Electronic structure of a magnetic quantum ring

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I. INTRODUCTION

For the last decades, there have been a great deal of interests in various confined systems of a two-dimensional electron gas (2DEG), which are usually fabricated on a mesoscopic scale through built-in electrostatic potentials. Of special interest is the use of these systems such as quantum wires, quantum dots, and quantum rings for future electronic and optical devices. Recently, with the application of spatially inhomogeneous magnetic fields, there have been proposed several alternative magnetic structures, such as magnetic quantum dots using a scanning tunneling microscope lithographic technique,\(^1\) magnetic superlattices by the patterning of ferromagnetic materials integrated by semiconductors,\(^2\) type-II superconducting materials deposited on conventional heterostructures,\(^3\) and nonplanar 2DEG systems grown by a molecular-beam epitaxy.\(^4\) Although these magnetic systems have a close analogy to conventional quantum structures confined by electrostatic potentials, they exhibit quite distinctive transport behavior due to different electronic structures.\(^5–11\)

For a 2DEG placed in inhomogeneous magnetic fields, which vary linearly across the plane, electron flow was shown to take place only in the direction perpendicular to the field gradient.\(^5\) A number of groups studied the quantum transport of a 2DEG in modulated magnetic fields along a particular direction.\(^7–9\) This system was also realized experimentally, and the predicted semiclassical commensurability effect was observed.\(^10\) A similar study on the magnetic structure was performed using the composite-fermion theory,\(^12\) the magnetotransport of composite fermions was calculated for the fractional quantum Hall system with a spatially varying electrostatic potential, for example, an antidot lattice.\(^13,14\) Since the effective magnetic field felt by composite fermions varies with the density of composite fermions, the antidot lattice modulates the effective magnetic field. Very recently, the electronic structure of a magnetic quantum dot with zero magnetic-field within a circular disk and constant field outside it was investigated.\(^11\) In this system, current-carrying states so-called magnetic edge states were found to exist along the boundary between two different magnetic domains, similar to the conventional edge states formed near sample edges. However, the magnetic edge state was shown to have quite different properties from the conventional one; a notable feature is that for a small conductor with a magnetic quantum dot at the center, magnetocconductances have aperiodic oscillations instead of the well-known Aharonov-Bohm-type periodic oscillations. It is interesting to see the formation of magnetic edge states in other magnetic quantum structures like magnetic quantum rings, which have different magnetic fields in the ring region and outside it. Since the magnetic edge states are related to the characteristics of the electronic-structure, the related physical properties are expected to be different from those of the magnetic quantum dot.

In this paper, we investigate the electronic structure and the magnetic edge states of magnetic quantum rings. We calculate exactly single electron energies by neglecting electron-electron interactions, and find that the energy spectra critically depend on the number of missing magnetic flux quanta rather than the geometry of the structure or the field abruptness. In contrast to the magnetic quantum dot,\(^11\) the angular momentum transitions in the ground state are found to occur as the magnetic field varies. The classical trajectories of the quantum states are obtained by using the general rules, which are derived from the energy and angular momentum conservation laws.

II. MAGNETIC QUANTUM RING

We first consider the 2DEG confined in a magnetic quantum ring formed by inhomogeneous magnetic fields; the magnetic field perpendicular to the plane is zero within a circular ring and constant \(B\) outside it. When electron-
\[
\frac{1}{2m^*}(\vec{p} + e\vec{A})^2 \psi(r) = E \psi(r),
\]

where \( m^* \) is the effective mass of an electron and \( e \) is the absolute value of the electron charge. In polar coordinates \((r, \theta)\) on the plane, the vector potential \( \vec{A} \) is chosen in a symmetric gauge such as

\[
\vec{A} = \begin{cases} 
\frac{1}{2} Br & (r < r_1) \\
\frac{1}{2} Br_1^2 & (r_1 < r < r_2) \\
\frac{1}{2} Br - \frac{1}{2r} B(r_2^2 - r_1^2) & (r_2 < r). 
\end{cases}
\]

Then, the wave functions are separable, i.e., \( \psi_{nm}(r) = R_{nm}(r) e^{im\theta} \), where \( m \) is the angular momentum quantum number and \( n(=0,1,2,\ldots) \) is the number of nodes in \( R_{nm}(r) \), and the equation for the radial part is written as

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + 2(E - V_{\text{eff}}) \right] R_{nm}(r) = 0.
\]

Here the effective potential \( V_{\text{eff}} \) is expressed as

\[
V_{\text{eff}} = \begin{cases} 
\frac{1}{2} r^2 + \frac{m_{\text{eff},1}^2}{2r^2} + m_{\text{eff},1} & (r < r_1) \\
\frac{m_{\text{eff},2}^2}{2r^2} & (r_1 < r < r_2), \\
\frac{1}{2} r^2 + \frac{m_{\text{eff},3}^2}{2r^2} + m_{\text{eff},3} & (r > r_2)
\end{cases}
\]

where \( m_{\text{eff},1} \), \( m_{\text{eff},2} \), and \( m_{\text{eff},3} \) are defined as \( m, m+s_1, \) and \( m-S \), respectively. All quantities are expressed in dimensionless units by letting \( \hbar \omega_0 = \hbar eB/(2m^*) \) and the inverse length \( \beta = \sqrt{m^*\omega_0/\hbar} \) to be 1. The dimensionless parameter \( S = (s_2 - s_1) \) represents the number of missing flux quanta, where \( s_1 = \pi r_1^2 B/\phi_0 \), \( s_2 = \pi r_2^2 B/\phi_0 \), and \( \phi_0 = (\hbar e) \) is the flux quantum. In these units \( \hbar^2/m^* = \hbar \omega_0/\beta^2 \rightarrow 1 \), \( r_1 \rightarrow \sqrt{s_1} \), and \( r_2 \rightarrow \sqrt{s_2} \), where \( r_1 \) and \( r_2 \) are the inner and outer radii of the ring, respectively. The solutions for \( R_{nm}(r) \) are found to be

\[
R_{nm}(r) = \begin{cases} 
C_1 J_{|m_{\text{eff},1}|} e^{-r^2/2M(a_1,b_1;r^2)} & (r < r_1) \\
C_2 J_{|m_{\text{eff},2}|} (\sqrt{2E}r) + C_3 N_{|m_{\text{eff},2}|} (\sqrt{2E}r) & (r_1 < r < r_2), \\
C_4 e^{r|m_{\text{eff},3}|} e^{-r^2/2U(a_2,b_2;r^2)} & (r > r_2)
\end{cases}
\]

where \( a_1 = (|m_{\text{eff},1}| + 1 - E + m_{\text{eff},1})/2 \), \( b_1 = |m_{\text{eff},1}| + 1 \), \( a_2 = (|m_{\text{eff},3}| + 1 - E + m_{\text{eff},3})/2 \), and \( b_2 = |m_{\text{eff},3}| + 1 \). Here, \( J \) and \( N \) denote the Bessel functions, and \( M \) and \( U \) are the confluent hypergeometric functions. The eigenenergies are determined by the continuity of the wave functions and their derivatives at the boundaries of the inner and outer circles.

From the effective potential \( V_{\text{eff}} \) in Eq. (4), we can obtain very useful information on the properties of the \((n,m)\) eigenstates without detailed calculations. For the states with \( m \leq 0 \), the minimum value of \( V_{\text{eff}} \) is always 0, and \(|m| \) magnetic flux quanta are enclosed at the minimum. For \( m > 0 \) and \( |m_{\text{eff},3}| > s_2 \), \( V_{\text{eff}} \) has the minimum value of \(|m_{\text{eff},3}| + m_{\text{eff},3} \) at \( r = \sqrt{|m_{\text{eff},3}|} \), which is located outside the outer circle of the magnetic ring. For \( m > 0 \) and \( |m_{\text{eff},3}| \leq s_2 \), the minimum of \( V_{\text{eff}} \) is always located at \( r = r_2 \), and the corresponding states are localized near the ring boundaries. The effective potentials for \( m = 0, -1, \) and 1 are drawn in Fig. 1.

In this case, we choose the parameters, \( s_1 = 2 \) and \( s_2 = 6 \), i.e., \( r_1 = \sqrt{s_1} \) and \( r_2 = \sqrt{3s_1} \), which give the ring size of about several hundreds \( \Lambda \) for magnetic fields of few teslas. As the magnetic field \( S \) increases, more states are populated in the ring region, with the energies deviated from the Landau levels. This deviation results in the formation of magnetic edge states, similar to the case of magnetic quantum dots. For the magnetic ring considered here, the calculated energies \( E_{nm} \) are plotted as a function of the magnetic field \( S \) in Fig. 2. For weak-magnetic fields, since the density of magnetic flux is low over the ring, the \((0,m<0)\) states must be localized in the region very far from the ring to enclose \(|m| \) flux quanta, if the magnitude of \( m \) is very large. In this case, the states resemble the Landau levels, which are normally formed by the uniform distribution of magnetic fields over the whole region. As \( B \) increases, the localized region of the states get closer to the ring, and these states feel the absence of magnetic fields inside the ring. Then, these states start to deviate from the Landau levels, with lower energies. When \( B \) is strong enough for the \((0,m<0)\) states to be prominent inside the inner circle, where uniform magnetic fields are present, the energies return to the Landau levels. The larger the magnitude of \( m \), the faster the recovery of the Landau levels takes place with increasing the magnetic field. Thus, the magnitude of \( m \) in the ground state continuously increases as \( B \) increases, i.e., the high-energy states with large values of \( m \) \((m<0)\) turn into the ground state. Such angular momentum transitions are mainly caused by the missing of flux quanta in the ring area, while these transitions were not seen in the magnetic quantum dot. In conventional quantum dots confined by electrostatic potentials, this type of angular momentum transitions in the ground state can occur...
only if electron-electron interactions are included. For a conventional quantum ring with a finite width confined by electrostatic potentials, the angular momentum transition was also observed with increasing magnetic field.

When the \((0,m<0)\) states have the quantum number of \(m = -s_1\), \(V_{\text{eff}}\) is zero over the ring region. Thus, for \(S=4\), the ground state is the \((0,-2)\) state instead of the \((0,0)\) state, as clearly shown in Fig. 2. In the limit of \(B \to \infty\), the whole energy spectra of the magnetic quantum ring become identical to those of the conventional quantum ring with the angular momentum quantum number shifted from \(m\) to \(m_{\text{eff}}\). This is because very high-magnetic fields outside the ring area act as an infinite barrier for electrons in the ring. In the case of \(m>0\), as expected from the behavior of \(V_{\text{eff}}\), the

FIG. 1. Effective potentials for the states \((m=0,-1,1)\) in the magnetic quantum ring with \(s_1=2\) and \(s_2=6\).

FIG. 2. Energy spectra of the magnetic quantum ring as a function of \(S\). Dotted lines represent the Landau levels, and \(E_{nm}\) represents the energies of the \((n,m)\) states, which are normalized by that for the lowest Landau level at \(S=4\), where \(n = 0,1,2,\ldots\) and \(m\) denote the radial and angular momentum quantum numbers, respectively.

FIG. 3. Dependence of \(E_{nm}\) on \(m\) in the magnetic quantum ring for \(S=4\), i.e., \(s_1=2\) and \(s_2=6\).

(a) \(n=0\), \(m=0\)

(b) \(n=0\), \(m=-1\)

(c) \(n=0\), \(m=1\)

FIG. 4. Classical trajectories of electrons and corresponding probability densities for the (a) \((0,0)\), (b) \((0,-1)\), and (c) \((0,1)\) states in the magnetic quantum ring.
states with \[|m_{\text{eff},3}| \ll s_2\] are mainly localized near the outer circle, while for \[|m_{\text{eff},3}| \gg s_2\] they are distributed outside the ring. Then, once the \((0,m>0)\) states deviate from the Landau levels, they never turn to the Landau levels again even for very high-magnetic fields, as shown in Fig. 2.

For a magnetic field given by \(s_1=2\) and \(s_2=6\), the dependence of \(E_{nm}\) on \(m\) is drawn in Fig. 3. If \(m_{\text{eff},3}\) in \(V_{\text{eff}}\) satisfies the condition \[|m_{\text{eff},3}| \gg s_2\], which gives \(m \gg 10\) or \(m \ll -2\), the \((0,m<0)\) states are distributed very far from the ring, and their energies becomes the Landau levels. As mentioned earlier, the \((0,m<0)\) states near \(m=-2\), where the ground state occurs, are perturbed by the missing of \(S\) flux quanta in the ring region. Since these states have lower energies than the Landau level, they have nonzero probability currents \(I_{nm}\), where \(I_{nm} = 1/\hbar \partial E_{nm}/\partial m\), forming magnetic edge states. Depending on the sign of \(\partial E_{nm}/\partial m\), the magnetic edge states carry currents circulating either clockwise or counterclockwise, while the degenerated Landau levels of \(m<0\) carry no probability currents.

In Fig. 4, the classical trajectories of electrons and their corresponding probability densities \(|R_{nm}(r)|^2\) are drawn for the \((0,0)\), \((-1,0)\), and \((0,1)\) states, which exhibit clearly the classical behavior of the magnetic edge states formed near the ring boundaries. In fact, these states represent the ensemble average of trajectories, which consist of straight-line paths in the ring region and cyclotron orbits with the radius \(r_i = \sqrt{E_{nm}/2}\) and the center located at \(r_j = \sqrt{r_i^2 - m_{\text{eff}}^2}\) outside the ring. Here, the value of \(m_{\text{eff}}\) depends on the region, as shown in Eq. (4). Thus, for given \(n\) and \(m\), the value of \(r_j\) is fixed whereas that for \(r_j\) varies with region. The general rules for \(r_i\) and \(r_j\) are derived from the conservation of both energy and angular momentum.\(^{11}\) Since the \((0,-2)\) state is the ground state, the \((0,0)\), \((0,-1)\), and \((0,1)\) states have the probability currents drifting along the counterclockwise direction, i.e., \(E_{nm} = 1/\hbar \partial E_{nm}/\partial m\) is positive (see Fig. 3). Besides the direction of classical motions, we find other interesting feature that although no tunneling is allowed between the classical trajectories which consist of separate sets of motions in general, the corresponding probability densities between the trajectories are connected smoothly due to quantum mechanical tunneling.

### III. Modified Magnetic Quantum Ring

We extend our study to the modified structure of the magnetic quantum ring, which has nonzero magnetic field \(B\) in the ring area \((r_1<r<r_2)\) and \(B\neq B^*\) elsewhere. The vector potential for this new structure is chosen to be

\[
\vec{A} = \hat{\theta} \left( \begin{array}{c}
\frac{1}{2} Br \\
\frac{1}{2} B^* r + \frac{1}{2r} (B-B^*) r_1^2 \\
\frac{1}{2} Br + \frac{1}{2r} B^*(r_2^2-r_1^2) - \frac{1}{2r} B (r_2^2-r_1^2)
\end{array} \right)
\]

Then, the effective angular momentum quantum numbers \(m_{\text{eff}}\) are modified such as

\[
m_{\text{eff},1} = m(r<r_1) \quad (r<r_1)
\]

\[
m_{\text{eff},2} = m + s_1^i (r_1<r<r_2) \quad (r_1<r<r_2)
\]

\[
m_{\text{eff},3} = m + s_2^i - s_1^i - (s_2-s_1)(r_2<r), \quad (r_2<r).
\]

where \(s_1 = \pi r_2^2 B/\hbar \), \(s_2 = \pi r_2^2 B/\hbar \), \(s_1^i = \pi r_1^2 B^*/\hbar \), and \(s_2^i = \pi r_2^2 B^*/\hbar \). Using the same dimensionless units as those in Sec. II, we can express \(r_1 = \sqrt{s_1}\) and \(r_2 = \sqrt{s_2}\). The radial function in the ring region is written as the combination of the confluent hypergeometric functions \(M\) and \(U\),

\[
\begin{align*}
R(r) &= \gamma(r) |m_{\text{eff},3}| e^{-1/2} r^2 \left[ C_2 M(a_3, b_3; \gamma^2 r^2) \
+ C_1 U(a_3, b_3; \gamma^2 r^2) \right] (r_1<r<r_2)
\end{align*}
\]

where \(a_3 = [m_{\text{eff},2}] + 1 - \gamma^2 E + \text{sign}(B^*) m_{\text{eff},2}/2\), \(b_3 = |m_{\text{eff},2}| + 1\), and \(\gamma^2 = B^*/|B|\). The wave functions outside the ring have the same forms as those of the magnetic quantum ring with the modified \(m_{\text{eff}}\)’s in Eq. (7). Here, we only consider a special case of \(B^* = -B\), which gives the effective potential \(V_{\text{eff}} = m_{\text{eff}}^2/(2r^2) + r^2/2 \pm m_{\text{eff}}\). The positive and negative signs of the last term corresponds to the regions of \(r>r_2\) and \(r<r_1\), respectively. In each region, \(V_{\text{eff}}\) has a minimum value of \(|m_{\text{eff}}| \pm m_{\text{eff}}\) at \(r^2 = |m_{\text{eff}}|\).

The calculated energies of the modified magnetic quantum ring are plotted as function of \(m\) for \(s_1=2\) and \(s_2=6\) in Fig. 5. Both the ground and excited states are found to deviate more severely from the Landau levels, as compared to the regular magnetic quantum ring considered in Sec. II. The results indicate that the energy levels and the angular momentum transitions can be modulated for both the ground and excited states by varying the ratio \(B^*/B\), i.e., the number of missing flux quanta in the ring region. For the states localized far outside the ring, which satisfy the condition \(|m_{\text{eff},3}| \gg s_2\), i.e., \(m \ll 2\) or \(m \gg 14\), their energies approach to the Landau levels under uniform magnetic fields with the quantum number shifted from \(m\) to \(m_{\text{eff},3}\).

For the \((n,m)\) eigenstates, the corresponding classical trajectories can also be constructed using the energy and angular momentum conservation laws such as \(r_i = |B| B^*/E_{nm}/2\) and \(r_j = |B| B^*/E_{nm}/2 - B^*/B m_{\text{eff}}\) inside the ring whereas \(r_i = \sqrt{E_{nm}/2}\) and \(r_j = \sqrt{E_{nm}/2 - m_{\text{eff}}^2}\) else-

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where $m_{\text{eff}}$'s are given in Eq. (7). In each region, the trajectories consist of circular orbits centered at $r_j$ with the radius $r_i$, and the resulting magnetic edge states circulate clockwise for $r_1<r<r_2$ while counterclockwise elsewhere. For $s_1=2$, $s_2=6$, and $B^x/B=-1$, the classical trajectories and the corresponding probability densities $|R_{nm}(r)|^2$ are drawn for the $(0,0)$, $(0,-1)$, and $(0,1)$ states in Fig. 6. These states exhibit wavelike trajectories unlike the magnetic quantum dot in Sec. II. Since the tunnelings between electrons are prohibited classically, some trajectories may consist of separated orbit sets. The radial distribution of the eigenstates obtained quantum mechanically shows rather smooth variations and agrees excellently with the corresponding trajectories. Although the $(0,0)$ and $(0,-1)$ states are almost degenerate as shown in Fig. 5, they have different probability currents, opposite to each other. The $(0,0)$ state circulates counterclockwise along the outer boundary of the ring, while the $(0,-1)$ state does clockwise along the inner boundary (see Fig. 6). For the $m=0$ states, the effective potential has three local minima of $V_{\text{eff}}=0$, in contrast to the magnetic quantum ring in Sec. II, where a minimum of $V_{\text{eff}}$ occurs only at $r=0$. This behavior illustrates why the probability density of the $(0,0)$ state in Fig. 6(a) is more reduced for $r<r_1$ than that for the regular magnetic ring in Fig. 4(a). The $(0,1)$ state is found to have no trajectories for $r<r_1$ and show a wavelike circulation along the outer boundary of $r=r_2$, which is also manifested in the radial distribution in Fig. 6(c).

IV. CONCLUSIONS

We have investigated the electronic structure of the magnetic quantum ring, which is formed by inhomogeneous distributions of magnetic fields. The eigenstates are found to deviate from the Landau levels due to the missing of magnetic flux quanta and form the magnetic edge states. These edge states carry nonzero probability currents and depend sensitively on the number of the enclosed magnetic flux quanta. We find that the magnetic quantum ring exhibits the angular momentum transitions in the ground state as the magnetic field increases. For extremely high-magnetic fields, the energy spectra resemble those for a conventional quantum ring without magnetic fields. For the modified magnetic quantum ring with nonzero but different magnetic fields inside the ring, the angular momentum transitions are also found in the ground state, which are enhanced by modifying the geometry of the quantum structure and varying the strength of magnetic field.

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