Q-TRIVIAL GENERALIZED BOTT MANIFOLDS

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Abstract

When the cohomology ring of a generalized Bott manifold with Q-coefficient is isomorphic to that of a product of complex projective spaces $\mathbb{C} P^{n_i}$, the generalized Bott manifold is said to be Q-trivial. We find a necessary and sufficient condition for a generalized Bott manifold to be Q-trivial. In particular, every Q-trivial generalized Bott manifold is diffeomorphic to a $\prod_{n_i=1}^{h} \mathbb{C} P^{n_i}$-bundle over a Q-trivial Bott manifold.

1. Introduction

A generalized Bott tower of height $h$ is a sequence of complex projective space bundles

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where $B_i = P(\mathbb{C} \oplus \xi_i)$, $\mathbb{C}$ is a trivial complex line bundle, $\xi_i$ is a Whitney sum of $n_i$ complex line bundles over $B_{i-1}$, and $P(\cdot)$ stands a projectivization. Each $B_i$ is called an $i$-stage generalized Bott manifold. When all $n_i$'s are 1 for $i = 1,\ldots,h$, the sequence (1.1) is called a Bott tower of height $h$ and $B_i$ is called an $i$-stage Bott manifold.

A ($h$-stage) generalized Bott manifold is said to be Q-trivial (respectively, Z-trivial) if $H^*(B_h;\mathbb{Q}) \cong H^*(\prod_{i=1}^{h} \mathbb{C} P^{n_i};\mathbb{Q})$ (respectively, $H^*(B_h;\mathbb{Z}) \cong H^*(\prod_{i=1}^{h} \mathbb{C} P^{n_i};\mathbb{Z})$). It is shown in [4] that if $B_h$ is Z-trivial, then every fiber bundle in the tower (1.1) is trivial so that $B_h$ is diffeomorphic to $\prod_{i=1}^{h} \mathbb{C} P^{n_i}$. Furthermore, Choi and Masuda show that every ring isomorphism between Z-cohomology rings of two Q-trivial Bott manifolds is induced by some diffeomorphism between them (see Theorem 3.1 and [2]).

We find a necessary and sufficient condition for a generalized Bott manifold to be Q-trivial. Namely, we have the following proposition.

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Proposition 1.1. An h-stage generalized Bott manifold $B_h$ is $\mathbb{Q}$-trivial if and only if each vector bundle $\xi_i$, $i = 1, \ldots, h$, satisfies

\[(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k\]

for $k = 1, \ldots, n_i + 1$, where $B_i = P(\mathbb{C} \oplus \xi_i)$.

Moreover, the following theorem says that a $\mathbb{Q}$-trivial generalized Bott manifold without $\mathbb{C}P^1$-fibration is weakly equivariantly diffeomorphic to a trivial generalized Bott manifold.

Theorem 1.2. Let $B_h$ be a generalized Bott manifold such that all $n_i$'s are greater than 1. Then the following are equivalent

1. $B_h$ is $\mathbb{Q}$-trivial,
2. total Chern class $c(\xi_i)$ is trivial for each $i = 1, \ldots, h$,
3. $B_h$ is $\mathbb{Z}$-trivial, and
4. $B_h$ is diffeomorphic to the product of projective spaces $\prod_{i=1}^h \mathbb{C}P^{n_i}$.

In the light of Theorem 1.2, we have a natural question.

Question 1.1. Let $B_h$ and $B'_h$ be generalized Bott manifolds with $n_i > 1$, $i = 1, \ldots, h$. Is $H^*(B_h; \mathbb{Z})$ isomorphic to $H^*(B'_h; \mathbb{Z})$ if $H^*(B_h; \mathbb{Q}) \cong H^*(B'_h; \mathbb{Q})$?

Unfortunately, Example 3.1 shows that the answer to the question is negative. From the proposition, we can deduce the following theorem.

Theorem 1.3. Every $\mathbb{Q}$-trivial generalized Bott manifold is diffeomorphic to a $\prod_{n_i>1} \mathbb{C}P^{n_i}$-bundle over a $\mathbb{Q}$-trivial Bott manifold.

The remainder of this paper is organized as follows. In Section 2, we recall general facts on a generalized Bott manifold and deal with its cohomology ring. In Section 3, we prove Proposition 1.1, Theorems 1.2 and 1.3.

2. Cohomology ring of a generalized Bott manifold

Let $B$ be a smooth manifold and let $E$ be a complex vector bundle over $B$. Let $P(E)$ denote the projectivization of $E$. Let $y \in H^2(P(E))$ be the negative of the first Chern class of the tautological line bundle over $P(E)$. Then $H^*(P(E))$ can be viewed as an algebra over $H^*(B)$ via $\pi^* : H^*(B) \rightarrow H^*(P(E))$, where $\pi : P(E) \rightarrow B$ denotes the projection. When $H^*(B)$ is finitely generated and torsion free (this is the case when
$B$ is a toric manifold), $\pi^*$ is injective and $H^*(P(E))$ as an algebra over $H^*(B)$ is known to be described as

\begin{equation}
H^*(P(E)) = H^*(B)[y] \Big/ \left( \sum_{k=0}^{n} c_k(E)y^{n-k} \right).
\end{equation}

where $n$ denotes the complex dimension of the fiber of $E$ (see [1]).

For a generalized Bott manifold $B_h$ in (1.1), since $\pi_j^*: H^*(B_{j-1}) \to H^*(B_j)$ is injective, we regard $H^*(B_{j-1})$ as a subring of $H^*(B_j)$ for each $j$ so that we have a filtration

$H^*(B_h) \supset H^*(B_{h-1}) \supset \cdots \supset H^*(B_1)$.

Let $x_j \in H^2(B_j)$ denote minus the first Chern class of the tautological line bundle over $B_j = P(\mathbb{C} \oplus \xi_j)$. We may think of $x_j$ as an element of $H^2(B_j)$ for $i \geq j$. Then the repeated use of (2.1) shows that the ring structure of $H^*(B_h)$ can be described as

$H^*(B_h) = \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h \rangle$.

Let $\xi_{2,1}$ be the tautological line bundle over $B_1 = \mathbb{C}P^{n_1}$ and let $\xi_{3,1} = \pi_{j}^*(\xi_{2,1})$ the pull-back bundle of the tautological line bundle over $B_1$ to $B_2$ via the projection $\pi_{2}: B_2 \to B_1$. In general, let $\xi_{i,j-1}$ be the tautological line bundle over $B_{j-1}$ and we define inductively

$\xi_{i,j-k} = \pi_{j-k+1}^* \circ \cdots \circ \pi_{j-1}^*(\xi_{i,j-1})$

for $k = 2, \ldots, j - 1$. Then one can see that the Whitney sum of complex line bundles $\xi_i$ over $B_{i-1}$ in the sequence (1.1) can be written as

$\xi_i := (\xi_{i,1}^{a_{i,1}} \otimes \cdots \otimes \xi_{i,1}^{a_{i,i-1}}) \oplus \cdots \oplus (\xi_{i,1}^{a_{i,1}} \otimes \cdots \otimes \xi_{i,1}^{a_{i,i-1}})$

for some integers $a_{i,1}, \ldots, a_{i,i-1}$. Note that $\xi_1 = (\mathbb{C})^{n_1}$. Hence, the total Chern class of $\xi_i$ is

\begin{equation}
c(\xi_i) = \prod_{j=1}^{n_i} \left( 1 + \sum_{k=1}^{i-1} a_{j,k}x_k \right).
\end{equation}

Therefore, the cohomology ring of $B_h$ is

$H^*(B_h; \mathbb{Z})$

$= \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h \rangle$

$= \mathbb{Z}[x_1, \ldots, x_h]/\left\langle \prod_{j=1}^{n_i} \left( \sum_{k=1}^{i-1} a_{j,k}x_k + x_i \right) \mid i = 1, \ldots, h \right\rangle$.  

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Remark 1. We can associate a generalized Bott manifold $B_h$ with an $h \times h$ vector matrix $A$ as follows:

\[
A^T = \begin{pmatrix}
1 & a_1^1 & a_1^2 & \cdots & a_1^n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a_1^1 & a_2^1 & \cdots & 1
\end{pmatrix},
\]

where

\[
a_k = \begin{pmatrix}
a_{1k}^j \\
\vdots \\
a_{nk}^j
\end{pmatrix}
\]

and

\[
1 = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}.
\]

Moreover we can consider $B_h$ as a quasitoric manifold over the product of simplices $\prod_{i=1}^{\bar{h}} \Delta^{n_i}$ with the reduced characteristic matrix $\Lambda_s = -A^T$.

3. $\mathbb{Q}$-trivial generalized Bott manifolds

As we mentioned in the introduction, Choi and Masuda classify $\mathbb{Q}$-trivial Bott manifolds as follows.

**Theorem 3.1** ([2]). (1) A Bott manifold $B_h$ is $\mathbb{Q}$-trivial if and only if for each $i = 1, \ldots, h$, each line bundle $\xi_i$ satisfies $c_1(\xi_i)^2 = 0$ in $H^*(B_h; \mathbb{Z})$.

(2) Every ring isomorphism $\varphi$ between two $\mathbb{Q}$-trivial Bott manifolds $B_h$ and $B'_h$ is induced by some diffeomorphism $B_h \rightarrow B'_h$.

In this section we shall prove Proposition 1.1 and Theorem 1.2. To prove them, we need the following lemmas.

**Lemma 3.2.** If a generalized Bott manifold $B_h$ is $\mathbb{Q}$-trivial, then there exist linearly independent primitive elements $z_1, \ldots, z_h$ in $H^2(B_h; \mathbb{Z})$ such that $z_i^{n_i}$ is not zero but $z_i^{n_i+1}$ is zero in $H^*(B_h; \mathbb{Z})$ for $i = 1, \ldots, h$.

Proof. Let $H^*(B_h; \mathbb{Z})$ be generated by $x_1, \ldots, x_h$ as in (2.3) and let

\[
H^*(\prod_{i=1}^{h} B_h; \mathbb{Q}) = \mathbb{Q}[y_1, \ldots, y_h]/(y_i^{n_i+1} \mid i = 1, \ldots, h).
\]
Since both \( \{x_1, \ldots, x_h\} \) and \( \{y_1, \ldots, y_h\} \) are sets of generators of \( H^2(B_h; \mathbb{Q}) \), we can write

\[
y_i = \sum_{j=1}^{h} c_{ij} x_j \quad \text{for} \quad i = 1, \ldots, h \quad \text{and} \quad c_{ij} \in \mathbb{Q},
\]

where the determinant of the matrix \( C = (c_{ij})_{h \times h} \) is non-zero. We may assume that \( c_{ij} \)'s are irreducible fractions. Multiplying \( (c_{i,1}, \ldots, c_{i,h}) \) by the least common denominator \( r_i \) of a set \( \{c_{i,1}, \ldots, c_{i,h}\} \), we can get a primitive element \( z_i = r_i y_i = r_i \sum_{j=1}^{h} c_{ij} x_j \) in \( H^2(B_h; \mathbb{Z}) \) such that \( z_i^{n_i+1} = r_i y_i^{n_i+1} \) is zero in \( H^*(B_h; \mathbb{Z}) \) for each \( i = 1, \ldots, h \).

Since \( z_i \) are linearly independent, the elements \( z_1, \ldots, z_h \) are also linearly independent. Since \( y_i^{n_i} \) is not zero in \( H^*(B_h; \mathbb{Q}) \), \( z_i^{n_i} \) cannot be zero in \( H^*(B_h; \mathbb{Z}) \). This proves the lemma. \( \square \)

**Lemma 3.3** ([4]). Let \( B_m \) be an \( m \)-stage generalized Bott manifold. Then the set

\[
\{ bx_m + w \in H^2(B_m) \mid 0 \neq b \in \mathbb{Z}, w \in H^2(B_{m-1}), (bx_m + w)^{n_m+1} = 0 \}
\]

lies in a one-dimensional subspace of \( H^2(B_m) \) if it is non-empty.

**Proof.** To satisfy \( (bx_m + w)^{n_m+1} = 0 \), we need \( bc_1(\xi_m) = (n_m + 1)w \). \( \square \)

**Lemma 3.4** ([4]). For an element \( z = \sum_{i=1}^{h} b_i x_i \in H^2(B_h) \), if \( b_i \) is non-zero, then \( z^{n_i} \) cannot be zero in \( H^*(B_h) \).

**Proof.** If we expand \( (\sum_{i=1}^{h} b_i x_i)^{n_i} \), there appears a non-zero scalar multiple of \( x_i^{n_i} \) because \( b_i \neq 0 \). Then, \( z^{n_i} \) cannot belong to the ideal generated by the polynomials \( x_i \prod_{j=1}^{i-1} (\sum_{k=1}^{i-1} a_{jk} x_k + x_i) \), hence it is not zero in \( H^*(B_h) \). \( \square \)

Now we can prove Proposition 1.1.

**Proof of Proposition 1.1.** If each vector bundle \( \xi_i \) satisfies the conditions (1.2), then \( (x_i + 1/(n_i + 1)c_1(\xi_i))^n_i+1 \) is zero in \( H^*(B_h; \mathbb{Q}) \). Since the set

\[
\left\{ x_i + \frac{1}{n_i + 1} c_1(\xi_i) \mid i = 1, \ldots, h \right\}
\]

generates \( H^*(B_h; \mathbb{Q}) \) as a graded ring, this shows that \( B_h \) is \( \mathbb{Q} \)-trivial.

Conversely, if a generalized Bott manifold is \( \mathbb{Q} \)-trivial, then there are linearly independent and primitive elements \( z_1, \ldots, z_h \) in \( H^2(B_h; \mathbb{Z}) \) such that \( z_i^{n_i+1} \) is zero but \( z_i^{n_i} \) is not zero in \( H^*(B_h) \) by Lemma 3.2. We can put \( z_i = \sum_{j=1}^{h} b_{ij} x_j \) with \( b_{ij} \in \mathbb{Z} \) for each \( i = 1, \ldots, h \).
Now, consider a map $\mu: \{1, \ldots, h\} \rightarrow \mathbb{N}$ given by $j \mapsto n_j$. Further assume that the image of $\mu$ is the set $\{N_1, \ldots, N_m\}$ with $N_1 < \cdots < N_m$. We will show inductively that each $z_i$ can be written as $r_i(x_i + 1/(\mu(i) + 1)c_1(\xi_i))$ for some $r_i \in \mathbb{Z} \setminus \{0\}$.

**Case 1:** Assume $i \in \mu^{-1}(N_1)$. Let $\mu^{-1}(N_1) := \{i_1, \ldots, i_\alpha\}$ with $i_1 < \cdots < i_\alpha$. We have $z_i^{N_1+1} = 0$. Then, by Lemma 3.4, we can see that

$$z_i = \sum_{j \in \mu^{-1}(N_1)} b_{ij}x_j,$$

that is, $b_{ij'} = 0$ for $j' \notin \mu^{-1}(N_1)$. Note that for each $i \in \mu^{-1}(N_1)$, one of $b_{ij}$'s is nonzero for $j \in \mu^{-1}(N_1)$ because the set $\{z_i \mid i \in \mu^{-1}(N_1)\}$ is linearly independent. For some $i_p \in \mu^{-1}(N_1)$, if $b_{ip} x_p$ is nonzero, then $z_{ip} \in H^2(B_{u_p})$ and $b_{ip} = 0$ for all $i \in \mu^{-1}(N_1) \setminus \{i_p\}$ by Lemma 3.3. Put $w_{ip} := z_{ip}$. If $b_{iq} x_q$ is nonzero for some $i_q \in \mu^{-1}(N_1) \setminus \{i_p\}$, then $z_{iq} \in H^2(B_{u_{iq}})$ and $b_{iq} = 0$ for all $i \in \mu^{-1}(N_1) \setminus \{i_p, i_q\}$. Now, put $w_{iq} := z_{iq}$. In this way, for each $i \in \mu^{-1}(N_1)$, we can obtain $w_i \in H^2(B_i)$ such that $w_i \notin H^2(B_{i-1})$ and $w_i^{N_1+1} = 0$ in $H^*(B_h)$. Moreover, from the proof of Lemma 3.3, we can write

$$w_i := r_i \left( x_i + \frac{1}{N_1 + 1} c_1(\xi_i) \right) \in H^2(B_i)$$

for each $i \in \mu^{-1}(N_1)$. In particular, if $N_1 = 1$, then $w_i$ is of the form either $\pm x_i$ or $\pm(2x_i + c_1(\xi_i))$ for $i \in \mu^{-1}(N_1)$. Furthermore, without loss of generality, we may assume that $z_i = w_i$ for $i \in \mu^{-1}(N_1)$.

**Case 2:** Assume that $z_k = r_k(x_k + 1/(\mu(k) + 1)c_1(\xi_k))$ for $N_1 \leq \mu(k) \leq N_{n-1}$ and let $l \in \mu^{-1}(N_n)$. Then we have $z_l^{N_n+1} = 0$. Then by Lemma 3.4, we can easily see that

$$z_l = \sum_{k \in \mu^{-1}(N_n)} b_{lk}x_k + \sum_{j \in \mu^{-1}(N_n)} b_{lj}x_j,$$

where $N_{n-1} = \{N_1, \ldots, N_{n-1}\}$. That is, $b_{lj'} = 0$ for $j' \notin \mu^{-1}(N_{n-1})$. Since $z_l^{N_n+1}$ is zero in $H^*(B_h)$, we have

$$\left( \sum_{k \in \mu^{-1}(N_n)} b_{lk}x_k + \sum_{j \in \mu^{-1}(N_n)} b_{lj}x_j \right)^{N_n+1}$$

(3.3) $$= \sum_{k \in \mu^{-1}(N_n)} f_k(x_1, \ldots, x_h)(x_k^{\mu(k)+1} + c_1(\xi_k)x_k^{\mu(k)} + \cdots + c_{\mu(k)}(\xi_k)x_k)$$

$$+ \sum_{j \in \mu^{-1}(N_n)} b_{lj}^{N_n+1}(x_j^{N_n+1} + c_1(\xi_j)x_j^{N_n} + \cdots + c_{N_n}(\xi_j)x_j)$$

as polynomials, where $f_k(x_1, \ldots, x_h)$ is a homogeneous polynomial of degree $N_n - \mu(k)$ for each $k \in \mu^{-1}(N_{n-1})$. Note that for each $l \in \mu^{-1}(N_n)$, one of $b_{lj}$'s is non-zero.
for \( j \in \mu^{-1}(N_n) \) from the linearly independency of the set \( \{ z_i \mid i \in \mu^{-1}(N_{\leq p}) \} \). Let \( \mu^{-1}(N_n) := \{ l_1, \ldots, l_p \} \) with \( l_1 < \cdots < l_p \). Assume \( b_{l_pl_p} \neq 0 \) for some \( l_p \in \mu^{-1}(N_n) \). Substituting \( l = l_p \) into (3.3) and comparing the monomials containing \( x_{l_p}^N \) as a factor on both sides of (3.3), we have

\[
(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_{\leq p})} b_{l_pl_p} x_k + \sum_{j \in \mu^{-1}(N_n) \setminus \{ l_p \}} b_{l_pj} x_j \right) = b_{l_pl_p} c_1(\xi_{l_p}).
\]

Since \( c_1(\xi_{l_p}) \) belongs to \( H^2(B_{l_p-1}) \), we can see that \( b_{l_pj} = 0 \) for \( k > l_p \). That is,

\[
z_{l_p} = \sum_{k \in \mu^{-1}(N_{\leq p})} b_{l_pl_p} x_k + \sum_{j \in \mu^{-1}(N_n) \setminus \{ l_p \}} b_{l_pj} x_j.
\]

Thus, we can see that \( z_{l_p} \in H^2(B_{l_p}) \) and \( b_{l_pj} = 0 \) for all \( l \in \mu^{-1}(N_n) \setminus \{ l_p \} \) by Lemma 3.3. Put \( w_{l_p} := z_{l_p} \). Now assume that \( b_{l_pj_{l_p-1}} \neq 0 \) for some \( l_{q} \in \mu^{-1}(N_n) \setminus \{ l_p \} \). Substituting \( l = l_{q} \) into (3.3) and comparing the monomials containing \( x_{l_{q}}^N \) as a factor on both sides of (3.3), we have

\[
(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_{\leq p})} b_{l_pl_p} x_k + \sum_{j \in \mu^{-1}(N_n) \setminus \{ l_{q} \}} b_{l_qj} x_j \right) = b_{l_pj_{l_{q}-1}} c_1(\xi_{l_{q}-1}).
\]

Since \( c_1(\xi_{l_{q}-1}) \) belongs to \( H^2(B_{l_{q}-1}) \), we can see that \( b_{l_qj} = 0 \) for \( k > l_{q}-1 \), and hence,

\[
z_{l_{q}} = \sum_{k \in \mu^{-1}(N_{\leq p})} b_{l_pl_p} x_k + \sum_{j \in \mu^{-1}(N_n) \setminus \{ l_{q} \}} b_{l_qj} x_j.
\]

Thus, we can see that \( z_{l_{q}} \in H^2(B_{l_{q}-1}) \) and \( b_{l_qj_{l_{q}-1}} = 0 \) for all \( l \in \mu^{-1}(N_n) \setminus \{ l_p, l_{q} \} \) by Lemma 3.3. Now, put \( w_{l_{q}-1} := z_{l_{q}} \). In this way, for each \( l \in \mu^{-1}(N_n) \), we can obtain \( w_l \in H^2(B_l) \) such that \( w_l \notin H^2(B_{l-1}) \) and \( w_l^{N_{l}+1} = 0 \) in \( H^2(B_l) \). Moreover, from the proof of Lemma 3.3, \( w_l \) can be written as \( r_l(x_l + 1/(N_n + 1) c_1(\xi_l)) \). Furthermore, without loss of generality, we may assume that \( z_l = w_l \) for \( l \in \mu^{-1}(N_n) \).

By Cases 1 and 2, we can see that, for each \( i = 1, \ldots, h \), we can write

\[
z_i = r_l \left( x_l + \frac{1}{n_i + 1} c_1(\xi_l) \right)
\]
for some \( r_i \in \mathbb{Z} \setminus \{0\} \). Therefore, \( \{(n_i + 1)x_i + c_1(\xi_i)\}^{n_i+1} \) is zero in \( H^*(B_h) \). From this, we can see

\[
(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k \quad \text{and} \quad c_1(\xi_i)^{n_i+1} = 0
\]

\( k = 1, \ldots, n_i \).

By using Proposition 1.1, we can prove Theorem 1.2.

Proof of Theorem 1.2. We first prove the implication (1) \( \Rightarrow \) (2). By Proposition 1.1, we have the relation

\[
(3.4) \quad (n_i + 1)^2 c_2(\xi_i) = \frac{n_i(n_i + 1)}{2} c_1(\xi_i)^2.
\]

If \( n_i = 2 \), from (2.2) and (3.4), we have

\[
(3.5) \quad (a_{i_1,j_1} + a_{i_2,j_2})^2 = 3a_{i_1,j_1}a_{i_2,j_2} \quad \text{whose integer solution is only}
\]

\( a_{i_1,j_1} = a_{i_2,j_2} = 0 \). If \( n_i = n > 2 \), then we have

\[
(3.6) \quad n((a_{i_1,j_1} + \cdots + a_{i_{n-1},j_{n-1}}) + \cdots + (a_{i_{n-1},j_{n-1}} + \cdots + a_{i_{n-1},j_{n-1}}))^2
\]

\[
= 2(n + 1)((a_{i_1,j_1} + \cdots + a_{i_{n-1},j_{n-1}})(a_{i_2,j_2} + \cdots + a_{i_{n-1},j_{n-1}}) + \cdots
\]

\[
+ (a_{n-1,j_{n-1}} + \cdots + a_{n-1,j_{n-1}})(a_{n-1,j_{n-1}} + \cdots + a_{n-1,j_{n-1}})).
\]

Since \( x_j^2 \neq 0 \) in \( H^*(B_i) \) for \( j = 1, \ldots, i - 1 \), by comparing the coefficients of \( x_j^2 \) on both sides of (3.6) we have

\[
(3.7) \quad n(a_{i_1,j_1} + \cdots + a_{i_{n-1},j_{n-1}})^2 = 2(n + 1) \sum_{1 \leq k < l \leq n} a_{k,l}^i a_{l,k}^j.
\]

The equation (3.7) is equivalent to

\[
\sum_{m=1}^{n}(a_{m,j}^i)^2 + \sum_{1 \leq k < l \leq n} (a_{k,j}^i - a_{l,j}^i)^2 = 0.
\]

Therefore, \( a_{i_1,j_1} = \cdots = a_{i_{n-1},j_{n-1}} = 0 \) for each \( j = 1, \ldots, i - 1 \), and hence, in any case, \( c(\xi_i) \) is trivial for all \( i = 1, \ldots, h \).
Let $y (3.9) 2$ we can see that as polynomials. So, we can see that

$$\text{line bundle over } B$$

isomorphism fiber bundle $P (3.8)$ $(\text{respectively}, P)$ defined by $\phi$ and $\phi$ are isomorphic to a generalized Bott manifold. But a $\mathbb{Q}$-trivial generalized Bott manifolds with $n_i > 1$ is diffeomorphic to $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$. Hence, $M$ is homeomorphic to $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$. □

The following is the counter-example of Question 1.1.

**Example 3.1.** Let $B$ be a fiber bundle $P(\mathbb{C}^3 \oplus \xi)$ over $\mathbb{C}P^2$ and let $B'$ be a fiber bundle $P(\mathbb{C}^3 \oplus \xi^2)$ over $\mathbb{C}P^2$, where $\xi$ is the tautological line bundle over $\mathbb{C}P^2$. Let $y$ (respectively, $Y$) denote the negative of the first Chern class of the tautological line bundle over $B_2$ (respectively, $B'_2$). Then their cohomology rings are

$$H^*(B) = \mathbb{Z}[x, y]/(x^3, y(y^3 + xy^2))$$

and

$$H^*(B') = \mathbb{Z}[X, Y]/(X^3, Y(Y^3 + 2XY^2)).$$

Then the map $\phi$ defined by $\phi(x) = 2X$ and $\phi(y) = Y$ is an isomorphism from $H^*(B; \mathbb{Q}) \rightarrow H^*(B'; \mathbb{Q})$. But this $\phi$ is not a $\mathbb{Z}$-isomorphism. Suppose that $\psi$ is an isomorphism $H^*(B; \mathbb{Z}) \rightarrow H^*(B'; \mathbb{Z})$. Then there exist $\alpha, \beta, \gamma, \delta$ in $\mathbb{Z}$ such that

$$\begin{pmatrix} \psi(x) \\ \psi(y) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and $\alpha \delta - \beta \gamma = \pm 1$. Since $\psi(x^3) = 0$ in $H^*(B'; \mathbb{Z})$, we have

$$(\alpha X + \beta Y)^3 = \alpha^3 X^3$$

as polynomials. So, we can see that $\beta$ is zero and $\alpha = \pm 1$, and hence $\delta = \pm 1$. Since $\psi(y(y^3 + xy^2))$ is zero in $H^*(B'; \mathbb{Z})$, we have

$$(3.8) \quad (\gamma X + \delta Y)^3((\alpha + \gamma)X + \delta Y) = (aX + bY)X^3 + cY(Y^3 + 2XY^2)$$

as polynomials in $\mathbb{Z}[X, Y]$. By comparing the coefficients of $XY^3$ on both sides of (3.8), we can see that

$$(3.9) \quad 2c = 3\gamma \delta^3 + (\alpha + \gamma)\delta^3 = \delta(\alpha + 4\gamma).$$
Since the right hand side of (3.9) is odd, there is no such an integer $c$. Hence, there is no such $\mathbb{Z}$-isomorphism $\psi$.

Now consider $\mathbb{Q}$-trivial generalized Bott manifolds $B_h$ which have $\mathbb{C}P^1$-fibers, that is, $n_k = 1$ for some $k \in [h]$.

**Lemma 3.6.** Let $B_h$ and $B'_h$ be two $h$-stage generalized Bott towers. If the associated vector matrices to them are

$$A = \begin{pmatrix}
1 \\
* & * & 1 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
a_1 & \cdots & a_{h-2} \\
b_1 & \cdots & b_{h-2} \\
\end{pmatrix}$$

and

$$A' = \begin{pmatrix}
1 \\
* & * & 1 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
b_1 & \cdots & b_{h-2} \\
a_1 & \cdots & a_{h-2} \\
\end{pmatrix},$$

respectively, then $B_h$ and $B'_h$ are equivariantly diffeomorphic.

Proof. Note that this lemma can be seen by the fact that $B_h$ and $B'_h$ are equivariantly diffeomorphic if two associated vector matrices are conjugated by a permutation matrix, see the paper [3]. It is obvious that

$$A' = E_{\sigma} A E_{\sigma}^{-1},$$

where $\sigma := (1, \ldots, h-2, h, h-1)$ is the permutation on $[h]$ which permutes only $h-1$ and $h$.

Now, we can prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $B_h$ be a $\mathbb{Q}$-trivial generalized Bott manifold whose associated matrix is of the form (2.4).

Consider a map $\mu: \{1, \ldots, h\} \to \mathbb{N}$ given by $j \mapsto n_j$ and assume that the image of $\mu$ is the set $\{N_1, \ldots, N_m\}$ with $1 = N_1 < N_2 < \cdots < N_m$.

For each $i \in \mu^{-1}(1)$, by Proposition 1.1, we have $c_1(\xi_i)^2 = 0$ in $H^*(B_h)$. Since $x_k^2 \neq 0$ in $H^*(B_h)$ for $k \notin \mu^{-1}(1)$, we can see that $a_{ik}^i = 0$ for $k \in [i-1]$ with $n_k > 1$. 

\qed
Now suppose that \( n_j > 1 \). Then by Proposition 1.1, we have the relation
\[
(n_j + 1)^2 c_2(\xi_j) = \frac{n_j(n_j + 1)}{2} c_1(\xi_j)^2.
\]

Since \( x_k^2 \neq 0 \) in \( H^*(B_h) \) for \( n_k > 1 \), we can show that \( a_k' = 0 \) by using the same argument to the proof of Theorem 1.2.

Since \( a_k' = 0 \) for all \( n_k > 1 \), by Lemma 3.6, \( B_h \) is diffeomorphic to the \( \mathbb{Q} \)-trivial generalized Bott manifold \( B' \) whose associated matrix is of the form
\[
(A')^T = \begin{pmatrix}
1 & a_{11}^2 & 1 & \cdots & 1 \\
\vdots & a_{11}^r & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & a_{11}^{r+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{11}^h & \cdots & a_{11}^r & 1 \\
\end{pmatrix},
\]

where \( r \) is the cardinality of the set \( \mu^{-1}(1) \), that is, \( r = |\mu^{-1}(1)| \). This proves the theorem.

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