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SOME TWO-WEIGHT AND THREE-WEIGHT LINEAR CODES

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Abstract. Let $\mathbb{F}_q$ be the finite field with $q = p^m$ elements, where p is an odd prime and m is a positive integer. For a positive integer t, let $D \subset \mathbb{F}_q^t$ and let Tr_m be the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. We define a p-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \{\mathbf{c}(a_1, a_2, \ldots, a_t) : a_1, a_2, \ldots, a_t \in \mathbb{F}_{p^m}\},$$

where

$$\mathbf{c}(a_1, a_2, \ldots, a_t) = \left(\mathrm{Tr}_m(a_1x_1 + a_2x_2 + \cdots + a_tx_t)\right)_{(x_1,x_2,\ldots,x_t)\in D}.$$

In this paper, we will present the weight enumerators of the linear codes $\mathcal{C}_D$ in the following two cases:

1. $D = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t \setminus \{(0, 0, \ldots, 0)\} : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 0\}$;
2. $D = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 1\}$.

It is shown that $\mathcal{C}_D$ is a two-weight code if tm is even and three-weight code if tm is odd in both cases. The weight enumerators of $\mathcal{C}_D$ in the first case generalize the results in [17] and [18]. The complete weight enumerators of $\mathcal{C}_D$ are also investigated.

1. Introduction

Let $\mathbb{F}_p$ be the finite field with p elements and let n be a positive integer, where p is an odd prime. An $[n, k, d]$ linear code $\mathcal{C}$ over $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$ with minimum distance d.

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Let A_i be the number of codewords with the Hamming weight i in the code $\mathcal{C}$ of length n. The weight enumerator of $\mathcal{C}$ is defined by

$$1 + A_1x + A_2x^2 + \cdots + A_nx^n.$$

The sequence $(1, A_1, A_2, \ldots, A_n)$ is called the weight distribution of the code $\mathcal{C}$. We call $\mathcal{C}$ an e-weight code if $|\{1 \le i \le n : A_i \ne 0\}| = e$.

Suppose that the elements of $\mathbb{F}_p$ are $0, 1, \ldots, p-1$. The composition of a vector $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{F}_p^n$ is defined to be $\mathrm{comp}(\mathbf{v}) = (t_0, t_1, \ldots, t_{p-1})$, where each $t_i = t_i(\mathbf{v})$ is the number of components $v_j (0 \le j \le n-1)$ of $\mathbf{v}$ that are equal to i. Clearly, we have

$$\sum_{i=0}^{p-1} t_i = n.$$

Definition 1.1. [23, 24]. Let $\mathcal{C}$ be a code over $\mathbb{F}_p$ and let $A(t_0, t_1, \ldots, t_{p-1})$ be the number of codewords $\mathbf{c} \in \mathcal{C}$ with $\mathrm{comp}(\mathbf{c}) = (t_0, t_1, \ldots, t_{p-1})$. Then the complete weight enumerator of $\mathcal{C}$ is the polynomial

$$\begin{aligned} W_{\mathcal{C}}(z_0, z_1, \ldots, z_{p-1}) &= \sum_{\mathbf{c} \in \mathcal{C}} z_0^{t_0(\mathbf{c})} z_1^{t_1(\mathbf{c})} \cdots z_{p-1}^{t_{p-1}(\mathbf{c})} \\ &= \sum_{(t_0, t_1, \ldots, t_{p-1}) \in B_n} A(t_0, t_1, \ldots, t_{p-1}) z_0^{t_0} z_1^{t_1} \cdots z_{p-1}^{t_{p-1}}, \end{aligned}$$

where $B_n = \{(t_0, t_1, \ldots, t_{p-1}) : 0 \le t_i \le n, \sum_{i=0}^{p-1} t_i = n\}$.

Cyclic codes are a special class of linear codes and their weight enumerators have been extensively investigated [10, 14, 19, 22, 26, 27, 28, 32]. In addition, some two-weight and three-weight cyclic codes were presented. The weight enumerators of the linear codes with a few weights were also given [6, 7, 11, 12, 16, 17, 25, 33] by using exponential sums in some cases. There are two survey articles on two weight codes [4] and three-weight cyclic codes [9]. In addition, linear codes with a few nonzero weights are of special interest in association schemes [3], strongly regular graphs [4], and secret sharing schemes [5, 31]. The complete weight enumerators of cyclic codes or linear codes over finite fields were studied in [1, 15, 21, 29, 30], which can be applied to compute the deception probabilities of certain authentication codes constructed from linear codes [8, 13].

We begin to recall a class of two-weight and three-weight linear codes which was proposed by K. Ding and C. Ding [17]. Let $q = p^m$ for a positive integer m and let Tr_m denote the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. Let $\mathcal{D} = \{x \in \mathbb{F}_q^* : \mathrm{Tr}_m(x^2) = 0\}$. Then a linear code of length $n = |\mathcal{D}|$ over $\mathbb{F}_p$ can be defined by

$$\mathcal{C}_{\mathcal{D}} = \{\mathbf{c}(a) = \big(\mathrm{Tr}_m(ax)\big)_{x \in \mathcal{D}} : a \in \mathbb{F}_q\}.$$

It was proved that $\mathcal{C}_{\mathcal{D}}$ is a two-weight code if m is even and a three-weight code if m is odd.

Motivated by the results given in [17], for $D \subset \mathbb{F}_q^t$, we define a p-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \{\mathbf{c}(a_1, a_2, \ldots, a_t) : a_1, a_2, \ldots, a_t \in \mathbb{F}_{p^m}\}, \tag{1}$$

where

$$\mathbf{c}(a_1, a_2, \ldots, a_t) = \big(\mathrm{Tr}_m(a_1x_1 + a_2x_2 + \cdots + a_tx_t)\big)_{(x_1, x_2, \ldots, x_t) \in D}. \tag{2}$$

In this paper, we shall present the weight enumerators of the linear codes $\mathcal{C}_D$ in the following two cases:

1. $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t \setminus \{(0, 0, \dots, 0)\} : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 0\}$;
2. $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 1\}$.

In both cases we show that $\mathcal{C}_D$ is a two-weight code if tm is even and a three-weight code if tm is odd. It should be remarked that the weight enumerators of $\mathcal{C}_D$ were presented when $t = 1$ [17] and $t = 2$ [18]. Thus the weight enumerators of $\mathcal{C}_D$ in the first case generalize these results. If $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = c\}$ for $c \in \mathbb{F}_p^*$, we point out that the weight enumerators of $\mathcal{C}_D$ can be similarly presented because $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(c^{-1}x_1^2 + c^{-1}x_2^2 + \cdots + c^{-1}x_t^2) = 1\}$. Moreover, the complete weight enumerators of $\mathcal{C}_D$ are also investigated.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries which are very useful to get our results. In Section 3, we present the weight enumerators of the linear codes $\mathcal{C}_D$ in the first case. In Section 4, we determine the weight enumerators of the linear codes $\mathcal{C}_D$ in the second case. In Section 5, we investigate the complete weight enumerators of the linear codes $\mathcal{C}_D$ in both cases. In Section 6, we conclude this paper.

2. Preliminaries

Suppose that $q = p^m$ for an odd prime p and a positive integer m. For $a \in \mathbb{F}_q$, an additive character of the finite field $\mathbb{F}_q$ can be defined as follows:

$$\psi_a : \mathbb{F}_q \to \mathbb{C}^*, \psi_a(x) = \zeta_p^{\mathrm{Tr}_m(ax)},$$

where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is a primitive p-th root of unity and Tr_m denotes the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. It is clear that $\psi_0(x) = 1$ for all $x \in \mathbb{F}_q$. Then ψ_0 is called the trivial additive character of $\mathbb{F}_q$. If $a = 1$, we call $\psi := \psi_1$ the canonical additive character of $\mathbb{F}_q$. It is easy to see that $\psi_a(x) = \psi(ax)$ for all $a, x \in \mathbb{F}_q$. The orthogonal property of additive characters which can be found in [20] is given by

$$\sum_{x \in \mathbb{F}_q} \psi_a(x) = \begin{cases} q, & \text{if } a = 0; \\ 0, & \text{if } a \in \mathbb{F}_q^*. \end{cases}$$

Let $\lambda : \mathbb{F}_q^* \to \mathbb{C}^*$ be a multiplicative character of $\mathbb{F}_q^*$. Now we define the Gauss sum over $\mathbb{F}_q$ by

$$G(\lambda) = \sum_{x \in \mathbb{F}_q^*} \lambda(x)\psi(x).$$

If λ is the trivial character λ_0 which is defined by $\lambda_0(x) = 1$ for all $x \in \mathbb{F}_q^*$, then it is clear that $G(\lambda_0) = -1$. In general, the explicit determination of Gauss sums is a difficult problem. However, they can be explicitly evaluated in a few cases [2, 20]. For future use, we state the quadratic Gauss sums in the following lemma.

Lemma 2.1. [2, 20] *Suppose that $q = p^m$ and η is the quadratic multiplicative character of $\mathbb{F}_q^*$, where p is an odd prime and $m \geq 1$. Then*

$$G(\eta) = (-1)^{m-1}\sqrt{(p^*)^m} = \begin{cases} (-1)^{m-1}\sqrt{q}, & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{m-1}(\sqrt{-1})^m\sqrt{q}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $p^* = \left(\frac{-1}{p}\right)p = (-1)^{\frac{p-1}{2}}p$.

The following exponential sums will be employed later.

Lemma 2.2. [20] *If q is odd and $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$, then*

$$\sum_{x\in\mathbb{F}_q} \zeta_p^{\mathrm{Tr}_m(f(x))} = \zeta_p^{\mathrm{Tr}_m(a_0 - a_1^2(4a_2)^{-1})}\eta(a_2)G(\eta),$$

where η is the quadratic character of $\mathbb{F}_q^*$.

3. Weight enumerators in the first case

In this section, we present the weight enumerators of the linear codes $\mathcal{C}_D$ defined by (1) and (2), where $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t \setminus \{(0, 0, \dots, 0)\} : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 0\}$.

Let η_p be the quadratic character of $\mathbb{F}_p^*$ and let $G(\eta_p)$ denote the quadratic Gauss sum over $\mathbb{F}_p$. For $z \in \mathbb{F}_p^*$, it is easy to check that $\eta(z) = \eta_p(z)$ if m is odd and $\eta(z) = 1$ if m is even (also see [17]), where η is the quadratic character of $\mathbb{F}_q^*$.

We have the following lemma which is important to get our results.

Lemma 3.1. *Denote $n_c = |\{x_1, x_2, \dots, x_t \in \mathbb{F}_q : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = c\}|$ for each $c \in \mathbb{F}_p$. Then*

$$n_c = \begin{cases} p^{tm-1} & \text{if } c = 0 \text{ and } tm \text{ is odd,} \\ p^{tm-1} + \frac{1}{p}\eta_p(-c)G(\eta)^tG(\eta_p) & \text{if } c \neq 0 \text{ and } tm \text{ is odd,} \\ p^{tm-1} + \frac{p-1}{p}G(\eta)^t & \text{if } c = 0 \text{ and } tm \text{ is even,} \\ p^{tm-1} - \frac{1}{p}G(\eta)^t & \text{if } c \neq 0 \text{ and } tm \text{ is even.} \end{cases}$$

Proof. By the orthogonal property of additive characters, we have

$$\begin{aligned} n_c &= \sum_{x_1,x_2,\cdots,x_t\in\mathbb{F}_q} \frac{1}{p}\sum_{y\in\mathbb{F}_p} \zeta_p^{y\left(\mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)-c\right)} \\ &= \frac{q^t}{p} + \frac{1}{p}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-yc}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2)} \\ &= \frac{q^t}{p} + \frac{1}{p}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-yc}G(\eta)^t\eta^t(y). \qquad \text{(by Lemma 2.2)} \end{aligned}$$

If tm is odd, then

$$\begin{aligned} n_c &= \frac{q^t}{p} + \frac{1}{p}G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-yc}\eta_p(y) \\ &= \begin{cases} p^{tm-1}, & \text{if } c = 0, \\ p^{tm-1} + \frac{1}{p}\eta_p(-c)G(\eta)^tG(\eta_p), & \text{if } c \neq 0. \end{cases} \end{aligned}$$

If tm is even, then

$$n_c = \frac{q^t}{p} + \frac{1}{p}G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-yc} = \begin{cases} p^{tm-1} + \frac{p-1}{p}G(\eta)^t, & \text{if } c = 0, \\ p^{tm-1} - \frac{1}{p}G(\eta)^t, & \text{if } c \neq 0. \end{cases}$$

This completes the proof. □

We are ready to determine the length $n = |D|$ of the code $\mathcal{C}_D$. By Lemma 3.1 we have

$$n = \begin{cases} p^{tm-1} - 1, & \text{if } tm \text{ is odd;} \\ p^{tm-1} + \frac{p-1}{p}G(\eta)^t - 1, & \text{if } tm \text{ is even.} \end{cases}$$

For a codeword $\mathbf{c}(a_1,\dots,a_t)$ of $\mathcal{C}_D$, let $N := N(a_1,\dots,a_t)$ denote the number of components $\mathrm{Tr}_m(a_1x_1+\cdots+a_2x_2)$ of $\mathbf{c}(a_1,\dots,a_t)$ which are equal to 0, i.e.,

$$\begin{aligned}
N+1 &= \sum_{x_1,\dots,x_t\in\mathbb{F}_q}\left(\frac{1}{p}\sum_{y\in\mathbb{F}_p}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)}\right)\left(\frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)}\right)\\
&= \frac{1}{p^2}\sum_{x_1,\dots,x_t\in\mathbb{F}_q}\left(1+\sum_{y\in\mathbb{F}_p^*}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)}\right)\left(1+\sum_{z\in\mathbb{F}_p^*}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)}\right)\\
&= p^{tm-2}+\frac{1}{p^2}\big(\Omega_1+\Omega_2+\Omega_3\big), \qquad (3)
\end{aligned}$$

where

$$\Omega_1=\sum_{y\in\mathbb{F}_p^*}\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2)}\right)\cdots\left(\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2)}\right),$$

$$\begin{aligned}
\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_tx_t)}\\
&= \begin{cases}(p-1)q^t, & \text{if } (a_1,\dots,a_t)=(0,\dots,0),\\ 0, & \text{otherwise},\end{cases}
\end{aligned}$$

and

$$\Omega_3=\sum_{y,z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2+za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2+za_tx_t)}.$$

Now we are going to compute the values of Ω_1 and Ω_3. By the proof of Lemma 3.1, it is easy to see that

$$\Omega_1=\begin{cases}0, & \text{if } tm \text{ is odd;}\\ (p-1)G(\eta)^t, & \text{if } tm \text{ is even.}\end{cases}$$

Moreover, by Lemma 2.2 we have

$$\begin{aligned}
\Omega_3 &= \sum_{y,z\in\mathbb{F}_p^*}\left(\zeta_p^{\mathrm{Tr}_m\left(-\frac{a_1^2z^2}{4y}\right)}\eta(y)G(\eta)\right)\cdots\left(\zeta_p^{\mathrm{Tr}_m\left(-\frac{a_t^2z^2}{4y}\right)}\eta(y)G(\eta)\right)\\
&= G(\eta)^t\sum_{y,z\in\mathbb{F}_p^*}\eta^t(y)\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}.
\end{aligned}$$

Now we consider the case that tm is odd. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then we have

$$\Omega_3=G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\eta_p(y)=0.$$

If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\neq 0$, then it follows from Lemma 2.2 that

$$\begin{aligned}\Omega_3 &= G(\eta)^t \sum_{y,z\in\mathbb{F}_p^*}\eta_p(y)\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}\\ &= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\eta_p(y)\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}\\ &= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\eta_p(y)\Big(\eta_p\big(-\frac{1}{4y}\big)\eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta_p)-1\Big)\\ &= (p-1)G(\eta)^tG(\eta_p)\eta_p\big(-\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big).\end{aligned}$$

Suppose that tm is even. Note that $\eta^t(y)=1$ for all $y\in\mathbb{F}_p^*$. Then we have

$$\begin{aligned}\Omega_3 &= G(\eta)^t\sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}\\ &= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{z^2\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)y}\\ &= \begin{cases}(p-1)^2G(\eta)^t, & \text{if } \mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0;\\ -(p-1)G(\eta)^t, & \text{if } \mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\neq 0.\end{cases}\end{aligned}$$

Theorem 3.2. *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2), where $D=\{(x_1,x_2,\ldots,x_t)\in\mathbb{F}_q^t\setminus\{(0,0,\ldots,0)\}:\mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)=0\}$.*

1. *If $tm>1$ is odd, then $\mathcal{C}_D$ is a $[p^{tm-1}-1,tm]$ three-weight linear code and its weight enumerator is given by Table 1.*
2. *If tm is even, then $\mathcal{C}_D$ is a $[p^{tm-1}+(-1)^{(\frac{m(p-1)}{4}+1)t}(p-1)p^{\frac{tm-2}{2}}-1,tm]$ two-weight linear code and its weight enumerator is given by Table 2.*

Table 1. Weight enumerators of Theorem 3.2 for odd tm

Weight	Frequency
0	1
$(p-1)p^{tm-2}$	$p^{tm-1}-1$
$(p-1)(p^{tm-2}-p^{\frac{tm-3}{2}})$	$\frac{p-1}{2}(p^{tm-1}+p^{\frac{tm-1}{2}})$
$(p-1)(p^{tm-2}+p^{\frac{tm-3}{2}})$	$\frac{p-1}{2}(p^{tm-1}-p^{\frac{tm-1}{2}})$

Table 2. Weight enumerators of Theorem 3.2 for even tm

Weight	Frequency
0	1
$(p-1)p^{tm-2}$	$p^{tm-1}+(-1)^{(\frac{m(p-1)}{4}+1)t}(p-1)p^{\frac{tm-2}{2}}-1$
$(p-1)\big(p^{tm-2}+(-1)^{(\frac{m(p-1)}{4}+1)t}p^{\frac{tm-2}{2}}\big)$	$(p-1)\big(p^{tm-1}-(-1)^{(\frac{m(p-1)}{4}+1)t}p^{\frac{tm-2}{2}}\big)$

Proof. (1) If $(a_1,\ldots,a_t)=(0,\ldots,0)$, then by (3) we have

$$N=p^{tm-1}-1.$$

If $(a_1,\ldots,a_t)\neq(0,\ldots,0)$ and $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then by (3) we have

$$N=p^{tm-2}-1.$$

It follows from Lemma 3.1 that the frequency of this value is equal to the length $n=p^{tm-1}-1$ of the code $\mathcal{C}_D$.

If $(a_1, \dots, a_t) \neq (0, \dots, 0)$ and $\mathrm{Tr}_m(a_1^2 + \cdots + a_t^2) = c \neq 0$, then by (3) we have

$$N = p^{tm-2} + \frac{1}{p^2}(p-1)G(\eta)^t G(\eta_p)\eta_p(-c) - 1.$$

It follows from Lemma 3.1 that the frequency of this value is equal to $p^{tm-1} + \frac{1}{p}G(\eta)^t G(\eta_p)\eta_p(-c)$.

By Lemma 2.1, it is easily checked that $G(\eta)^t G(\eta_p) = (-1)^{\frac{(p-1)(tm+1)}{4}} p^{\frac{tm+1}{2}}$ if tm is odd. Note that the Hamming weight of $\mathbf{c}(a_1, \dots, a_t)$ defined as (2) is equal to

$$W_H(\mathbf{c}(a_1, \dots, a_t)) = n - N(a_1, \dots, a_t).$$

It is easy to see that $W_H(\mathbf{c}(a_1, \dots, a_t)) = 0$ if and only if $a_1 = \cdots = a_t = 0$, so the dimension of $\mathcal{C}_D$ is tm. Then we can immediately obtain the desired results.

(2) By Lemma 2.1, we have $G(\eta)^t = (-1)^{(\frac{m(p-1)}{4}+1)t} p^{\frac{tm}{2}}$ if tm is even. The proof of Part 2 is very similar to that of Part 1 and we omit the details. □

Example 1. (1) Let $p = 3$, $m = 2$, and $t = 3$. Then $q = 9$ and $n = 260$. By Theorem 3.2, the code $\mathcal{C}_D$ is a $[260, 6, 162]$ linear code and its weight enumerator is

$$1 + 260x^{162} + 468x^{180},$$

which is consistent with numerical computation by Magma.

(2) Let $p = 3$, $m = 3$, and $t = 3$. Then $q = 27$ and $n = 6560$. By Theorem 3.2, the code $\mathcal{C}_D$ is a $[6560, 9, 4320]$ linear code and its weight enumerator is

$$1 + 6642x^{4320} + 6560x^{4374} + 6480x^{4428},$$

which is consistent with numerical computation by Magma.

It should be remarked that the weight enumerators of $\mathcal{C}_D$ were presented when $t = 1$ [17] and $t = 2$ [18]. Thus Theorem 3.2 generalizes these results. In Tables 1 and 2, we observe that the weights of $\mathcal{C}_D$ have a common divisor $p - 1$. Let $\overline{D}$ be a subset of D such that

$$D = \mathbb{F}_p^* \overline{D} = \{y(x_1, \dots, x_t) = (yx_1, \dots, yx_t) : y \in \mathbb{F}_p^*, (x_1, \dots, x_t) \in \overline{D}\}.$$

Then the weight enumerators of linear codes $\mathcal{C}_{\overline{D}}$ can be obtained from Theorem 3.2 and more two-weight and three-weight linear codes can be presented.

In fact, if $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t \setminus \{(0, 0, \dots, 0)\} : \mathrm{Tr}_m(\beta_1 x_1^2 + \beta_2 x_2^2 + \cdots + \beta_t x_t^2) = 0\}$, where $\beta_1, \beta_2, \dots, \beta_t \in \mathbb{F}_q^*$, the weight enumerators can be similarly determined and the details are omitted here.

4. Weight enumerators in the second case

In this section, we present the weight enumerators of the linear codes $\mathcal{C}_D$ defined by (1) and (2), where $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 1\}$.

To this end, we begin to determine the length n of the code $\mathcal{C}_D$. Note that $\eta_p(-1) = (-1)^{\frac{p-1}{2}}$. Then by Lemma 3.1 we have

$$n = |D| = \begin{cases} p^{tm-1} + \frac{1}{p}(-1)^{\frac{p-1}{2}} G(\eta)^t G(\eta_p), & \text{if } tm \text{ is odd,} \\ p^{tm-1} - \frac{1}{p}G(\eta)^t, & \text{if } tm \text{ is even.} \end{cases}$$

For a codeword $\mathbf{c}(a_1,\ldots,a_t)$ of $\mathcal{C}_D$, let N denote the number of components $\mathrm{Tr}_m(a_1x_1+\cdots+a_2x_2)$ of $\mathbf{c}(a_1,\ldots,a_t)$ which are equal to 0, i.e.,

$$\begin{aligned} N &= \sum_{x_1,\ldots,x_t\in\mathbb{F}_q}\Big(\frac{1}{p}\sum_{y\in\mathbb{F}_p}\zeta_p^{y\left(\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)-1\right)}\Big)\Big(\frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)}\Big)\\ &= \frac{1}{p^2}\sum_{x_1,\ldots,x_t\in\mathbb{F}_q}\Big(1+\sum_{y\in\mathbb{F}_p^*}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)-y}\Big)\Big(1+\sum_{z\in\mathbb{F}_p^*}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)}\Big)\\ &= p^{tm-2}+\frac{1}{p^2}\big(\Omega_1+\Omega_2+\Omega_3\big), \end{aligned} \tag{4}$$

where

$$\Omega_1=\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2)}\Big)\cdots\Big(\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2)}\Big),$$

$$\begin{aligned}\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_tx_t)}\\ &= \begin{cases}(p-1)q^t, & \text{if } (a_1,\ldots,a_t)=(0,\ldots,0),\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\Omega_3=\sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2+za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2+za_tx_t)}.$$

By the proof of Lemma 3.1, it is easy to check that

$$\Omega_1=\begin{cases}(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p), & \text{if } tm \text{ is odd;}\\ -G(\eta)^t, & \text{if } tm \text{ is even.}\end{cases}$$

Moreover, by Lemma 2.2 we have

$$\begin{aligned}\Omega_3 &= \sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\zeta_p^{\mathrm{Tr}_m\left(-\frac{a_1^2z^2}{4y}\right)}\eta(y)G(\eta)\Big)\cdots\Big(\zeta_p^{\mathrm{Tr}_m\left(-\frac{a_t^2z^2}{4y}\right)}\eta(y)G(\eta)\Big)\\ &= G(\eta)^t\sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\eta^t(y)\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}.\end{aligned}$$

Now we consider the case that tm is odd. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then we have

$$\Omega_3=G(\eta)^t\sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(y)=(-1)^{\frac{p-1}{2}}(p-1)G(\eta)^tG(\eta_p).$$

If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\neq 0$, then it follows from Lemma 2.2 that

$$\begin{aligned}\Omega_3 &= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(y)\Big(\eta_p\big(-\frac{1}{4y}\big)\eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta_p)-1\Big)\\ &= G(\eta)^tG(\eta_p)\eta_p\big(-\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}-G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(y)\\ &= -G(\eta)^tG(\eta_p)\eta_p\big(-\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)-(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p)\\ &= -(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p)\Big(\eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)+1\Big).\end{aligned}$$

Suppose that tm is even. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then

$$\Omega_3 = G(\eta)^t(p-1)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y} = -(p-1)G(\eta)^t.$$

If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\neq 0$, then by Lemma 2.2 we have

$$\begin{aligned}
\Omega_3 &= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-\frac{\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}{4y}z^2}\\
&= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\eta_p\big(-\frac{1}{4y}\big)\eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta_p)-1\Big)\\
&= \eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta)^tG(\eta_p)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p\big(-\frac{y}{4y^2}\big)+G(\eta)^t\\
&= \eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta)^tG(\eta_p)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(-y)+G(\eta)^t\\
&= \eta_p\big(\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\big)G(\eta)^tG(\eta_p)^2+G(\eta)^t.
\end{aligned}$$

Theorem 4.1. *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2), where $D=\{(x_1,x_2,\ldots,x_t)\in\mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)=1\}$.*

1. *Suppose that $tm>1$ is odd. Then $\mathcal{C}_D$ is a $[p^{tm-1}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}},tm]$ three-weight linear code and its weight enumerator is given by Table 3.*
2. *Suppose that tm is even. If t is odd and $8\mid m(p-1)$, then $\mathcal{C}_D$ is a $[p^{tm-1}+p^{\frac{tm-2}{2}},tm]$ two-weight linear code with weight enumerator given by Table 4; otherwise, $\mathcal{C}_D$ is a $[p^{tm-1}-p^{\frac{tm-2}{2}},tm]$ two-weight linear code with weight enumerator given by Table 5.*

Table 3. Weight enumerators of Theorem 4.1 for odd tm

Weight	Frequency
0	1
$(p-1)p^{tm-2}$	$p^{tm-1}-1$
$(p-1)p^{tm-2}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}}+p^{\frac{tm-3}{2}}$	$\frac{p-1}{2}(p^{tm-1}+p^{\frac{tm-1}{2}})$
$(p-1)p^{tm-2}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}}-p^{\frac{tm-3}{2}}$	$\frac{p-1}{2}(p^{tm-1}-p^{\frac{tm-1}{2}})$

Table 4. Weight enumerators of Theorem 4.1 for even tm
$2\nmid\big(\frac{m(p-1)}{4}+1\big)t$

Weight	Frequency
0	1
$(p-1)p^{tm-2}$	$\frac{p+1}{2}p^{tm-1}-\frac{p-1}{2}p^{\frac{tm-2}{2}}-1$
$(p-1)p^{tm-2}+2p^{\frac{tm-2}{2}}$	$\frac{p-1}{2}\big(p^{tm-1}+p^{\frac{tm-2}{2}}\big)$

Table 5. Weight enumerators of Theorem 4.1 for even tm
$2\mid\big(\frac{m(p-1)}{4}+1\big)t$

Weight	Frequency
0	1
$(p-1)p^{tm-2}$	$\frac{p+1}{2}p^{tm-1}+\frac{p-1}{2}p^{\frac{tm-2}{2}}-1$
$(p-1)p^{tm-2}-2p^{\frac{tm-2}{2}}$	$\frac{p-1}{2}\big(p^{tm-1}-p^{\frac{tm-2}{2}}\big)$

Proof. (1) If $(a_1, \dots, a_t) = (0, \dots, 0)$, then by (4) we have

$$N = p^{tm-1} + \frac{1}{p}(-1)^{\frac{p-1}{2}} G(\eta)^t G(\eta_p).$$

If $(a_1, \dots, a_t) \neq (0, \dots, 0)$ and $\mathrm{Tr}_m(a_1^2 + \cdots + a_t^2) = 0$, then by (4) we have

$$N = p^{tm-2} + \frac{1}{p}(-1)^{\frac{p-1}{2}} G(\eta)^t G(\eta_p).$$

It follows from Lemma 3.1 that the frequency of this value is equal to $p^{tm-1} - 1$.

If $(a_1, \dots, a_t) \neq (0, \dots, 0)$ and $\mathrm{Tr}_m(a_1^2 + \cdots + a_t^2) = c \neq 0$, then by (4) we have

$$N = p^{tm-2} - \frac{1}{p^2}(-1)^{\frac{p-1}{2}} \eta_p(c) G(\eta)^t G(\eta_p).$$

It follows from Lemma 3.1 that the frequency of this value is equal to

$$p^{tm-1} + \frac{1}{p}(-1)^{\frac{p-1}{2}} \eta_p(c) G(\eta)^t G(\eta_p).$$

By Lemma 2.1, it is easy to see that $(-1)^{\frac{p-1}{2}} G(\eta)^t G(\eta_p) = (-1)^{\frac{(p-1)(tm+3)}{4}} p^{\frac{tm+1}{2}}$ if tm is odd. Note that the Hamming weight of $\mathbf{c}(a_1, \dots, a_t)$ defined as (2) is equal to

$$W_H(\mathbf{c}(a_1, \dots, a_t)) = n - N(a_1, \dots, a_t).$$

It is easy to see that $W_H(\mathbf{c}(a_1, \dots, a_t)) = 0$ if and only if $a_1 = \cdots = a_t = 0$, so the dimension of $\mathcal{C}_D$ is tm. Then we can immediately obtain the desired results.

(2) The proof of Part 2 is very similar to that of Part 1 and we omit the details. □

Example 2. (1) Let $p = 3$, $m = 3$, and $t = 3$. Then $q = 27$ and $n = 6642$. By Theorem 4.1, the code $\mathcal{C}_D$ is a $[6642, 9, 4374]$ linear code and its weight enumerator is

$$1 + 6560x^{4374} + 6480x^{4428} + 6642x^{4482},$$

which is consistent with numerical computation by Magma.

(2) Let $p = 3$, $m = 2$, and $t = 3$. Then $q = 9$ and $n = 234$. By Theorem 4.1, the code $\mathcal{C}_D$ is a $[234, 6, 144]$ linear code and its weight enumerator is

$$1 + 234x^{144} + 494x^{162},$$

which is consistent with numerical computation by Magma.

(3) Let $p = 5$, $m = 2$, and $t = 3$. Then $q = 25$ and $n = 3150$. By Theorem 4.1, the code $\mathcal{C}_D$ is a $[3150, 6, 2500]$ linear code and its weight enumerator is

$$1 + 9324x^{2500} + 6300x^{2550},$$

which is consistent with numerical computation by Magma.

In fact, if $D = \{(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(\beta_1 x_1^2 + \beta_2 x_2^2 + \cdots + \beta_t x_t^2) = c\}$, where $c \in \mathbb{F}_p^*$ and $\beta_1, \beta_2, \dots, \beta_t \in \mathbb{F}_q^*$, we can similarly present the weight enumerators of $\mathcal{C}_D$ and we omit the details here.

5. Complete weight enumerators of $\mathcal{C}_D$

In this section, we investigate the complete weight enumerators of the linear codes $\mathcal{C}_D$ in the two cases.

We begin to consider the first case. For a codeword $\mathbf{c}(a_1,\dots,a_t)$ of $\mathcal{C}_D$ and $\rho\in\mathbb{F}_p^*$, let $N_\rho := N_\rho(a_1,\dots,a_t)$ be the number of components $\mathrm{Tr}_m(a_1x_1+\cdots+a_2x_2)$ of $\mathbf{c}(a_1,\dots,a_t)$ which are equal to ρ, where $D=\{(x_1,x_2,\dots,x_t)\in\mathbb{F}_q^t\setminus\{(0,0,\dots,0)\}:\mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)=0\}$. Then by the orthogonal property of additive characters we have

$$\begin{aligned}
& N_\rho \\
= & \sum_{\substack{x_1,\dots,x_t\in\mathbb{F}_q\\(x_1,x_2,\dots,x_t)\neq(0,0,\dots,0)}}\left(\frac{1}{p}\sum_{y\in\mathbb{F}_p}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)}\right)\left(\frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\left(\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)-\rho\right)}\right)\\
= & \frac{1}{p^2}\sum_{x_1,\dots,x_t\in\mathbb{F}_q}\left(1+\sum_{y\in\mathbb{F}_p^*}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)}\right)\left(1+\sum_{z\in\mathbb{F}_p^*}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)-z\rho}\right)\\
(5)\quad = & p^{tm-2}+\frac{1}{p^2}\left(\Omega_1+\Omega_2+\Omega_3\right),
\end{aligned}$$

where

$$\Omega_1=\sum_{y\in\mathbb{F}_p^*}\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2)}\right)\cdots\left(\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2)}\right),$$

$$\begin{aligned}
\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_tx_t)}\\
&= \begin{cases}-q^t, & \text{if } (a_1,\dots,a_t)=(0,\dots,0),\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_3 &= \sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2+za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2+za_tx_t)}\\
&= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta^t(y)\zeta_p^{-\frac{z^2}{4y}\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}.
\end{aligned}$$

Suppose that tm is odd. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then

$$\Omega_3=G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta_p(y)=0.$$

If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\neq 0$, then

$$\begin{aligned}
\Omega_3 &= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta_p(y)\zeta_p^{-\frac{z^2}{4y}\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}\\
&= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta_p(-\frac{1}{4y})\zeta_p^{-z^2y\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}\\
&= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta_p(-\frac{y}{4y^2})\zeta_p^{-z^2y\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}\\
&= G(\eta)^t\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{y\in\mathbb{F}_p^*}\eta_p(-y)\zeta_p^{-z^2y\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)}\\
&= -G(\eta)^tG(\eta_p)\eta_p\left(-\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)\right).
\end{aligned}$$

Thus Ω_3 is independent of $\rho \in \mathbb{F}_p^*$ when $(a_1, \ldots, a_t)$ runs over $\mathbb{F}_q^t$.

Suppose that tm is even. If $\mathrm{Tr}_m(a_1^2 + \cdots + a_t^2) = 0$, then

$$\Omega_3 = -(p-1)G(\eta)^t.$$

If $\mathrm{Tr}_m(a_1^2 + \cdots + a_t^2) \neq 0$, then

$$\begin{aligned} \Omega_3 &= G(\eta)^t \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-\frac{z^2}{4y}\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)} \\ &= G(\eta)^t \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{yz^2\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)} = G(\eta)^t. \end{aligned}$$

Thus Ω_3 is independent of $\rho \in \mathbb{F}_p^*$ when $(a_1, \ldots, a_t)$ runs over $\mathbb{F}_q^t$.

It is clear that Ω_1 and Ω_2 both are independent of $\rho \in \mathbb{F}_p^*$ when $(a_1, \ldots, a_t)$ runs over $\mathbb{F}_q^t$. Then by (5) we have

$$N_{\rho_1}(a_1, \ldots, a_t) = N_{\rho_2}(a_1, \ldots, a_t)$$

for $\rho_1, \rho_2 \in \mathbb{F}_p^*$. Then by Theorem 3.2 we can get directly the following theorem on the complete weight enumerators of $\mathcal{C}_D$.

Theorem 5.1. *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2), where $D = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t \setminus \{(0, 0, \ldots, 0)\} : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 0\}$.*

1. *If $tm > 1$ is odd, then the complete weight enumerator of $\mathcal{C}_D$ is given by Table 6.*
2. *If tm is even, then the complete weight enumerator of $\mathcal{C}_D$ is given by Table 7.*

Table 6. Complete weight enumerators of Theorem 5.1 for odd tm

$N_0 = n - \sum_{\rho \in \mathbb{F}_p^*} N_\rho$

$N_\rho(\rho \in \mathbb{F}_p^*)$	Frequency
0	1
p^{tm-2}	$p^{tm-1} - 1$
$p^{tm-2} - p^{\frac{tm-3}{2}}$	$\frac{p-1}{2}(p^{tm-1} + p^{\frac{tm-1}{2}})$
$p^{tm-2} + p^{\frac{tm-3}{2}}$	$\frac{p-1}{2}(p^{tm-1} - p^{\frac{tm-1}{2}})$

Table 7. Complete weight enumerators of Theorem 5.1 for even tm

$N_0 = n - \sum_{\rho \in \mathbb{F}_p^*} N_\rho$

$N_\rho(\rho \in \mathbb{F}_p^*)$	Frequency
0	1
p^{tm-2}	$p^{tm-1} + (-1)^{(\frac{m(p-1)}{4}+1)t}(p-1)p^{\frac{tm-2}{2}} - 1$
$p^{tm-2} + (-1)^{(\frac{m(p-1)}{4}+1)t}p^{\frac{tm-2}{2}}$	$(p-1)\big(p^{tm-1} - (-1)^{(\frac{m(p-1)}{4}+1)t}p^{\frac{tm-2}{2}}\big)$

Now we are ready to consider the second case. For a codeword $\mathbf{c}(a_1, \ldots, a_t)$ of $\mathcal{C}_D$ and $\rho \in \mathbb{F}_p^*$, let $N_\rho := N_\rho(a_1, \ldots, a_t)$ be the number of components $\mathrm{Tr}_m(a_1x_1 + \cdots + a_2x_2)$ of $\mathbf{c}(a_1, \ldots, a_t)$ which are equal to ρ, where $D = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1^2 + x_2^2 + \cdots + x_t^2) = 1\}$. Then by the orthogonal property of additive characters we have

$$
\begin{aligned}
&N_\rho\\
=&\sum_{\substack{x_1,\dots,x_t\in\mathbb{F}_q\\(x_1,x_2,\dots,x_t)\neq(0,0,\dots,0)}}\Big(\frac{1}{p}\sum_{y\in\mathbb{F}_p}\zeta_p^{y\left(\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)-1\right)}\Big)\Big(\frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\left(\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)-\rho\right)}\Big)\\
=&\frac{1}{p^2}\sum_{x_1,\dots,x_t\in\mathbb{F}_q}\Big(1+\sum_{y\in\mathbb{F}_p^*}\zeta_p^{y\mathrm{Tr}_m(x_1^2+\cdots+x_t^2)-y}\Big)\Big(1+\sum_{z\in\mathbb{F}_p^*}\zeta_p^{z\mathrm{Tr}_m(a_1x_1+\cdots+a_tx_t)-z\rho}\Big)\\
(6)&\\
=&p^{tm-2}+\frac{1}{p^2}(\Delta_1+\Delta_2+\Delta_3),
\end{aligned}
$$

where

$$
\Delta_1=\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2)}\Big)\cdots\Big(\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2)}\Big),
$$

$$
\begin{aligned}
\Delta_2 &= \sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_tx_t)}\\
&= \begin{cases}-q^t, & \text{if } (a_1,\dots,a_t)=(0,\dots,0),\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_3 &= \sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\zeta_p^{-z\rho}\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_1^2+za_1x_1)}\cdots\sum_{x_t\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(yx_t^2+za_tx_t)}\\
&= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta^t(y)\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-\frac{z^2}{4y}\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)-z\rho}.
\end{aligned}
$$

In Section 4, it is shown that

$$
\Delta_1=\begin{cases}(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p), & \text{if } tm \text{ is odd;}\\ -G(\eta)^t, & \text{if } tm \text{ is even.}\end{cases}
$$

Suppose that tm is odd. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then

$$
\Delta_3=G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(y)\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}=-(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p).
$$

If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=c\neq 0$, then

$$
\begin{aligned}
\Delta_3 &= G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\eta_p(y)\Big(\eta_p\big(-\frac{c}{4y}\big)\zeta_p^{\frac{y\rho^2}{c}}G(\eta_p)-1\Big)\\
&= \eta_p(-c)G(\eta)^tG(\eta_p)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{(\frac{\rho^2}{c}-1)y}-\eta_p(-1)G(\eta)^tG(\eta_p)\\
&= \begin{cases}(-1)^{\frac{p-1}{2}}\big(\eta_p(c)(p-1)-1\big)G(\eta)^tG(\eta_p), & \text{if } \rho^2=c;\\ (-1)^{\frac{p-1}{2}}\big(-\eta_p(c)-1\big)G(\eta)^tG(\eta_p), & \text{if } \rho^2\neq c.\end{cases}
\end{aligned}
$$

Theorem 5.2. *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2), where $D=\{(x_1,x_2,\dots,x_t)\in\mathbb{F}_q^t:\mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)=1\}$. If $tm>1$ is odd, then the complete weight enumerators of $\mathcal{C}_D$ is given as follows:*

1. *If $(a_1,\dots,a_t)=(0,\dots,0)$, then*

$$
N_0=p^{tm-1}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}} \text{ and } N_\rho=0 \text{ for } \rho\in\mathbb{F}_p^*.
$$

2. *If $(a_1,\dots,a_t)\neq(0,\dots,0)$ and $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then*
$$N_0=p^{tm-2}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}} \text{ and } N_\rho=p^{tm-2} \text{ for } \rho\in\mathbb{F}_p^*.$$
The frequency of this composition is equal to $p^{tm-1}-1$.
3. *Suppose that $(a_1,\dots,a_t)\neq(0,\dots,0)$ and $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=c\neq 0$.*
   - *If $\eta_p(c)=1$, let $\rho_1(c),\rho_2(c)$ be two solutions of the equation $\rho^2=c$, then*
$$N_0=p^{tm-2}-(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-3}{2}},$$
$$N_{\rho_1(c)},N_{\rho_1(c)}=p^{tm-2}+(-1)^{\frac{(p-1)(tm+3)}{4}}(p-1)p^{\frac{tm-3}{2}},$$
$$\text{and } N_\rho=p^{tm-2}-(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-3}{2}} \text{ for } \rho\in\mathbb{F}_p^*,\rho\neq\rho_1(c),\rho_2(c).$$
The frequency of this composition is equal to $p^{tm-1}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}}$.
   - *If $\eta_p(c)=-1$, then*
$$N_\rho=p^{tm-2}+(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-3}{2}} \text{ for } \rho\in\mathbb{F}_p.$$
The frequency of this composition is equal to $p^{tm-1}-(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm-1}{2}}$.

Proof. Note that $(-1)^{\frac{p-1}{2}}G(\eta)^tG(\eta_p)=(-1)^{\frac{(p-1)(tm+3)}{4}}p^{\frac{tm+1}{2}}$ if tm is odd. By (6) and the proof of Theorem 4.1, we can similarly get the desired results and we omit the details here. □

Example 3. Let $p=3$, $m=3$, and $t=3$. Then $q=27$ and $n=6642$. By Theorem 5.2, the code $\mathcal{C}_D$ is a $[6642,9,4374]$ linear code and its complete weight enumerator is
$$z_0^{6642}+6560z_0^{2268}(z_1z_2)^{2187}+6480(z_0z_1z_2)^{2214}+6642z_0^{2160}(z_1z_2)^{2241},$$
which is consistent with numerical computation by Magma.

Suppose that tm is even. If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then
$$\Delta_3=G(\eta)^t\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\sum_{z\in\mathbb{F}_p^*}\zeta_p^{-z\rho}=G(\eta)^t.$$
If $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=c\neq0$, then we similarly have
$$\begin{aligned}\Delta_3 &= G(\eta)^tG(\eta_p)\eta_p(-c)\sum_{y\in\mathbb{F}_p^*}\eta_p(y)\zeta_p^{(\frac{\rho^2}{c}-1)y}+G(\eta)^t\\ &= \begin{cases} G(\eta)^t, & \text{if } \rho^2=c;\\ \eta_p(c-\rho^2)G(\eta)^tG(\eta_p)^2+G(\eta)^t, & \text{if } \rho^2\neq c.\end{cases}\end{aligned}$$
By (6) and Lemma 3.1, we can similarly get the following theorem on the complete weight enumerators of $\mathcal{C}_D$ if tm is even.

Theorem 5.3. *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2), where $D=\{(x_1,x_2,\dots,x_t)\in\mathbb{F}_q^t:\mathrm{Tr}_m(x_1^2+x_2^2+\cdots+x_t^2)=1\}$. Suppose that tm is even. If t is odd and $8\mid m(p-1)$, then the complete weight enumerators of $\mathcal{C}_D$ is given as follows:*
1. *If $(a_1,\dots,a_t)=(0,\dots,0)$, then*
$$N_0=p^{tm-1}+p^{\frac{tm-2}{2}} \text{ and } N_\rho=0 \text{ for } \rho\in\mathbb{F}_p^*.$$
2. *If $(a_1,\dots,a_t)\neq(0,\dots,0)$ and $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$, then*
$$N_0=p^{tm-2}+p^{\frac{tm-2}{2}} \text{ and } N_\rho=p^{tm-2} \text{ for } \rho\in\mathbb{F}_p^*.$$
The frequency of this composition is equal to $p^{tm-1}-(p-1)p^{\frac{tm-2}{2}}-1$.

3. Suppose that $(a_1,\ldots,a_t)\neq(0,\ldots,0)$ *and* $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=c\neq 0$.

- *If* $\eta_p(c)=1$*, then*

$$N_0=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}},$$

$$N_\rho=p^{tm-2} \textit{ for } c-\rho^2=0,$$

$$N_\rho=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=1,$$

$$\textit{and } N_\rho=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=-1.$$

The frequency of this composition is equal to $p^{tm-1}+p^{\frac{tm-2}{2}}$.

- *If* $\eta_p(c)=-1$*, then*

$$N_0=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}},$$

$$N_\rho=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=1,$$

$$\textit{and } N_\rho=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=-1.$$

The frequency of this composition is equal to $p^{tm-1}+p^{\frac{tm-2}{2}}$.

In other cases, the complete weight enumerators of $\mathcal{C}_D$ *is given as follows:*

1. If $(a_1,\ldots,a_t)=(0,\ldots,0)$*, then*

$$N_0=p^{tm-1}-p^{\frac{tm-2}{2}} \textit{ and } N_\rho=0 \textit{ for } \rho\in\mathbb{F}_p^*.$$

2. If $(a_1,\ldots,a_t)\neq(0,\ldots,0)$ *and* $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=0$*, then*

$$N_0=p^{tm-2}-p^{\frac{tm-2}{2}} \textit{ and } N_\rho=p^{tm-2} \textit{ for } \rho\in\mathbb{F}_p^*.$$

The frequency of this composition is equal to $p^{tm-1}+(p-1)p^{\frac{tm-2}{2}}-1$.

3. Suppose that $(a_1,\ldots,a_t)\neq(0,\ldots,0)$ *and* $\mathrm{Tr}_m(a_1^2+\cdots+a_t^2)=c\neq 0$.

- *If* $\eta_p(c)=1$*, then*

$$N_0=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}},$$

$$N_\rho=p^{tm-2} \textit{ for } c-\rho^2=0,$$

$$N_\rho=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=1,$$

$$\textit{and } N_\rho=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=-1.$$

The frequency of this composition is equal to $p^{tm-1}-p^{\frac{tm-2}{2}}$.

- *If* $\eta_p(c)=-1$*, then*

$$N_0=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}},$$

$$N_\rho=p^{tm-2}+(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=1,$$

$$\textit{and } N_\rho=p^{tm-2}-(-1)^{\frac{p-1}{2}}p^{\frac{tm-2}{2}} \textit{ for } \eta_p(c-\rho^2)=-1.$$

The frequency of this composition is equal to $p^{tm-1}-p^{\frac{tm-2}{2}}$.

Note that $c,\rho\in\mathbb{F}_p$. It is not difficult to compute $c-\rho^2$ for small odd prime p. For fixed c, the number of ρ such that $\eta_p(c-\rho^2)=1$ or -1 can be determined by cyclotomic numbers of order 2, we do not consider it here.

Example 4. (1) Let $p = 3$, $m = 2$, and $t = 3$. Then $q = 9$ and $n = 234$. By Theorem 5.3, the code $\mathcal{C}_D$ is a $[234, 6, 144]$ linear code and its complete weight enumerator is

$$z_0^{234} + 234z_0^{90}(z_1z_2)^{72} + 494z_0^{72}(z_1z_2)^{81},$$

which is consistent with numerical computation by Magma.

(2) Let $p = 5$, $m = 2$, and $t = 3$. Then $q = 25$ and $n = 3150$. By Theorem 5.3, the code $\mathcal{C}_D$ is a $[3150, 6, 2500]$ linear code and its complete weight enumerator is

$$\begin{aligned} z_0^{3150} &+ 3150z_0^{600}(z_1z_4)^{625}(z_2z_3)^{650} + 3150z_0^{600}(z_1z_4)^{650}(z_2z_3)^{625} \\ &+ 3024z_0^{650}(z_1z_2z_3z_4)^{625} \\ &+ 3150z_0^{650}(z_1z_4)^{600}(z_2z_3)^{650} + 3150z_0^{650}(z_1z_4)^{650}(z_2z_3)^{600}, \end{aligned}$$

which is consistent with numerical computation by Magma.

6. Concluding remarks

In this paper, we employed exponential sums to present the weight enumerators of the linear codes $\mathcal{C}_D$ in the two cases. It was proved that $\mathcal{C}_D$ is a two-weight code if tm is even and three-weight code if tm is odd. It should be remarked that the weight enumerators of $\mathcal{C}_D$ in the first case generalize the results in [17] and [18]. The complete weight enumerators of the linear codes $\mathcal{C}_D$ were also investigated. Linear codes with two and three weights are closely related to strongly regular graphs and association schemes. It would be nice if more two-weight and three-weight linear codes can be found.

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