Multiparticle Interference, Greenberger-Horne-Zeilinger Entanglement, and Full Counting Statistics

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We investigate the quantum transport in a generalized $N$-particle Hanbury Brown–Twiss setup enclosing magnetic flux, and demonstrate that the $N$th-order cumulant of current cross correlations exhibits Aharonov-Bohm oscillations, while there is no such oscillation in all the lower-order cumulants. The multiparticle interference results from the orbital Greenberger-Horne-Zeilinger entanglement of $N$ indistinguishable particles. For sufficiently strong Aharonov-Bohm oscillations the generalized Bell inequalities may be violated, proving the $N$-particle quantum nonlocality.

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The Aharonov-Bohm (AB) effect [1], being a most remarkable manifestation of quantum coherence, is at the heart of quantum mechanics. It is essentially a topological effect, because it requires a multiple-connected physical system, e.g., a quantum ring, and consists in a periodic variation of physical observables as a function of the magnetic field threading the loop. It is also a nonlocal effect, since no local physical observable is sensitive to the field. Originally introduced for a single particle [1], it can be generalized as a two-particle AB effect in the average current [2], or in the noise power [3], if only two particles are able to enclose the loop.

The last effect [3], being implemented in the Hanbury Brown–Twiss (HBT) geometry [4], shows the two-particle character in a most dramatic way [5], because single-particle observables, such as the average current, do not contain AB oscillations. In contrast to the single-particle AB effect, which may have a classical analog [6], the two-particle AB effect is essentially quantum, because it originates from quantum indistinguishability of particles. Moreover, it is strongly related to the orbital entanglement in HBT setup [5], and to the quantum nonlocality as expressed via the violation of Bell inequalities [7,8].

In this Letter, we investigate the generalized HBT setup, exemplified in Fig. 1, and introduce the $N$-particle AB effect for $N \geq 3$. We start with analyzing the full counting statistics (FCS) [9,10] and demonstrating that in the $N$-particle HBT setup the $N$th-order cumulant of current cross correlations may exhibit AB oscillations, while there is no such oscillation in all the lower-order cumulants. Next we show that the $N$-particle AB effect originates from the orbital $N$-particle entanglement. Namely, we prove that a many-particle state injected into the HBT setup contains the Greenberger-Horne-Zeilinger-type (GHZ) state (6) [11], which via the postselection [12] contributes to the particle transport. We remark here that the concept of the FCS has been used in earlier works in the context of a two- [13] and three-particle [14] entanglement. We further demonstrate that the Svetlichny inequality [15] may be formulated in terms of the joint detection probability and then violated, if the visibility of the AB oscillations exceeds the value $1/\sqrt{2}$. This inequality, being most restrictive in the whole family of generalized $N$-particle Bell inequalities [15–18], discriminates quantum mechanics and all hybrid local-nonlocal theories [19], proving the $N$-particle quantum nonlocality in the generalized HBT setup. We note that in contrast to the two-particle case, the overall sign of the $N$th-order cumulant cannot be uniquely associated with statistics of particles in the cases of $N \geq 3$.

FIG. 1. Left: Hanbury Brown–Twiss (HBT) interferometer with $N = 3$ independent electron sources (reservoirs 2, 6, 10), $2N$ beam splitters $M_i$, $2N$ chiral current paths, each connecting two beam splitters, and $2N$ detector reservoirs, 3, 4, 7, 8, 11, and 12. Arrows indicate electron trajectories. In the setup, any single electron cannot enclose magnetic flux $\Phi$, while $N$ electrons emitted from $N$ sources can do so. For example, electrons incident from the reservoir 2 move towards the splitters $M_5$ or $M_4$ and disappear in the detectors 3, 4, 11, or 12. This HBT setup can be easily generalized to the case of arbitrary $N$. Right: in the HBT interferometer $N$-electron states incident from the sources are decomposed into $2^N$ states, shown for $N = 3$. The states (a) and (b) result in Aharonov-Bohm (AB) oscillations in the $N$th-order cumulant of the current cross correlations at detectors. These two states together form a Greenberger-Horne-Zeilinger-type (GHZ) state of $N$ pseudospins. All the others, (c)–(h), do not contribute to the AB effect.
Generalized HBT interferometer.—We consider the coherent electron transport in the mesoscopic multiterminal conductor consisting of $4N$ electron reservoirs, $i = 1, 2, \ldots, 4N$ (enumerated clockwise), $2N$ beam splitters $M_i$, $i = 1, 2, \ldots, 2N$, and $2N$ chiral current paths enclosing magnetic flux $\Phi$ (see Fig. 1). The $(2i - 1)$th and the $2i$th reservoirs couple to the splitter $M_i$, at which electrons can be transmitted with probability $T_i$ or reflected with probability $R_i = 1 - T_i$. Among $4N$ reservoirs, $N$ reservoirs, $2, 6, \ldots, 4N - 2$, behave as independent electron sources, and $N$ pairs of reservoirs with indexes $\alpha_i = \{3, 4\}$, $\alpha_i = \{7, 8\}$, $\alpha_i = \{4N - 1, 4N\}$, act as detectors. All sources are biased with voltage $eV$, while the rest of $3N$ reservoirs are grounded. Neighboring beam splitters $M_i$ and $M_{i+1}$ are connected via a one-dimensional spinless current path, where phase difference $\phi_i$ is accumulated.

The total scattering matrix of the generalized HBT setup consists of two $2N$-dimensional unitary chiral blocks $\hat{U}$ and $\hat{V}$. The block $\hat{U}$ is determined as follows: the electron propagation from source $4i + 2$ to detectors $4i - 1, 4i, 4i + 3$, or $4i + 4$ gives the matrix elements $\exp(-i\phi_{2i})\chi_{4i+1}^{4i+3}, \exp(-i\phi_{2i})\chi_{4i+1}^{4i+4}$, and $\exp(-i\phi_{2i+1})\chi_{4i+1}^{4i+4}$, respectively. The block $\hat{V}$ describes noiseless outgoing currents from detector reservoirs. It is not relevant for the following discussion; therefore, the requirement of the coherence in this sector may be relaxed. All the other matrix elements are zero, thus no single electron can enclose the flux $\Phi$ in the generalized HBT setup. However, $N$ electrons can do so, and this leads to the $N$-particle interference in the FCS.

N-particle Aharonov-Bohm effect.—In the long measurement time limit $t \gg \tau_C$, where $\tau_C = 2\pi eV/|eV|$ is the correlation time, the electron transport is a Markovian random process. The characteristic property of such a process is that the irreducible correlators (cumulants) of the number of electrons arriving at detectors are proportional to the number of transmission attempts, $t/\tau_C$. Therefore, we normalize the cumulants to $t/\tau_C$, so that the cumulant generating function of the FCS at the detector reservoirs takes the form [9]:

$$S(\chi) = \frac{\tau_C}{2\pi\hbar} \int dE Tr \ln[1 - \hat{f} + \hat{f} \hat{U} \hat{\Lambda} \hat{U} \hat{\Lambda}^{-1}].$$

(1)

Here $\hat{f}$ is the diagonal matrix with elements $\hat{f}_{ij}$ being Fermi-Dirac occupations: $\hat{f}_j(E) = f_E(E - eV)$ at the sources, and $\hat{f}_j(E) = f_E(E)$ in the rest of reservoirs. The matrix $\hat{\Lambda}$ is diagonal with elements $\hat{\Lambda}_{ij} = \delta_{ij} \exp(i\chi_{ij})$, where $\chi = \{\chi_{ij}\}$ is the set of counting variables in the detector reservoirs. The cumulants may be obtained by evaluating the derivatives of $S(\chi)$ and setting $\chi = 0$.

The generating function (1) is simplified by introducing matrices $(\hat{A}_\alpha)_{nm} = f^{*}_n \hat{U}_{n,m} \hat{A}_{\alpha,m} [20]:$

$$S(\chi) = \frac{\tau_C}{2\pi\hbar} \int dE Tr \ln[1 + \sum_{\alpha} (e^{i\chi_{\alpha}} - 1)\hat{A}_{\alpha}].$$

(2)

where the sum runs over all the detectors $\alpha$, and $n, m = 1, 2, 5, 6, \ldots, 4N - 3, 4N - 2$. In order to generalize the two-particle interference to the cases of $N \geq 3$, we consider the lowest-order cross-correlation functions with all detectors being different. It turns out that the number of such cross correlators is limited, because between all the possible lowest-order products of the form $\prod_{\alpha} \hat{A}_{\alpha}$, where all $\alpha$ are different, only few products give nonvanishing traces: $\text{Tr}\hat{A}_{\alpha_1} \hat{A}_{\alpha_4} \hat{A}_{\alpha_3} \hat{A}_{\alpha_2}$ for all $i$, and $\hat{A}_{\alpha_1} \hat{A}_{\alpha_2} \hat{A}_{\alpha_3} \hat{A}_{\alpha_4}$ and $\hat{A}_{\alpha_1} \hat{A}_{\alpha_3} \hat{A}_{\alpha_4} \hat{A}_{\alpha_2}$. Expanding the logarithm in the right-hand side of (2) and collecting nonzero traces we obtain the first-order cumulant $Q_{\alpha i} = c(t\alpha_i)\Gamma_{\alpha_i} (i.e.,$ the normalized average current) and the second-order cumulant $Q_{\alpha i,\alpha j}(\text{the normalized zero-frequency noise power})$,

$$Q_{\alpha i} = P_{\alpha_i} R_{2i+1} + (1 - P_{\alpha_i}) T_{2i+1} + (3a)$$

$$Q_{\alpha i,\alpha j} = -P_{\alpha_i} (1 - P_{\alpha_j}) T_{2i+1} R_{2j+1},$$

(3b)

where $P_{\alpha_i} = R_{2i} (P_{\alpha_i} = T_{2i})$ for $\alpha_i$ being odd (even).

The next nonzero cumulant is the $N$th-order cross-correlation function

$$Q_{\alpha_1,\alpha_2,\ldots,\alpha_N} = 2\text{sgn} (\alpha_1, \alpha_2, \ldots, \alpha_N) \Gamma_{\text{BS}} \cos (\phi_{\text{FL}}).$$

(4)

where the sign depends on the choice of the detectors, $\text{sgn} (\alpha_1, \alpha_2, \ldots, \alpha_N) = (-1)^{N-1+\sum_{\alpha i}}$, the factor $\Gamma_{\text{BS}} = \prod_{l=1}^{N} \frac{\Gamma_{\alpha_i}}{\sqrt{R_{\alpha_i} T_{\alpha_i}}} \frac{1}{\sqrt{R_{\alpha_i} T_{\alpha_i}}}$ characterizes the transmission of beam splitters, and the total phase accumulated around the HBT loop is $\phi_{\text{FL}} = 2\pi\Phi/\hbar$, where $\Phi$ is Fermi velocity. Second, every cumulant in Eqs. (3a), (3b), and (4), has the prefactor $(\tau_C/2\pi\hbar) \int dE (f - f_0)^k$, where $k = 1, 2, N$. We consider the zero temperature limit and set this prefactor to 1. Next, the sign function in the equation (4) has two contributions: the term $(-1)^{\sum_{\alpha_i}}$ comes from the detector phases, while the term $(-1)^{N-1}$ originates from the fermionic exchange effect. Thus, in contrast to the case of the second cumulant (3b), the overall sign of high-order cumulants is not universal. Finally, only the $N$th-order cross correlator shows oscillations as a function of the magnetic flux threading the HBT loop. We stress that these oscillations appear in the FCS of electrons injected from $N$ uncorrelated sources, and thus they can be regarded as a $N$-particle AB effect. Below we connect this effect with $N$-particle GHZ entanglement.

GHZ entanglement.—To clarify the origin of the AB oscillations in the cumulant $Q_{\alpha_1}$ in Eq. (4), we analyze the multipartite state injected from the sources,
\[ |\Psi\rangle = \prod_{0 < E \leq V} \left( \sum_{i} C_{i}^{\dagger}(E) c_{i}(E) \right) \langle 0 |. \] Here, |0\rangle denotes the filled Fermi sea below \( E = 0 \), and the operator \( c_{i}^{\dagger}(E) \) creates an electron with energy \( E \) in the source reservoir \( 4i - 2 \). Using the scattering matrix \( \tilde{U} \), we write
\[ c_{4i-2}^{\dagger} = -i\sqrt{R_{2i-1}} c_{i+1}^{\dagger} + \sqrt{R_{2i-1}} b_{i}. \]
where \( a_{i}^{\dagger} \) creates an electron moving clockwise from the beam splitter \( M_{2i-1} \) to \( M_{2i} \), and \( b_{i}^{\dagger} \) creates an electron moving counterclockwise from \( M_{2i} \) to \( M_{2i-1} \). Then up to the overall constant prefactor the total state becomes
\[ |\Psi\rangle = \prod_{0 < E < V} \left[ \gamma_{a} C_{a}^{\dagger}(E) + \gamma_{b} C_{b}^{\dagger}(E) + D(E) \right]|0\rangle, \tag{5} \]
where \( \gamma_{a} = \prod_{k=1}^{N} (-i\sqrt{R_{2i-1}}) \), and \( \gamma_{b} = \prod_{k=1}^{N} \sqrt{R_{2i-1}} \). The operator \( C_{a}^{\dagger} = \prod_{k=1}^{N} a_{k}^{\dagger} \) creates \( N \) electrons moving clockwise [see Fig. 1(a), and \( C_{b}^{\dagger} = \prod_{k=1}^{N} b_{k}^{\dagger} \) creates the \( N \) electrons moving counterclockwise [Fig. 1(b)], while \( D^{\dagger} \) creates the other possible states [Fig. 1(c)–(h)].

It follows from Eq. (5) that the total state |\Psi\rangle is a product of \( N \)-particle states where all \( N \) particles are taken at same energy. It is obvious that the zero-frequency \( N \)th-order cumulant \( Q_{(a)} \) originates from the independent contribution of such states, more precisely, from the part |\psi_{N}\rangle = (\gamma_{a} C_{a}^{\dagger} + \gamma_{b} C_{b}^{\dagger})|0\rangle. Introducing orbital pseudospin notations \( \mid \uparrow \rangle = a_{i}^{\dagger} \) for electrons moving clockwise and \( \mid \downarrow \rangle = b_{i}^{\dagger} \) for electrons moving counterclockwise, we rewrite the relevant \( N \)-particle state as
\[ |\psi_{N}\rangle = p e^{i\theta_{N}} \mid \uparrow \downarrow \downarrow \downarrow \downarrow \rangle \] for \( \theta_{N} = -\pi N / 2 \), and \( \theta_{a} = \pi \). This state is nothing but the \( N \)-particle GHZ-type entangled state \[ 11 \]. We thus arrive at the important result that in fact GHZ entanglement is responsible for the \( N \)-particle AB effect, and that the measurement of the \( N \)th-order cumulant \( Q_{(a)} \) effectively postselects \[ 12 \] the \( N \)-particle entangled state.

There exists an equivalent representation of the total state |\Psi\rangle in the wave packet basis \[ 9 \], which obviously leads to the same physical results but allows slightly different interpretation that we will use below. In this representation, electrons fully occupy the stream of wave packets regularly approaching, with the rate \( eV / (2\pi \hbar) \), the HBT interferometer from the sources \( 4i - 2 \), \( i = 1, 2, \ldots, N \). After being injected to the interferometer, the wave packets form an \( N \)-particle state that contains the entangled state |\psi_{N}\rangle. The entangled state then moves towards the detectors \( \alpha_{i} = (4i - 1, 4i) \), where each electron accumulates the phase \( \pm \phi_{i} \), is rotated by the beam splitters \( M_{2i} \), \( a_{i}^{\dagger} = -i\sqrt{R_{2i-1}} c_{i+1}^{\dagger} + \sqrt{R_{2i-1}} b_{i} \), and \( b_{i}^{\dagger} = \sqrt{T_{2i}} c_{i+1}^{\dagger} - i\sqrt{R_{2i-1}} c_{i}^{\dagger} \), and then |\psi_{N}\rangle is detected in one of the sets \( \{\alpha\} = \{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{N}\} \) of \( N \) detectors.

Following Ref. \[ 21 \] we introduce the probability \( P_{(a)} \) of joint detection (JDP) of \( N \) particles in the detector reservoirs defined as
\[ P_{(a)} \propto \langle \Psi| I_{\alpha_{1}} I_{\alpha_{2}} \ldots I_{\alpha_{N}} |\Psi\rangle, \] where \( I_{\alpha} = (e^{2i\pi \hbar} \int dE \langle c_{\alpha}^{\dagger}(E) c_{\alpha}(E') \rangle \) are the current operators in reservoirs \( \alpha \), taken at the same time \( t = 0 \). Then the straightforward calculation gives
\[ P_{(a)} \propto \prod_{i=1}^{N} \left( \sum_{\alpha} P_{\alpha} \right) + \prod_{i=1}^{N} (1 - P_{\alpha}) + Q_{(a)} \tag{7} \]
with the prefactor determined by the normalization \( \sum_{\alpha} P_{(a)} = 1 \). This result can be easily understood after looking closely at Fig. 1. The wave packets moving as shown in Figs. 1(c)–1(h) do not contribute to \( P_{\alpha} \), because one of the currents \( I_{\alpha} \) is exactly zero. Thus, \( P_{(a)} \) originates from the states (a) and (b), i.e., from the GHZ-type state (6).

For the GHZ-type state (6) we have
\[ E_{N} = g(p, q) \sum_{i=1}^{N} \cos \phi_{i} + 2pq \cos \theta, \tag{8} \]
where \( g(p, q) = p^{2} + (-1)^{N} q^{2} \) and \( \theta = \theta_{p} - \theta_{q} + \sum_{i=1}^{N} \theta_{i} \). Turning now to the generalized HBT setup, we note that the detector beam splitters \( M_{2i} \) implement the orbital pseudospin rotation, while the reservoirs 4i and 4i - 1 detect pseudospins in z direction. The pseudospin correlation function can now be found as
\[ E_{N} = \sum_{\alpha} (-1)^{N_{\alpha}} P_{(a)}, \tag{9} \]
generalizing two-particle cases \[ 5 \]. After identifying \( \cos \phi_{i} = T_{2i} - R_{2i} \) and \( \theta_{i} = \phi_{2i-1} + \phi_{2i} - \pi / 2 \) we find that \( E_{N} \) takes the same form as (8) with \( \theta \) being replaced with \( \phi_{0a} - \pi (N - 1) \).

For a particular choice of two sets of directions \( \{\vec{n}_{1}, \ldots, \vec{n}_{N}\} \) and \( \{\vec{n}'_{1}, \ldots, \vec{n}'_{N}\} \), the expectation value of the \( N \)-spin operator introduced via iterations, \( M_{N} = \frac{1}{2} (\sigma_{n_{1}} + \sigma_{n_{2}}) \otimes M_{N-1} + \frac{1}{2} (\sigma_{n_{1}} - \sigma_{n_{2}}) \otimes M'_{N-1} \) (with \( M' \) obtained from \( M_{N} \) by exchanging all \( \vec{n}_{i} \) and \( \vec{n}'_{i} \)), may violate the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities (\( M_{N} \approx 1 \) \[ 16 \] that discriminate local variable theories and quantum mechanics. These inequalities generalize the well known Clauser-Horne-Shimony-Holt inequalities.

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inequality [8], \( \langle M_2 \rangle = (1/2)[E(\vec{n}_1, \vec{n}_2) + E(\vec{n}_1', \vec{n}_2') + E(\vec{n}_1, \vec{n}_2') - E(\vec{n}_1', \vec{n}_2)] \leq 1 \). The violation of more restrictive Svetlichny inequalities \( \langle S_N \rangle \leq 2^{N-2}/2 \) [15], where \( S_N = (1/2)(M_N + M_N') \) for \( N \) being odd and \( S_N = M_N \) otherwise, rules out all hybrid local-nonlocal models [19].

Here we focus on the sufficient condition for the violation of generalized Bell inequalities, which can be found as follows: after fixing \( T_{2j} = R_{2j} = 1/2 \) for all detectors the maximum values \( \langle M_N \rangle_{\text{max}} = 2pq2^{N-1}/2 \) [22] and \( \langle S_N \rangle_{\text{max}} = 2pq2^{N-1}/2 \) [19] can be reached for a particular choices of detector phases \( \phi_j \). This will violate MABK inequalities if \( 2pq > 1/2^{N-1}/2 \), and Svetlichny inequalities if \( 2pq > 1/\sqrt{2} \). We note that according to Eqs. (4) and (7) the value \( 2pq \) is nothing but the visibility \( V_{AB} = (P_{\alpha} - P_{\min})/(P_{\alpha} + P_{\min}) \) of AB oscillations in the JDP for a fixed set \( \{\alpha\} \) of detectors. Thus, we come to the important practical conclusion that the observation of sufficiently strong AB oscillations in \( P_{\alpha} \),

\[
V_{AB} > 1/\sqrt{2},
\]

will guarantee the possibility of the violation of Svetlichny inequalities [23]. This is also true in the case of a weak dephasing, since in our HBT setup, where single pseudo-spin flips are not allowed [5], its only effect is to suppress the second term in the correlator (8). Summarizing this discussion we conclude that the \( N \)-particle AB effect may be viewed as a manifestation of genuine quantum non-locality in the generalized HBT setup.

Feasibility of experimental realization.—The mesoscopic implementation of two-particle HBT setup proposed in Ref. [5] may be well utilized in the cases of \( N \geq 3 \). It relies on the quantum Hall edge states as chiral channels, and quantum point contacts as beam splitters, and generalizes the electronic Mach-Zehnder interferometer, which has been recently experimentally realized [24]. All limitations not specific to \( N \geq 3 \) can be found in [5].

Recent experiments [25] revealed a number of specific difficulties in measuring FCS. First of all, the detection of current fluctuations on long time scale \( t \gg \tau_c = 2\pi\hbar/[eV] \) reduces the signal-to-noise ratio for the \( N \)th-order cumulants by the factor \((\tau_c/t)^{N/2-1}\), the consequence of the central limit theorem. This may dramatically increase the total measurement time for high-order cumulants. Second, experimentally measurable high-order cumulants contain nonuniversal low-order corrections from electrical circuit, which makes it difficult to extract intrinsic noise. We believe, however, that all these difficulties should not be that severe in our case, because low-order cumulants do not contain AB oscillations and appear merely as a background contribution.

Finally, in contrast to the two-particle case in Ref. [5], for the demonstration of the quantum nonlocality in the generalized HBT setup the measurement on the short time scale \( t < \tau_c \) is preferable. This is because the nonlocality condition (10) may require a more accurate determination of the visibility \( V_{AB} \) via measuring the JDP. Such high-frequency “quantum noise” detection techniques have recently become available [26].

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[20] Because of the chiral nature of transport in the HBT setup, the average current defined by (2) is divergent. This divergence is canceled by the outgoing noiseless current from detectors, which is taken into account in Eq. (3a).
[23] MABK inequalities lead to a less restrictive condition, \( V_{AB} > 1/2^{N-1}/2 \). Moreover, for \( N = 3 \) [18] and for \( N \approx 4 \) being even [17], they can be generalized and violated for arbitrary weak visibility, ruling out all local variable theories. However, none of these discriminate hybrid local-nonlocal theories and quantum mechanics.