ups, an advertiser cannot make more profit with start-ups because with start-ups the advertiser must pay pulsing costs, and the path \((u, x)\) may deviate from \((u^*, x^*)\). In this specific case, \((u^{**}, x^{**})\) with no start-up is feasible so that the effect of pulsing costs virtually vanishes. Thus, \((u^{**}, x^{**}) = (u^*, x^*)\) for all \(t \geq 0\), implying that there is no start-up at all.

(ii) \(u(0) = 0\) \((x(0)\) may be bigger than, equal to, or smaller than \(x^*_t\)): Let \(t^{**}_1\) and \(t^{**}_1\) be the time that the first advertising start-up occurs for \(u^*\) and \(u^{**}\), respectively. The effect of pulsing costs being positive, if anything, is to delay the start-up (that occurs just once in case \(P = 0\)). Thus, \(t^{**}_1 \geq t^*_1\) and \(x^{**}(t^{**}_1) \leq x^*(t^*_1) = x^*_t\).
Define a new problem (OP') such that \( x(0) = x^* (t_1^* ) \) instead of \( x_0 \) in (OP), and let \( (\bar{u}^*, \bar{x}^*) \) by the optimal path of (OP') for the case \( P = 0 \). By definition, \( u^* (t_1^* ) > 0 \). From Case 1, we know that there is no advertising start-up for \( (\bar{u}^*, \bar{x}^*) \). In addition, from (i), \( (\bar{u}^*, \bar{x}^*) \) is optimal also for (OP') with \( P > 0 \). Therefore, for \( t \geq t_1^* \), \( (u^*, x^*) = (\bar{u}^*, \bar{x}^*) \), implying that there is no start-up after \( t_1^* \).

(iii) \( u(0) > 0 \) and \( x(0) > x^* \). If \( P = 0 \), from Case 1 we know that the maximum profit is attained by an MRAP. Suppose \( u^*(t) \) is at a level close to zero, until \( x(t) \) reaches \( x^* \), and then at a level that maintains \( x^* \) from then on. Then, \( u^*(t) \) which avoids a start-up is feasible. In addition, \( u^*(t) \) attain almost the same profit as \( u(t) \). Hence, it is optimal not to start up at all.

It follows from the analysis of Case 1 and 2 that whether \( P = 0 \) or \( P > 0 \), the optimal advertising has at most one start-up. Q.E.D.

2. Proof of Lemma 2

For the time interval not containing switching points, the optimal control must satisfy the necessary conditions of Pontryagin's maximum principle. Thus, there exists a continuous function \( \lambda(t) \) such that

\[
H = bx - [F_0(u) + cu] + \lambda(u(N - x) - qx],
\]

\[
dx/dt = u(N - x) - qx,
\]

\[
d\lambda/dt = \lambda x' - b + \lambda u x + \lambda q,
\]

\[
H_u = -c + \lambda(N - x) = 0,
\]

for all admissible controls. Here, \( H \) is the Hamiltonian and \( H_u \) is the first derivative of \( H \) with respect to \( u \). It is easy to show that the maximizing condition for the Hamiltonian yields \( u \in \{0, \bar{u}, u_i\} \) for any given \( t \), where \( u_i \) is a singular control. Obviously, \( u(t) = 0 \) or \( u(t) = \bar{u} \) for all \( t \geq 0 \) are not optimal.

We first investigate the monotonic solution such that once advertising brings \( x(t) \) to a specific level, it never stops, avoiding pulsing costs. Then nonconvexity due to fixed costs vanishes so that the optimal is the even policy; in the long run, \( u(t) = q_0 ((N - x_0) \) for \( t \geq \tau_m \) where \( \tau_m \) is the end of the start-up period for the monotonic solution, and \( x_i \) is the state path corresponding to the singular control \( u_i \).

Next, we turn to periodic solutions. Consider the pulsing policy such that

\[
u(t) = \begin{cases} 
\bar{u} & \text{if } t \in [t_p + n\theta, t_p + n\theta + a], \quad n = 0, 1, 2, \ldots, \\
0 & \text{if } t \in (t_p + n\theta + t_p + n\theta + \theta), \quad n = 0, 1, 2, \ldots, 
\end{cases}
\]

where \( t_p \) is the end of the start-up period for the periodic solution. During the advertising period, the state path is

\[x(t) = [x(t_p + n\theta) - M]e^{-\theta/(M^2)} + M,
\]

where \( M = \bar{u}N/(\bar{u} + q) \) as defined in the text. By definition, \( x_p \) and \( x_h \) are such that \( x_0 = x(t_p + n\theta) \) and \( x_0 = x(t_p + n\theta + a) \). The awareness level \( x(t) \) during the period without advertising is \( x(t) = x_0 e^{-\theta} \).

Let \( \tau \) be the time when the first start-up occurs since max \( \{\tau_m, \tau_p\} \) (see Figure 1). Then, the optimal discounted long-run profit starting from \( \tau \) in the monotonic case \( J_m \) is

\[J_m = \int_0^\infty (bx_i - cu_i - F)e^{-\theta} dt = D_4 - \int_0^\infty Fe^{-\theta} dt,
\]

where \( D_4 \) is as defined in the text.

Let \( [J(x_i, x_{i+1}, \theta_i) - P] \) be the optimal discounted profit during the \( i \)th advertising cycle from \( \tau \), starting at \( x_i \) and ending at \( x_{i+1} \). Then, the optimal long-run discounted profit starting from \( \tau \) in the periodic case \( J_p \) is

\[J_p = \sum_{i=0}^\infty e^{-\theta} [J(x_i, x_{i+1}, \theta_i) - P], \quad \text{where} \quad \tau_i = \tau.
\]

From the periodicity stated in Sorger's result on p. 10, we know that

\[J(x_i, x_{i+1}, \theta_i) = J(x_i, x_0, \theta_i)
\]

\[= \int_{\tau}^{\tau+\theta} \left[ b(x_i - M) e^{-\theta/(M^2)} + M \right] - F - c\bar{u} \] \[e^{-\theta} dt + \int_{\tau+\theta}^{\tau+2\theta} bx_0 e^{-\theta} dt.
\]

It follows from Sorger's result that \( J_m \) is less than \( J_p \) if and only if the periodic pulsing policy is optimal in the long run. Thus
\[ J_m - J_p = D_4 - \int_0^\infty Fe^{-n} dt - \sum_{i=1}^\infty e^{-n_i} \left[ (D_2 + D_3) - P - \int_0^\infty Fe^{-n} dt \right] \]

\[ = -R[D_1 + D_2 + D_3] + D_4 + R \cdot P < 0 \]

where \( R, D_1, D_2, \) and \( D_3 \) are as defined in the text. Therefore, we get the sufficient condition, \( D(F) > R \cdot P. \) Q.E.D.

References


