Siegel families with application to class fields

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We investigate certain families of meromorphic Siegel modular functions on which
Galois groups act in a natural way. By using Shimura’s reciprocity law we construct
some algebraic numbers in the ray class fields of CM-fields in terms of special values
of functions in these Siegel families.

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CM-fields; Siegel modular functions

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1. Introduction

For a positive integer N let \( \mathfrak{F}_N \) be the field of meromorphic modular functions of
level N (defined on \( \mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \)) whose Fourier coefficients belong to
the Nth cyclotomic field. As is well known, \( \mathfrak{F}_N \) is a Galois extension of \( \mathfrak{F}_1 \) whose
Galois group is isomorphic to \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle \pm I_2 \rangle \) (see [11, §6.1–6.2]). Now, let
\( N \geq 2 \) and consider a set

\[ V_N = \{ v \in \mathbb{Q}^2 \mid Nv \in \mathbb{Z}^2 \} \]

as the index set. We call a family \( \{ f_v(\tau) \}_{v \in V_N} \) of functions in \( \mathfrak{F}_N \) a Fricke family
of level N if each \( f_v(\tau) \) depends only on \( \pm v \) (mod \( \mathbb{Z}^2 \)) and satisfies

\[ f_v(\tau)^\alpha = f_{\alpha v}(\tau) \quad (\alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle \pm I_2 \rangle), \]

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where $\alpha^T$ means the transpose of $\alpha$. For example, Siegel functions of one variable form such a Fricke family of level $N$ [8, ch. 2, proposition 1.3] (see also [4] or [6]).

Let $K$ be an imaginary quadratic field with the ring of integers $\mathcal{O}_K$, and let $f$ be a proper non-trivial ideal of $\mathcal{O}_K$. We denote by $\text{Cl}(f)$ and $K_f$ the ray class group modulo $f$ and its corresponding ray class field modulo $f$, respectively. If $\{f_{\nu}(\tau)\}_\nu$ is a Fricke family of level $N$ in which every $f_{\nu}(\tau)$ is holomorphic on $\mathbb{H}$, then we can assign to each ray class $C \in \text{Cl}(f)$ an algebraic number $f_t(C)$ as a special value of a function in $\{f_{\nu}(\tau)\}_\nu$. Furthermore, we attain by Shimura’s reciprocity law that $f_t(C)$ belongs to $K_f$ and satisfies

$$f_t(C)^{\sigma(D)} = f_t(CD) \quad (D \in \text{Cl}(f)),$$

where $\sigma$ is the Artin reciprocity map for $f$ (see [8, ch. 11, theorem 1.1]).

In this paper we shall define a Siegel family $\{h_{M}(Z)\}_M$ of level $N$ consisting of meromorphic Siegel modular functions of (higher) genus $g$ and level $N$, which is a generalization of a Fricke family of level $N$ in the case when $g = 1$ (definition 3.1). It turns out that every Siegel family of level $N$ is induced from a meromorphic Siegel modular function for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients (theorem 3.5).

Let $K$ be a CM-field and let $f = NO_K$. Given a Siegel family $\{h_{M}(Z)\}_M$ of level $N$, we shall introduce a number $h_t(C)$ as a special value of a function in $\{h_{M}(Z)\}_M$ for each ray class $C \in \text{Cl}(f)$ (definition 5.4). Under certain assumptions on $K$ (assumption 5.1) we shall prove that if $h_t(C)$ is finite, then it lies in the ray class field $K_f$ whose Galois conjugates are of the same form (theorem 7.2 and corollary 7.3). To this end, we assign a principally polarized abelian variety to each non-trivial ideal of $\mathcal{O}_K$, and apply Shimura’s reciprocity law to $h_t(C)$.

On the other hand, we note that there is a remarkable paper by Grant [2] in which he generalized a classical formula of Eisenstein and obtained classes of $S$-units by evaluating abelian functions at the intersections of divisors on the Jacobian of the curve $y^2 = x^3 + 1$. We hope that our invariant $h_t(C)$ obtained from a Siegel family in theorem 4.3 will contribute further towards finding a higher-dimensional analogue of an elliptic unit.

2. Actions on Siegel modular functions

First, we shall describe the Galois group between fields of meromorphic Siegel modular functions in a concrete way.

Let $g$ be a positive integer and let

$$\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

For every commutative ring $R$ with unity we define

\[
\begin{align*}
\text{GSp}_{2g}(R) &= \{\alpha \in \text{GL}_{2g}(R) \mid \alpha^T \eta_g \alpha = \nu(\alpha) \eta_g \text{ with } \nu(\alpha) \in R^\times\}, \\
\text{Sp}_{2g}(R) &= \{\alpha \in \text{GSp}_{2g}(R) \mid \nu(\alpha) = 1\}.
\end{align*}
\]

Let

$$G = \text{GSp}_{2g}(\mathbb{Q})$$
and let $G_h$ be the adelization of $G$, let $G_0$ be its non-Archimedean part and let $G_\infty$ be its Archimedean part. One can extend the multiplier map $\nu : G \to \mathbb{Q}^\times$ continuously to the map $\nu : G_h \to \mathbb{Q}_h^\times$ and set

$$G_\infty+ = \{ \alpha \in G_\infty \mid \nu(\alpha) > 0 \}, \quad G_{h+} = G_0 G_\infty+, \quad G_+ = G \cap G_{h+}.$$  

Furthermore, let

$$\Delta = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \mid s \in \prod_p \mathbb{Z}_p^\times \right\},$$

$$U_1 = \prod_p \text{GSp}_{2g}(\mathbb{Z}_p) \times G_\infty+,$$

$$U_N = \{ x \in U_1 \mid x_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p \}$$

for every positive integer $N$. Then we have

$$U_N \leq U_1 \leq G_{h+} \quad \text{and} \quad G_{h+} = U_N \Delta G_+$$

(see [13, lemma 8.3(1)]).

Note that the group $G_\infty+$ acts on the Siegel upper half-space

$$\mathbb{H}_g = \{ Z \in M_g(\mathbb{C}) \mid Z^\top = Z, \text{Im}(Z) \text{ is positive definite} \}$$

by

$$\alpha(Z) = (AZ + B)(CZ + D)^{-1} \quad (\alpha \in G_\infty+, \ Z \in \mathbb{H}_g),$$

where $A, B, C, D$ are $g \times g$ block matrices of $\alpha$. Let $\mathcal{F}_N$ be the field of meromorphic Siegel modular functions of genus $g$ for the congruence subgroup

$$\Gamma(N) = \{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})} \}$$

of the symplectic group $\text{Sp}_{2g}(\mathbb{Z})$ whose Fourier coefficients belong to the $N$th cyclotomic field $\mathbb{Q}(\zeta_N) = e^{2\pi i/N}$. That is, if $f \in \mathcal{F}_N$, then

$$f(Z) = \frac{\sum_h c(h) e(\text{tr}(hZ)/N)}{\sum_h d(h) e(\text{tr}(hZ)/N)} \quad \text{for some } c(h), d(h) \in \mathbb{Q}(\zeta_N),$$

where the denominator and numerator of $f$ are Siegel modular forms of the same weight, $h$ runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$ [5, § 4, theorem 1]. Let

$$\mathcal{F} = \bigcup_{N=1}^\infty \mathcal{F}_N.$$  

**PROPOSITION 2.1.** There exists a homomorphism $\tau : G_{h+} \to \text{Aut}(\mathcal{F})$ satisfying the following properties. Let

$$f(Z) = \frac{\sum_h c(h) e(\text{tr}(hZ)/N)}{\sum_h d(h) e(\text{tr}(hZ)/N)} \in \mathcal{F}_N.$$

(i) If $\alpha \in G_+ = \{ \alpha \in G \mid \nu(\alpha) > 0 \}$, then

$$f^{\tau(\alpha)} = f \circ \alpha.$$
(ii) If
\[
\beta = \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \in \Delta
\]
and \( t \) is a positive integer such that \( t \equiv s_p \pmod{N\mathbb{Z}_p} \) for all rational primes \( p \), then
\[
f^{\tau(\beta)} = \sum_h c(h)^s e(\text{tr}(h^{\mathbb{Z}})/N) = \sum_h d(h)^s e(\text{tr}(h^{\mathbb{Z}})/N),
\]
where \( \sigma \) is the automorphism of \( \mathbb{Q}(\zeta_N) \) given by \( \zeta_N^\sigma = \zeta_N^t \).

(iii) For every positive integer \( N \) we have
\[
\mathcal{F}_N = \{ f \in \mathcal{F} \mid f^{\tau(x)} = f \text{ for all } x \in U_N \}.
\]

(iv) We have \( \ker(\tau) = \mathbb{Q}^\times G_{\infty^+} \).

**Proof.** See [13, theorem 8.10]. \( \square \)

Since
\[
U_N(\mathbb{Q}^\times G_{\infty^+})/\mathbb{Q}^\times G_{\infty^+} \simeq U_N/(U_N \cap \mathbb{Q}^\times G_{\infty^+}) \simeq \begin{cases} U_1/\pm G_{\infty^+} & \text{if } N = 1, \\ U_N/G_{\infty^+} & \text{if } N > 1, \end{cases}
\]
we see by proposition 2.1(iii) and (iv) that \( \mathcal{F}_N \) is a Galois extension of \( \mathcal{F}_1 \) with
\[
\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N. \tag{2.1}
\]

**Proposition 2.2.** We have
\[
\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2g\}.
\]

**Proof.** Let \( \alpha \in U_1 \). Take a matrix \( A \) in \( M_{2g}(\mathbb{Z}) \) for which \( A \equiv \alpha_p \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \) for all rational primes \( p \). Define a matrix \( \psi(\alpha) \in M_{2g}(\mathbb{Z}/N\mathbb{Z}) \) by the image of \( A \) under the natural reduction \( M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Then, by the Chinese remainder theorem, \( \psi(\alpha) \) is well defined and independent of the choice of \( A \). Furthermore, let \( t \) be an integer relatively prime to \( N \) such that \( t \equiv \nu(\alpha_p) \pmod{N \mathbb{Z}_p} \) for all rational primes \( p \). We then derive that
\[
t \eta_\alpha \equiv \nu(\alpha_p) \eta_\alpha \equiv \alpha_p^T \eta_\alpha \equiv A^T \eta_\alpha \equiv \psi(\alpha)^T \eta_\alpha \psi(\alpha) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}
\]
for all rational primes \( p \), and hence \( \psi(\alpha) \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Thus, we obtain a group homomorphism
\[
\psi: U_1 \to \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}).
\]

Let \( \beta \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and take a preimage \( B \) of \( \beta \) under the natural reduction \( M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Since \( \nu(\beta) \in (\mathbb{Z}/N\mathbb{Z})^\times \) and
\[
B^T \eta_\beta \equiv \beta^T \eta_\beta \equiv \nu(\beta) \eta_\beta \pmod{N \cdot M_{2g}(\mathbb{Z})},
\]
\( B \) belongs to \( \text{GSp}_{2g}(\mathbb{Z}_p) \) for every rational prime \( p \) dividing \( N \). Let \( \alpha_p = (\alpha_p)_p \) be the element of \( \prod_p \text{GSp}_{2g}(\mathbb{Z}_p) \) given by
\[
\alpha_p = \begin{cases} B & \text{if } p|N, \\ I_2g & \text{otherwise}. \end{cases}
\]
We then see that \( \alpha \in U_1 \) and \( \psi(\alpha) = \beta \). Thus, \( \psi \) is surjective.
Clearly, $U_N$ is contained in $\ker(\psi)$. Let $\gamma \in \ker(\psi)$. Since $\gamma_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes $p$, we get $\gamma \in U_N$, and hence $\ker(\psi) = U_N$. Therefore, $\psi$ induces an isomorphism $U_1/U_N \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, from which we achieve, by (2.1),

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \frac{U_1}{\pm U_N} \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$ 

\[ \Box \]

**Remark 2.3.** We have the decomposition

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where

$$G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \mid \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$ 

By proposition 2.1 one can describe the action of $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\mathcal{F}_N$ as follows.

Let

$$f(Z) = \frac{\sum_h e(c(h) e(\text{tr}(hZ)/N))}{\sum_h d(h) e(\text{tr}(hZ)/N)} \in \mathcal{F}_N.$$ 

(i) An element

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$$

of $G_N$ acts on $f$ by

$$f^\beta = \frac{\sum_h e(c(h)^\sigma e(\text{tr}(hZ)/N))}{\sum_h d(h)^\sigma e(\text{tr}(hZ)/N)},$$

where $\sigma$ is the automorphism of $\mathbb{Q}(\zeta_N)$ satisfying $\zeta_N^\sigma = \zeta_N'$.

(ii) An element $\gamma$ of $\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on $f$ by

$$f^\gamma = f \circ \gamma',$$

where $\gamma'$ is any preimage of $\gamma$ under the natural reduction

$$\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

3. Siegel families of level $N$

By making use of the description of $\text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ in § 2 we shall introduce a generalization of a Fricke family in higher dimensional cases.

Let $N \geq 2$. For $\alpha, \beta \in M_{2g}(\mathbb{Z})$ we denote by $\tilde{\alpha}$ its reduction modulo $N$. Define a set

$$V_N = \left\{ \frac{1}{N} \begin{bmatrix} A^T \\ B^T \end{bmatrix} \mid \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \text{ such that } \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right\}.$$ 

Let $\alpha, \beta$ be elements of $M_{2g}(\mathbb{Z})$ satisfying $\alpha, \beta \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. If $M$ is an element of $V_N$ induced from $\alpha$, then it is straightforward that $\beta^T M$ is also an element of $V_N$ given by the product $\alpha \beta$. 

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DEFINITION 3.1. We call a family \( \{ h_M(Z) \}_{M \in \mathcal{V}_N} \) a Siegel family of level \( N \) if it satisfies the following:

(S1) each \( h_M(Z) \) belongs to \( \mathcal{F}_N \);

(S2) \( h_M(Z) \) depends only on \( \pm M \pmod{M_{2g \times g}(\mathbb{Z})} \);

(S3) \( h_M(Z)^\sigma = h_{\sigma \tau M}(Z) \) for all \( \sigma \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_{2g} \} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \).

By \( \mathcal{S}_N \) we mean the set of such Siegel families of level \( N \).

REMARK 3.2. Let \( \{ h_M(Z) \}_{M \in \mathcal{S}_N} \).

(i) The property (S3) yields a right action of the group \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_{2g} \} \) on \( \{ h_M(Z) \}_{M} \).

(ii) We let \( M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix} \in \mathcal{V}_N \), and so there is a matrix \( \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \) such that \( \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Considering \( \tilde{\alpha} \) as an element of \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_{2g} \} \) we obtain

\[
h_{\frac{1}{(1/N)}[I_g \ O_g]}(Z)^{\tilde{\alpha}} = h_{\frac{1}{(1/N)}[I_g \ O_g]}(Z) = h_M(Z).
\]

Thus, the action of \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_{2g} \} \) on \( \{ h_M(Z) \}_{M} \) is transitive.

Let
\[
\Gamma^1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},
\]
and let \( \mathcal{F}^1_N(\mathbb{Q}) \) be the field of meromorphic Siegel modular functions for \( \Gamma^1(N) \) with rational Fourier coefficients.

LEMMA 3.3. If \( \{ h_M(Z) \}_{M \in \mathcal{S}_N} \), then

\[
h_{\frac{1}{(1/N)}I_g}(Z) \in \mathcal{F}^1_N(\mathbb{Q}).
\]

Proof. For any \( \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^1(N) \), we deduce by (S2) and (S3) that

\[
h_{\frac{1}{(1/N)}I_g}(\gamma(Z)) = h_{\frac{1}{(1/N)}I_g}(Z)^{\tilde{\gamma}} = h_{\frac{1}{(1/N)}I_g}(Z)
\]

\[
= h_{\frac{1}{(1/N)}[A^T \ B^T]}(Z) = h_{\frac{1}{(1/N)}[I_g \ O_g]}(Z).
\]
because
\[ A \equiv I_g, \quad B \equiv O_g \pmod{N \cdot M_g(Z)}. \]
Thus, \( h\left[ \frac{(1/N)I_g}{O_g} \right](Z) \) is modular for \( \Gamma^1(N) \).

For every \( \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \) we see by (S2) and (S3) that
\[
h\left[ \frac{(1/N)I_g}{O_g} \right](Z) \left[ O_g \nu I_g \right] = h\left[ \frac{I_g}{O_g} \right](Z) = h\left[ \frac{(1/N)I_g}{O_g} \right](Z),
\]
which implies that \( h\left[ \frac{(1/N)I_g}{O_g} \right](Z) \) has rational Fourier coefficients. This proves the lemma. \( \square \)

One can consider \( S_N \) as a field under the binary operations
\[
\begin{align*}
\{ h_M(Z) \}_M + \{ k_M(Z) \}_M &= \{ (h_M + k_M)(Z) \}_M, \\
\{ h_M(Z) \}_M \cdot \{ k_M(Z) \}_M &= \{ (h_M k_M)(Z) \}_M.
\end{align*}
\]
By lemma 3.3 we get the ring homomorphism
\[
\phi_N : S_N \to \mathcal{F}_N^1(\mathbb{Q})
\]
\[
\{ h_M(Z) \}_M \mapsto h\left[ \frac{(1/N)I_g}{O_g} \right](Z).
\]

**Lemma 3.4.** If \( M \in V_N \), then there exists
\[
\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})
\]
such that \( \tilde{\gamma} \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and
\[
M = \left( \begin{array}{cc} A & B \\ 1/N & B^T \end{array} \right).
\]

**Proof.** Let
\[
\alpha = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \in M_{2g}(\mathbb{Z})
\]
such that \( \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and
\[
M = \left( \begin{array}{cc} A^T \\ 1/N & B^T \end{array} \right).
\]
In \( M_{2g}(\mathbb{Z}/N\mathbb{Z}) \), decompose \( \tilde{\alpha} \) as
\[
\tilde{\alpha} = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix} \quad \text{with} \quad \nu = \nu(\tilde{\alpha}) \in (\mathbb{Z}/N\mathbb{Z})^\times
\]
so that
\[
\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}
\]
belongs to $\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since the reduction $\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ is surjective (see [10]), we can take $\gamma \in M_{2g}(\mathbb{Z})$ satisfying

$$\tilde{\gamma} = \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}.$$ 

\[\square\]

**Theorem 3.5.** $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via $\phi_N$.

**Proof.** Since $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are fields, it suffices to show that $\phi_N$ is surjective.

Let $h(Z) \in \mathcal{F}_N^1(\mathbb{Q})$. For each $M \in \mathcal{V}_N$, take any $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma} \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}$$

by using lemma 3.4. We set

$$h_M(Z) = h(Z)^{\tilde{\gamma}}.$$ 

We claim that $h_M(Z)$ is independent of the choice of $\gamma$. Indeed, if $\gamma' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma}' \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, then we attain in $M_{2g}(\mathbb{Z}/N\mathbb{Z})$ that

$$\tilde{\gamma}' \tilde{\gamma}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix} = \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}$$

by the fact $\tilde{\gamma}, \tilde{\gamma}' \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $\delta$ be an element of $\text{Sp}_{2g}(\mathbb{Z})$ such that $\tilde{\delta} = \tilde{\gamma}' \tilde{\gamma}^{-1}$. We then achieve

$$h(Z)^{\tilde{\gamma}} = (h(Z)^{\tilde{\gamma}'})^{\tilde{\gamma}} = h(\delta(Z))^{\tilde{\gamma}} = h(Z)^{\tilde{\gamma}}$$

because $h(Z)$ is modular for $\Gamma^1(N)$ and $\delta \in \Gamma^1(N)$.

Now, for any $\sigma = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ with $\nu = \nu(\sigma)$ we derive that

$$h_M(Z)^{\sigma} = h(Z)^{\tilde{\gamma}^\sigma} = h(Z)^{\tilde{\gamma}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = h(Z)^{\tilde{\gamma}} \begin{bmatrix} I_g & O_g \\ O_{\nu^{-1}(CP+DR)} & \nu^{-1}(CP+DS) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = h(Z)^{\tilde{\gamma}} \begin{bmatrix} AP+BR & AQ+BS \\ CP+DR & CQ+DS \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = h(Z)^{\tilde{\gamma}} \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$
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\[ h(Z) = h\left[ (AP+BR)^{(Z)} \right] = h\left[ (AQ+BS)^{(Z)} \right] \]

since \( h(Z) \) has rational Fourier coefficients

\[ h(Z) = h\left[ (AP+BR)^{(Z)} \right] = h\left[ (AQ+BS)^{(Z)} \right] \]

This shows that the family \( \{ h_{M}(Z) \}_M \) belongs to \( \mathcal{S}_N \). Furthermore, since

\[ \phi_N(\{ h_{M}(Z) \}_M) = h\left[ (1/N)I \right](Z) = h(Z) \]

\( \phi \) is surjective as desired. \( \square \)

**Remark 3.6.**

(i) By proposition 2.2 and remark 2.3 we obtain

\[ \text{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \cong G_N \cdot \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \mid \gamma = \pm \left[ \begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] \right\} \]

(ii) Let \( \mathcal{F}_{1,N}(\mathbb{Q}) \) be the field of meromorphic Siegel modular functions for

\[ \Gamma_1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \left[ \begin{array}{cc} I_g & * \\ O_g & I_g \end{array} \right] \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\} \]

with rational Fourier coefficients. If we set

\[ \omega = \left[ \begin{array}{cc} (1/\sqrt{N})I_g & O_g \\ O_g & \sqrt{N}I_g \end{array} \right] \]

then we know that \( \omega \in \text{Sp}_{2g}(\mathbb{R}) \) and

\[ \omega \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \omega^{-1} = \left[ \begin{array}{cc} A & (1/N)B \\ NC & D \end{array} \right] \text{ for all } \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \in \text{Sp}_{2g}(\mathbb{R}). \]

This implies

\[ \omega \Gamma_1(N) \omega^{-1} = \Gamma_1(N), \]

and so \( \mathcal{F}_{1,N}(\mathbb{Q}) \) and \( \mathcal{F}_N^1(\mathbb{Q}) \) are isomorphic via

\[ \mathcal{F}_{1,N}(\mathbb{Q}) \rightarrow \mathcal{F}_N^1(\mathbb{Q}) \\
\text{h}(Z) \mapsto (h \circ \omega)(Z) = h(1/N)Z. \]

**4. An example of a Siegel family**

In this section, we shall give a concrete example of a Siegel family by means of theta constants.

Let \( g \) be a positive integer. For

\[ \mathbf{v} = \left[ \begin{array}{c} \mathbf{v}_u \\ \mathbf{v}_v \end{array} \right] \in \mathbb{Q}^{2g} \]
Remark 4.2

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with \(v_u, v_\ell \in \mathbb{Q}^g\), the theta constant \(\theta_v(Z)\) is given by

\[
\theta_v(Z) = \sum_{n \in \mathbb{Z}^g} e(\frac{1}{2}(n + v_u)^T Z(n + v_u) + (n + v_u)^T v_\ell) \quad (Z \in \mathbb{H}_g).
\]

It was shown by Igusa (see [3, theorem 2]) that \(\theta_v(Z)\) is identically zero if and only if every entry of the vector \(v\) is in \((1/2)\mathbb{Z}\) and \(e(2v_u^T v_\ell) = -1\). Let

\[
S_\pm = \left\{ a = \begin{bmatrix} a_u \\ a_\ell \end{bmatrix} \in \{0, 1/2\}^{2g} \mid e(2a_u^T a_\ell) = -1 \right\} \quad \text{and} \quad S_+ = \{0, 1/2\}^{2g} \backslash S_-.
\]

Now, let \(v \in \mathbb{Q}^{2g}\) with exact denominator \(N \geq 2\). We define

\[
\Theta_v(Z) = 2^{4N} e(-2^g N(2^g - 1)(2^g + 1)v_u^T v_\ell) \prod_{a \in S_-} \theta_{a-v}(Z) \prod_{b \in S_+} \theta_{b}(Z) \quad (Z \in \mathbb{H}_g)
\]

(see [7, definition 4.2]).

**Proposition 4.1.** The function \(\Theta_v(Z)\) depends only on \(\pm v \pmod{\mathbb{Z}^{2g}}\). Moreover, it belongs to \(\mathcal{F}_N\) and satisfies that

\[
\Theta_v(Z) = \Theta_{\sigma^g v}(Z)
\]

for every \(\sigma \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)\).

**Proof.** See [7, lemma 4.4]. \(\square\)

**Remark 4.2.** One can readily verify that if \(g \geq 2\), then \(\Theta_v(Z)\) is identically zero if and only if \(N = 2\).

**Theorem 4.3.** If \(r \in \mathbb{Q}^g\) with exact denominator \(N \geq 3\), then \(\{\Theta_{M(N r)}\}_{M \in \mathcal{V}_N}\) is a Siegel family of level \(N\).

**Proof.** For any \(\gamma \in \Gamma^1(N)\) we derive by proposition 4.1 that

\[
\Theta_{\gamma r}([0]) (Z) = \Theta_{[0]}(Z) \gamma = \Theta_{\gamma^* r}([0]) (Z) = \Theta_{[I_{2g} O_{2g} \nu I_{2g}]}([0]) (Z) = \Theta_{[0]}(Z).
\]

This shows that \(\Theta_{[0]}(Z)\) is modular for \(\Gamma^1(N)\). Furthermore, for any \(\nu \in (\mathbb{Z}/N\mathbb{Z})^\times\), by proposition 4.1 we see that

\[
\Theta_{[r]}(Z) [I_{2g} O_{2g} \nu I_{2g}] = \Theta_{[I_{2g} O_{2g} \nu I_{2g}]}([0]) (Z) = \Theta_{[0]}(Z).
\]

Thus, \(\Theta_{[r]}(Z)\) has rational Fourier coefficients, and hence \(\Theta_{[r]}(Z)\) belongs to \(\mathcal{F}_N^1(\mathbb{Q})\).

For each \(M \in \mathcal{V}_N\), we can take an element

\[
\gamma_M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

of \(M_{2g}(\mathbb{Z})\) such that \(\tilde{\gamma}_M \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\) and

\[
M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}
\]
by lemma 3.4. Then, by the proof of theorem 3.5, the family \( \{ \Theta_{[r]}(Z)^{\gamma_M} \}_{M \in V_N} \) turns out to be a Siegel family of level \( N \). Lastly, we obtain by proposition 4.1 that

\[
\Theta_{[r]}(Z)^{\gamma_M} = \Theta_{[r]}(Z)^{\gamma_M} = \Theta(\gamma_T)^{\gamma_M}(Z) = \Theta_M(Nr)(Z).
\]

This completes the proof.

5. Special values associated with a Siegel family

As an application of a Siegel family of level \( N \) we shall construct a number associated with each ray class modulo \( N \) of a CM-field.

Let \( n \) be a positive integer, \( K \) be a CM-field with \( [K : \mathbb{Q}] = 2n \) and \( \{ \varphi_1, \ldots, \varphi_n \} \) be a set of embeddings of \( K \) into \( \mathbb{C} \) such that \( (K, \{ \varphi_i \}_{i=1}^n) \) is a CM-type. We fix a finite Galois extension \( L \) of \( \mathbb{Q} \) containing \( K \), and set

\[
S = \{ \sigma \in \text{Gal}(L/\mathbb{Q}) \mid \sigma|_K = \varphi_i \text{ for some } i \in \{1, 2, \ldots, n\} \},
\]

\[
S^* = \{ \sigma^{-1} \mid \sigma \in S \},
\]

\[
H^* = \{ \gamma \in \text{Gal}(L/\mathbb{Q}) \mid \gamma S^* = S^* \}.
\]

Let \( K^* \) be the subfield of \( L \) corresponding to the subgroup \( H^* \) of \( \text{Gal}(L/\mathbb{Q}) \), and let \( \{ \psi_1, \ldots, \psi_g \} \) be the set of all embeddings of \( K^* \) into \( \mathbb{C} \) arising from the elements of \( S^* \). Then we know that \( (K^*, \{ \psi_j \}_{j=1}^g) \) is a primitive CM-type and

\[
K^* = \mathbb{Q}\left( \sum_{i=1}^n a^{\varphi_i} \mid a \in K \right)
\]

(see [12, §8.3, proposition 28]). We call this CM-type \( (K^*, \{ \psi_j \}_{j=1}^g) \) the reflex of \( (K, \{ \varphi_i \}_{i=1}^n) \).

Using this CM-type we define an embedding

\[
\Psi : K^* \to \mathbb{C}^g
\]

\[
a \mapsto \left[ \begin{array}{c}
a^{\psi_1} \\
\vdots \\
a^{\psi_g}
\end{array} \right].
\]

For each purely imaginary element \( c \) of \( K^* \) we associate an \( \mathbb{R} \)-bilinear form

\[
E_c : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}
\]

\[
(u, v) \mapsto \sum_{j=1}^g c^{\psi_j} (u_j \bar{v}_j - \bar{u}_j v_j)
\]

\[
\begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix}
\]

Then, one can readily check that

\[
E_c(\Psi(a), \Psi(b)) = \text{Tr}_{K^*/\mathbb{Q}}(cab) \quad \text{for all } a, b \in K^* \quad (5.1)
\]

by using the fact \( a^{\psi_j} = \bar{a}^{\psi_j} \) for all \( a \in K^* \) (\( 1 \leq j \leq g \)).
Assumption 5.1. In what follows we assume the following conditions.

(i) $(K^*)^* = K$.

(ii) There is a purely imaginary element $\xi$ of $K^*$ and a $\mathbb{Z}$-basis $\{a_1, \ldots, a_{2g}\}$ of the lattice $\mathcal{P}(\mathcal{O}_{K^*})$ in $\mathbb{C}^g$ for which

$$[E_\xi(a_i, a_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$ 

In this case, we say that the complex torus $(\mathbb{C}^g / \mathcal{P}(\mathcal{O}_{K^*}), E_\xi)$ is a principally polarized abelian variety with a symplectic basis $\{a_1, \ldots, a_{2g}\}$. See [12, § 6.2].

(iii) $f = NO_K$ for an integer $N \geq 2$.

Remark 5.2. The assumption 5.1(i) is equivalent to saying that $(K, \{\varphi_i\}_{i=1}^n)$ is a primitive CM-type, namely, the abelian varieties of this CM-type are simple [12, § 8.2, proposition 26].

By assumption 5.1(i) one can define a group homomorphism

$$g: K^* \to (K^*)^*$$

$$d \mapsto \prod_{i=1}^n d^{\varphi_i},$$

and extend it continuously to the homomorphism $g: K_{\mathfrak{a}}^* \to (K^*)_{\mathfrak{a}}^*$ of idele groups.

It is also known that for each fractional ideal $\mathfrak{a}$ of $K$ there is a fractional ideal $\mathcal{G}(\mathfrak{a})$ of $K^*$ such that [12, § 8.3]

$$\mathcal{G}(\mathfrak{a})\mathcal{O}_L = \prod_{i=1}^n (\mathfrak{a}\mathcal{O}_L)^{\varphi_i}.$$ 

Let $\mathcal{C}$ be a given ray class in $\text{Cl}(f)$. Take any integral ideal $\mathfrak{c}$ in $\mathcal{C}$, and let

$$\mathcal{N}(\mathfrak{c}) = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{c}) = |\mathcal{O}_K/\mathfrak{c}|.$$ 

Lemma 5.3. $(\mathbb{C}^g / \mathcal{P}(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ is also a principally polarized abelian variety.

Proof. It follows from (5.1) that

$$E_\xi(\mathcal{N}(\mathfrak{c})^{-1}) = \text{Tr}_{K^*/\mathbb{Q}}(\mathcal{N}(\mathfrak{c})\mathcal{G}(\mathfrak{c})^{-1}\mathcal{G}(\mathfrak{c})^{-1})$$

$$= \text{Tr}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathcal{O}_{K^*}))$$

$$= E_\xi(\mathcal{G}(\mathcal{O}_{K^*}), \mathcal{G}(\mathcal{O}_{K^*}))$$

$$\subseteq \mathbb{Z}$$

because $E_\xi$ is a Riemann form on $\mathbb{C}^g / \mathcal{P}(\mathcal{O}_{K^*})$. Thus, $E_{\xi\mathcal{N}(\mathfrak{c})}$ defines a Riemann form on $\mathbb{C}^g / \mathcal{P}(\mathcal{G}(\mathfrak{c})^{-1})$.

Now, let $\{b_1, \ldots, b_{2g}\}$ be a symplectic basis of the abelian variety $(\mathbb{C}^g / \mathcal{P}(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ so that

$$\mathcal{P}(\mathcal{G}(\mathfrak{c})^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}b_j$$

and

$$[E_{\xi\mathcal{N}(\mathfrak{c})}(b_i, b_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix},$$
By taking the determinant we get

$$\mathcal{E} = \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_g \end{bmatrix}$$

is a $g \times g$ diagonal matrix for some positive integers $\varepsilon_1, \ldots, \varepsilon_g$ satisfying $\varepsilon_1 \cdots \varepsilon_g$. Furthermore, let $b_1, \ldots, b_{2g}$ be elements of $G(\mathcal{O})^{-1}$ such that $b_j = \Psi(b_j)$ ($1 \leq j \leq 2g$). Since $\mathcal{O}_K^* \subseteq G(\mathcal{O})^{-1}$, we have

$$[a_1 \cdots a_{2g}] = [b_1 \cdots b_{2g}] \alpha \quad \text{for some } \alpha \in M_{2g}(\mathbb{Z}) \cap \text{GL}_{2g}(\mathbb{Q}), \quad (5.2)$$

and hence

$$\begin{bmatrix} a_1^\psi & \cdots & a_{2g}^\psi \\ \vdots & \ddots & \vdots \\ a_1^\psi & \cdots & a_{2g}^\psi \end{bmatrix} = \begin{bmatrix} b_1^\psi & \cdots & b_{2g}^\psi \\ \vdots & \ddots & \vdots \\ b_1^\psi & \cdots & b_{2g}^\psi \end{bmatrix} \alpha.$$

Taking determinant and squaring gives rise to the identity

$$\Delta_{K^*/\mathbb{Q}}(a_1, \ldots, a_{2g}) = \Delta_{K^*/\mathbb{Q}}(b_1, \ldots, b_{2g}) \det(\alpha)^2.$$

It then follows that

$$\det(\alpha)^2 = \frac{|\Delta_{K^*/\mathbb{Q}}(a_1, \ldots, a_{2g})|}{|\Delta_{K^*/\mathbb{Q}}(b_1, \ldots, b_{2g})|} = \frac{d_{K^*/\mathbb{Q}}(\mathcal{O}_K^*)}{d_{K^*/\mathbb{Q}}(G(\mathcal{O}))^{-1}} = N_{K^*/\mathbb{Q}}(G(\mathcal{O}))^2 = N_{K^*/\mathbb{Q}}(G(\mathcal{O}) \bar{G}(\mathcal{O})) = N(\mathcal{O})^{2g}, \quad (5.3)$$

where $d_{K^*/\mathbb{Q}}$ stands for the discriminant of a fractional ideal of $K^*$ [9, ch. III, proposition 13]. Furthermore, we deduce by (5.2) that

$$N(c) \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} = [N(c)E_{\xi}(a_i, a_j)]_{1 \leq i, j \leq 2g} = [E_{\xi N(c)}(a_i, a_j)]_{1 \leq i, j \leq 2g} = E_T[E_{\xi N(c)}(b_i, b_j)]_{1 \leq i, j \leq 2g} \alpha = E_T \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix} \alpha.$$

By taking the determinant we get $N(c)^{2g} = \det(\alpha)^2(|\varepsilon_1| \cdots |\varepsilon_g|)^2$, which, by (5.3), yields that $\varepsilon_1 = \cdots = \varepsilon_g = 1$, and so $\mathcal{E} = I_g$. Therefore, $(\mathbb{C}^g/\mathbb{Q}(\mathcal{O})^{-1}), E_{\xi N(c)}$ becomes a principally polarized abelian variety.
In this section we shall show that the value $h$ which belongs to $\tilde{\mathcal{C}}$.

**Proof.** Let $b_{1}, \ldots, b_{2g}$ be elements of $G(\mathbf{c})^{-1}$ such that $b_{j} = \Psi(b_{j}) (1 \leq j \leq 2g)$. We then have

$$[a_{1} \cdots a_{2g}] = [b_{1} \cdots b_{2g}] \alpha \quad \text{for some} \quad \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \cap \text{GSp}_{2g}(\mathbb{Q}).$$

(5.4)

Since $\nu(\alpha) = N(\mathbf{c})$ is relatively prime to $N$, the reduction $\tilde{\alpha}$ of $\alpha$ modulo $N$ belongs to $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $Z_{\mathbf{c}}^{*}$ be the CM-point associated with the symplectic basis $\{b_{1}, \ldots, b_{2g}\}$, namely

$$Z_{\mathbf{c}}^{*} = [b_{g+1} \cdots b_{2g}]^{-1} [b_{1} \cdots b_{g}]$$

which belongs to $\mathbb{H}_{g}$ [1, proposition 8.1.1].

**Definition 5.4.** Let $\{h_{M}(Z)\}_{M} \in \mathcal{S}_{N}$. For a given ray class $\mathcal{C} \in \text{Cl}(f)$ we define

$$h_{1}(\mathcal{C}) = h_{(1/N)[B]}(Z_{\mathbf{c}}^{*}).$$

**Remark 5.5.** Here, the index matrix

$$(1/N) \begin{bmatrix} B \\ D \end{bmatrix}$$

is obtained using the fact that

$$\left( \begin{bmatrix} O_{g} & -I_{g} \\ I_{g} & O_{g} \end{bmatrix} \right)^{T} \alpha = \begin{bmatrix} B^{T} & D^{T} \\ -A^{T} & -C^{T} \end{bmatrix}.$$

**6. Well-definedness of $h_{1}(\mathcal{C})$**

In this section we shall show that the value $h_{1}(\mathcal{C})$ given in definition 5.4 depends only on the ray class $\mathcal{C}$, and hence it is independent of the choice of a symplectic basis and an integral ideal in $C$.

**Proposition 6.1.** The value $h_{1}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{b_{1}, \ldots, b_{2g}\}$ of $(\mathbb{C}^{g}/\Psi(G(\mathbf{c}))^{-1}, E_{\xi N(\mathbf{c}))}$.

**Proof.** Let $\{\hat{b}_{1}, \ldots, \hat{b}_{2g}\}$ be another symplectic basis of $(\mathbb{C}^{g}/\Psi(G(\mathbf{c}))^{-1}, E_{\xi N(\mathbf{c}))}.$ Thus,

$$[\hat{b}_{1} \cdots \hat{b}_{2g}] = [b_{1} \cdots b_{2g}] \beta \quad \text{for some} \quad \beta = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \text{GL}_{2g}(\mathbb{Z}).$$

(6.1)

We then derive

$$\begin{bmatrix} O_{g} & -I_{g} \\ I_{g} & O_{g} \end{bmatrix} = [E_{\xi N(\mathbf{c}}(\hat{b}_{i}, \hat{b}_{j})]_{1 \leq i, j \leq 2g}$$

$$= \beta^{T} [E_{\xi N(\mathbf{c}}(b_{i}, b_{j})]_{1 \leq i, j \leq 2g} \beta$$

$$= \beta^{T} \begin{bmatrix} O_{g} & -I_{g} \\ I_{g} & O_{g} \end{bmatrix} \beta.$$
which shows that $\beta \in \text{Sp}_{2g}(Z)$. Since

$$[\alpha_1 \ldots \alpha_{2g}] = [\beta_1 \ldots \beta_{2g}] = \begin{bmatrix} \tilde{b}_1 & \ldots & \tilde{b}_{2g} \end{bmatrix} \beta^{-1} \alpha$$

by (5.4) and (6.1), the special value obtained by $\{\tilde{b}_1, \ldots, \tilde{b}_{2g}\}$ is

$$h_{(1/N)\beta^{-1}[B]_D}(\tilde{Z}_\xi^*),$$

where $\tilde{Z}_\xi^*$ is the CM-point corresponding to $\{\tilde{b}_1, \ldots, \tilde{b}_{2g}\}$. On the other hand, we attain that

$$\tilde{Z}_\xi^* = \begin{bmatrix} \tilde{b}_{g+1} & \ldots & \tilde{b}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{b}_1 & \ldots & \tilde{b}_g \end{bmatrix}$$

Thus, we deduce that

$$h_{(1/N)\beta^{-1}[B]_D}(\tilde{Z}_\xi^*) = h_{(1/N)\beta^{-1}[B]_D}(\beta^T(Z_\xi^*)) \quad \text{by (6.2)}$$

$$= (h_{(1/N)\beta^{-1}[B]_D}(Z))^\beta_{|Z=Z_\xi^*}$$

$$= h_{(1/N)\beta^{T}\beta^{-1}[B]_D}(Z_\xi^*) \quad \text{by the property (S3) of } \{h_M(Z)\}_M$$

$$= h_{(1/N)[B]_D}(Z_\xi^*).$$

This proves that the value $h_{\xi}(C)$ is independent of the choice of a symplectic basis of $(\mathcal{C}/\mathcal{P}S^{-1}(\xi^{-1}), E_{\xi N(\xi)})$.

**Remark 6.2.** One can analogously readily show that $h_{\xi}(C)$ does not depend on the choice of a symplectic basis $\{\alpha_1, \ldots, \alpha_{2g}\}$ of $(\mathcal{C}/\mathcal{P}S(\mathcal{O}_K), E_\mathcal{C})$.

**Proposition 6.3.** $h_{\xi}(C)$ does not depend on the choice of an integral ideal $\xi$ in $C$.

**Proof.** Let $\xi'$ be another integral ideal in the class $C$, and hence

$$\xi' \xi^{-1} = (1 + a)\mathcal{O}_K \quad \text{for some } a \in \mathfrak{a}^{-1},$$

(6.3)
where \( a \) is an integral ideal of \( K \) relatively prime to \( \mathfrak{f} \). Since \( 1 \in \mathfrak{c}^{-1} \) and \( (1 + a) \in \mathfrak{c}'\mathfrak{c}^{-1} \subseteq \mathfrak{c}^{-1} \), we get \( a \in \mathfrak{c}^{-1} \). Thus, we derive that
\[
\mathfrak{a} \mathfrak{c} \subseteq \mathfrak{f} \cap \mathfrak{a} \quad \text{by the facts that } a \in \mathfrak{f}^{-1} \text{ and } a \in \mathfrak{c}^{-1}
\subseteq \mathfrak{f} \cap \mathfrak{a} \\
= \mathfrak{f} \mathfrak{a} \quad \text{because } \mathfrak{f} \text{ and } \mathfrak{a} \text{ are relatively prime},
\]
from which it follows that \( a \in \mathfrak{f}^{-1} \). Using the fact that \( \mathfrak{f} = \mathcal{O}_K \) yields
\[
\mathfrak{g}(1 + a) = \prod_{i=1}^{n}(1 + a)^{r_i} \subseteq K^* \cap \prod_{i=1}^{n}(1 + N(\mathfrak{c}^{-1} \mathcal{O}_L)) \subseteq K^* \cap (1 + N\mathcal{G}(\mathfrak{c}^{-1} \mathcal{O}_L)) \text{.}
\]

Furthermore, we get that
\[
c \subseteq \mathfrak{a}
\]
\[
\text{from which it follows that } a \in \mathfrak{c}^{-1}.
\]

Let
\[
b_j' = \mathfrak{g}(1 + a)^{-1}b_j \quad \text{and} \quad b_j' = \Psi(b_j') \quad (1 \leq j \leq 2g).
\]

We know that \( \{b_1', \ldots, b_{2g}'\} \) is a \( \mathbb{Z} \)-basis of the lattice \( \Psi(\mathcal{G}(\mathfrak{c}')^{-1}) \) in \( \mathbb{C}^g \) and
\[
b_j' = T b_j \quad \text{with } T = \begin{bmatrix}
(\mathfrak{g}(1 + a)^{-1})^{\psi_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (\mathfrak{g}(1 + a)^{-1})^{\psi_g}
\end{bmatrix}. \quad (6.6)
\]

Furthermore, we get that
\[
[E_{\xi \mathcal{N}(\mathfrak{c}')} (b_i', b_j')]_{1 \leq i, j \leq 2g}
= \begin{bmatrix}
\text{Tr}_{K^*/Q}(\xi \mathcal{N}(\mathfrak{c}')b_i' b_j')
\end{bmatrix}_{1 \leq i, j \leq 2g} \quad \text{by (5.1)}
= \begin{bmatrix}
\text{Tr}_{K^*/Q}(\xi \mathcal{N}(\mathfrak{c}')\mathfrak{g}(1 + a)^{-1}b_i\mathfrak{g}(1 + a)^{-1}b_j)
\end{bmatrix}_{1 \leq i, j \leq 2g} \quad \text{by (6.5)}
= \begin{bmatrix}
\text{Tr}_{K^*/Q}(\xi \mathcal{N}(\mathfrak{c}')\mathfrak{g}(1 + a)^{-1}b_i b_j)
\end{bmatrix}_{1 \leq i, j \leq 2g} \quad \text{by (6.3) and the fact that } N_{K/Q}(1 + a) > 0
= \begin{bmatrix}
O_g & -I_g \\
I_g & O_g
\end{bmatrix}. \quad (6.7)
\]

Thus, \( \{b_1', \ldots, b_{2g}'\} \) is a symplectic basis of \( (\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}')^{-1}), E_{\xi \mathcal{N}(\mathfrak{c}')} ) \), and its associated CM-point \( Z_{\bar{c}}^* \) is given by
\[
Z_{\bar{c}}^* = \begin{bmatrix}
b_{g+1}' & \cdots & b_{2g}'
\end{bmatrix}^{-1} \begin{bmatrix}
b_1' & \cdots & b_g'
\end{bmatrix}
= \begin{bmatrix}
T b_{g+1} & \cdots & T b_{2g}
\end{bmatrix}^{-1} \begin{bmatrix}
T b_1 & \cdots & T b_g
\end{bmatrix} \quad \text{by (6.6)}
= Z_{\bar{c}}^*. \quad (6.7)
\]

Let \( \alpha = [a_{ij}] \), \( \alpha' = [a_{ij}'] \in M_{2g}(\mathbb{Z}) \) such that
\[
[a_1 \cdots a_{2g}] = [b_1 \cdots b_{2g}] \alpha = [b_1' \cdots b_{2g}'] \alpha'. \quad (6.8)
\]
For each $1 \leq i \leq 2g$ we obtain that
\[
\sum_{j=1}^{2g} a'_j b_j = g(1 + a) \sum_{j=1}^{2g} a'_j b'_j \quad \text{by (6.5)}
\]
\[= a_i g(1 + a) \quad \text{by (6.8)}
\]
\[\in a_i (1 + NG(c)^{-1}) \quad \text{by (6.4)}
\]
\[\subseteq a_i + NG(c)^{-1} \quad \text{because } a_i \in \mathcal{O}_K
\]
\[= \sum_{j=1}^{2g} a_j b_j + N \sum_{j=1}^{2g} Z b_j \quad \text{by (6.8)}.
\]

This yields $\alpha \equiv \alpha' \pmod{N \cdot M_{2g}(\mathbb{Z})}$, and hence
\[
(1/N)\alpha \equiv (1/N)\alpha' \pmod{M_{2g}(\mathbb{Z})}.
\]

Now, the result follows from (6.7), (6.9) and the property (S2) of $\{h_M(Z)\}_M$. \qed

7. Galois actions on $h_1(C)$

Finally, we shall show that if $h_1(C)$ is finite, then it lies in the ray class field $K_1$ and satisfies the natural transformation formula under the Artin reciprocity map for $f$.

Let $r: K^* \to M_{2g}(\mathbb{Q})$ be the regular representation with respect to the ordered basis $\{a_1, \ldots, a_{2g}\}$ of $K^*$ over $\mathbb{Q}$ given by
\[
a \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} = r(a) \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \quad (a \in K^*).
\]

Then it can be extended to the map $r: (K^*)_\Lambda \to M_{2g}(\mathbb{Q}_\Lambda)$ of adele rings.

Lemma 7.1 (Shimura’s reciprocity law). Let $f$ be an element of $\mathcal{F}$ that is finite at $Z^*_\epsilon$.

(i) The special value $f(Z^*_\epsilon)$ lies in $K_{ab}$.

(ii) For every $s \in K^*_\Lambda$ we have $r(g(s)) \in G_{K^+}$ and
\[
f(Z^*_\epsilon)^{[s,K]} = f^r(r(g(s)^{-1}))(Z^*_\epsilon).
\]

Proof. See [13, lemma 9.5 and theorem 9.6]. \qed

Theorem 7.2. If $h_1(C)$ is finite, then it belongs to $K_1$. Furthermore, it satisfies
\[h_1(C)^{\sigma_1(D)} = h_1(CD) \quad \text{for every } D \in \text{Cl}(f),\]

where $\sigma_1$ is the Artin reciprocity map for $f$.

Proof. Since $h_1(C)$ belongs to $K_{ab}$ by lemma 7.1(i), there is a sufficiently large positive integer $M$ so that $N|M$ and $h_1(C) \in K_m$ with $m = M\mathcal{O}_K$. Take an integral
ideal $\mathfrak{d}$ in $\mathcal{D}$ relatively prime to $\mathfrak{m}$ by using the surjectivity of the natural map $\text{Cl}(\mathfrak{m}) \to \text{Cl}(\mathfrak{f})$. Let $\{\mathbf{d}_1, \ldots, \mathbf{d}_{2g}\}$ be a symplectic basis of the principally polarized abelian variety $(\mathbb{C}/\Psi(\mathcal{G}(\mathfrak{d})^{-1}), E_{\mathbb{C}N(\mathfrak{d})})$, and let $\mathbf{d}_1, \ldots, \mathbf{d}_{2g}$ be elements of $\mathcal{G}(\mathfrak{d})^{-1}$ such that $\mathbf{d}_j = \Psi(\mathbf{d}_j) (1 \leq j \leq 2g)$. Since $\mathcal{G}(\mathfrak{c})^{-1} \subseteq \mathcal{G}(\mathfrak{d})^{-1}$, we get

$$[\mathbf{b}_1 \ldots \mathbf{b}_{2g}] = [\mathbf{d}_1 \ldots \mathbf{d}_{2g}] \delta$$

for some $\delta \in M_{2g} (\mathbb{Z}) \cap \text{GL}_{2g}(\mathbb{Q})$. (7.2)

We then have that

$$\begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} = \left[ E_{\mathbb{C}N(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j) \right]_{1 \leq i, j \leq 2g}$$

$$= \delta^T \left[ E_{\mathbb{C}N(\mathfrak{c})}(\mathbf{d}_i, \mathbf{d}_j) \right]_{1 \leq i, j \leq 2g} \delta$$

$$= \delta^T \left[ \mathcal{N}(\mathfrak{c}) \mathcal{N}(\mathfrak{d})^{-1} E_{\mathbb{C}N(\mathfrak{d})}(\mathbf{d}_i, \mathbf{d}_j) \right]_{1 \leq i, j \leq 2g} \delta$$

$$= \mathcal{N}(\mathfrak{d})^{-1} \delta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \delta.$$ (7.3)

This claims that

$$\delta \in M_{2g}(\mathbb{Z}) \cap G_+$$

with $\nu(\delta) = \mathcal{N}(\mathfrak{d})$. (7.4)

Furthermore, if we let $Z_{\mathcal{C}^b}$ be the CM-point associated with $\{\mathbf{d}_1, \ldots, \mathbf{d}_{2g}\}$, then we obtain

$$Z_{\mathfrak{c}^b}^* = (\delta^{-1})^T (Z_{\mathfrak{c}}^*)$$

in a similar way to the argument in the proof of proposition 6.1.

Let $s = (s_p)_p$ be an idele of $K$ such that

$$s_p = \begin{cases} 1 & \text{if } p | M, \\ p & \text{if } p \nmid M \end{cases}$$

(7.5)

If we set $\mathcal{D}$ to be the ray class in $\text{Cl}(\mathfrak{m})$ containing $\mathfrak{d}$, then by (7.5) we attain

$$[s, K]_{|K_m} = \sigma_m(\mathcal{D}),$$

$$\mathfrak{g}(s)^{-1} \mathcal{O}_{K^*} = \mathcal{G}(\mathfrak{d})^{-1}$$

for all rational primes $p$. (7.6)

(7.7)

It then follows from (7.1)–(7.7) that for every rational prime $p$, the entries of each of the vectors

$$r(\mathfrak{g}(s)^{-1})_p \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix} \quad \text{and} \quad (\delta^{-1})^T \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix}$$

form a basis of $\mathcal{G}(\mathfrak{d})^{-1} = \mathcal{G}(\mathfrak{c})^{-1} \mathcal{G}(\mathfrak{d})^{-1}$. So, there exists a matrix $u = (u_p)_p \in \prod_p \text{GL}_{2g}(\mathbb{Z}_p)$ satisfying

$$r(\mathfrak{g}(s)^{-1}) = u(\delta^{-1})^T.$$ (7.8)

Since $\delta^T$ and

$$\begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\mathfrak{d}) I_g \end{bmatrix}$$

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by (7.5), we get

\[ u \sigma \]

Then

\[ D \]

owing to the surjectivity of the reduction \( \text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/\mathbb{M}) \). Since

\[ r(g(s)^{-1})_p = I_{2g} \quad \text{for all } p \mid M \]

by (7.5), we get \( u_p = \delta^T \) for all \( p \mid M \) by (7.8). Hence, we deduce using (7.9) that

\[ u_p \gamma^{-1} \equiv \left[ \begin{array}{cc} I_g & O_g \\ O_g & N(\delta)I_g \end{array} \right] \quad \text{(mod } M \cdot M_{2g}(\mathbb{Z}_p)) \quad \text{for all rational primes } p. \quad (7.10) \]

On the other hand, we have by (5.4) and (7.2) that

\[
\begin{bmatrix}
a_1 & \cdots & a_{2g}
\end{bmatrix}
= \left( \begin{bmatrix}
b_1 & \cdots & b_{2g}
\end{bmatrix} \right) \delta^{-1}(\delta \alpha)
= \left[ \begin{array}{cc}
d_1 & \cdots \\
c_2g
\end{array} \right] (\delta \alpha).
\]

(7.11)

Letting

\[ \alpha = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \]

we induce the following:

\[ h_1(\mathcal{C})^{\sigma_m(\mathcal{D})} = h_1(\mathcal{C})^{[s,K]} \quad \text{by (7.6)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D](Z^T) \quad \text{by definition 5.4} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{by lemma 7.1(ii)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{by (7.8)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{by (7.9) and (S3)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{by (S3)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{by (7.9) and (S2)} \]

\[ = h_{(1/N)^{\frac{1}{2}}}[B_D] \quad \text{due to the fact that } \delta \in G_+ \text{ and by (A1)} \]

\[ = h_1(\mathcal{CD}) \quad \text{by (7.4), (7.11) and definition 5.4.} \]

In particular, suppose that \( \mathfrak{d} = dO_K \) for some \( d \in O_K \) such that \( d \equiv 1 \pmod{f} \).

Then \( \mathcal{D} \) is the identity class of \( \text{Cl}(f) \), and so the above observation implies that \( \sigma_m(\mathcal{D}) \) leaves \( h_1(\mathcal{C}) \) fixed. Therefore, we conclude that \( h_1(\mathcal{C}) \) lies in \( K_f \).  \( \square \)
Corollary 7.3. Let $H$ be a subgroup of $\text{Cl}(f)$ defined by

$$H = \langle \mathcal{D} \in \text{Cl}(f) \mid \mathcal{D} \text{ contains an integral ideal } \mathfrak{d} \text{ of } K \text{ for which } \mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \text{ for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}} \rangle,$$

and let $K_f^H$ be the fixed field of $H$. If $h_f(C)$ is finite, then it belongs to $K_f^H$.

Proof. Let $C_0$ be the identity class of $\text{Cl}(f)$. Since $h_f(C_0) \in K_f$ by theorem 7.2, $K(h_f(C_0))$ is a Galois extension of $K$ as a subfield of $K_f$. Furthermore, since $h_f(C_0) \sigma_f(C) = h_f(C_0)$ by theorem 7.2, $K(h_f(C_0))$ contains $h_f(C)$. Thus, it suffices to show that $h_f(C_0)$ belongs to $K_f^H$.

To this end, let $\mathcal{D}$ be an element of $\text{Cl}(f)$ containing an integral ideal $\mathfrak{d}$ of $K$ for which

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \text{ for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}.$$

Now that

$$(\mathcal{G}/\Psi(\mathcal{G}(\mathfrak{d})^{-1}), E_{\mathcal{G}(\mathfrak{d})}) = (\mathcal{G}/\Psi(\mathfrak{g}(d)^{-1}\mathcal{O}_{K^*}), E_{\mathcal{G}(\mathfrak{d})}) = (\mathcal{G}/\Psi(\mathfrak{d}\mathcal{O}_{K^*})^{-1}, E_{\mathcal{G}(\mathfrak{d})}) = (\mathcal{G}/\Psi(\mathfrak{d}\mathcal{O}_{K^*})^{-1}, E_{\mathcal{G}(\mathfrak{d})})$$

we obtain

$$h_f(C_0)^{\sigma_i(\mathcal{D})} = h_f(\mathcal{D}) = h_f([d\mathcal{O}_K])$$

where $[a]$ is the ray class containing $a$ for a fractional ideal $a$ of $K$. Moreover, since $\mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}$, we obtain

$$h_f([d\mathcal{O}_K]) = h_f([\mathcal{O}_K]) = h_f(C_0)$$

analogously to the proof of proposition 6.3. This proves that $h_f(C_0)$ belongs to $K_f^H$. \qed

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References


