AN OBSERVER DESIGN FOR TIME-DELAY CONTROL
AND ITS APPLICATION TO DC SERVO MOTOR

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Abstract. This paper addresses the estimation problem of states and their derivatives for the Time Delay Control (TDC), a robust control technique for nonlinear systems. To this end, an observer design method is presented. Then, in simulations, the controller/observer has been applied to a nonlinear plant, with satisfactory results. Finally, experiments were undertaken on a DC servo motor subject to substantial inertia variations and external disturbances. The results showed that the controller/observer performs quite robustly under those variations and disturbances, and is much less sensitive to sensor noise than the controller using numerical differentiations.

Key Words. Observer; Time Delay Control; DC motor; robust control; sensor noise

1. INTRODUCTION

The main purpose of using robust control methods is to assure control performances (such as accuracy, stability, speed, etc) in the presence of significant plant uncertainties. The plant uncertainties, in general, include external disturbances, unpredictable parameter variations, and unmodeled plant nonlinear dynamics.

So far, several approaches for robust control have been proposed and considerable progress has been made in this area (Dorato, 1987). Among the popular techniques primarily intended for linear systems, there are LQG/LTR technique (Athans, 1986; Kazerooni and Houpt, 1986), $H^\infty$-related theory (Francis $et$ $al$, 1984; Kimura, 1984), and adaptive or self-tuning control (Craig $et$ $al$, 1986). Among the techniques used mainly for nonlinear systems, the sliding-mode control has been noted for its effectiveness (Utkin, 1977; Slotine and Sastry, 1983).

Recently, Time Delay Control (TDC) has also attracted attention as an excellent robust nonlinear control algorithm (Youcef-Toumi and Ito, 1990). To explain briefly, TDC uses the time-delayed values of control inputs and derivatives of state variables at the previous time step to cancel the nominal nonlinear dynamics and the aforementioned uncertainties. Thus, TDC does not require any real-time computation of nonlinear dynamics, nor does it use the parameter estimations as in adaptive control. As a result, TDC shows quite robust responses, adaptively canceling the uncertainties; yet it is computationally much more efficient than the aforementioned methods. These merits have been clearly demonstrated in the successful applications of TDC to a robot (Hsia and Gao, 1990; Youcef-Toumi and Shortlidge, 1991) and a magnetic bearing (Youcef-Toumi and Bobbett, 1991).

In order to apply TDC to a plant, it is necessary to be able to measure all of the state variables and their derivatives. Unfortunately, this is not always the case in practice. In many plants, even state variables are not always available, not to mention their derivatives. Even if all the state variables are measurable, additional sensors or numerical differentiators for their derivatives are needed, at the cost of further disadvantages: The use of derivative sensors makes the overall system more complex and expensive; the use of differentiators makes the system more sensitive to measurement noise. Hence, the measurability requirement presents a serious limitation on the implementation of TDC to real plants.

Regarding numerical differentiators, on the one hand, this can be alleviated by using the convolution method put forward by Youcef-Toumi and Shortlidge (1991), which requires one to have all the state variables available. As to the measurability requirement, on the other hand, the input/output linearization method of Youcef-Toumi and Wu (1991) enables one to apply TDC only
with output variables and their $r$-th derivatives, where $r$ denotes the relative degree of freedom. In this case, one needs a way to estimate the $r$-th derivatives, which may probably be done by numerical differentiation.

To avoid noise problems, by the way, a low-pass filter may be used in conjunction with the differentiation. From the authors' experience, using a filter often turns out to be effective. Nevertheless, low pass-filtering is sometimes observed to fail in reducing noise, no matter how well tuned it may be. Instead, it induces even larger overshoots — especially when nonlinearity effects are substantial.

In the context described so far, the problem may be stated as: How can TDC be applied to systems, where only a part of state variables are available, yet alleviating the noise problems due to numerical differentiations? As a solution to this problem, this paper proposes an observer design method that can stably reconstruct state variables and their derivatives, thereby trying to contribute to the degree of perfection of TDC.

In the following section, the control problem is defined and the TDC algorithm is reviewed. Section 3 presents an observer design method for TDC. In Section 4, the stability of overall system is analyzed and discussed. In Section 5, the effectiveness of the proposed observer and its sensitivity to sensor noise is evaluated through simulations. Section 6 presents the results of experimentations on a DC servo motor which is subject to substantial inertia variation and external disturbances. Finally in Section 7, the results are summarized and conclusions are drawn.

2. REVIEW OF TIME DELAY CONTROL LAW

For the sake of completeness, the TDC algorithm will be briefly reviewed. The nonlinear plant in question is described as follows:

$$\begin{align*}
\dot{x} &= f(x, t) + Bu + d(t) \\
y &= Cx
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is the control input, $y \in \mathbb{R}^m$ the plant output, $f(x, t)$ represents the plant dynamics, which is unknown, $d(t)$ is the unknown disturbance, $B(x, t)$ is the control distribution matrix, the ranges (instead of exact values) of which are known, and $C$ is the output distribution matrix. Note by the way that equation (1) covers a broad range of nonlinear dynamic plants.

Rearranging equation (1) into two terms — known part and unknown part — leads to

$$\begin{align*}
\dot{x} &= f(x, t) + \hat{B}u \\
y &= Cx
\end{align*}$$

(2)

where $\hat{B}$ denotes a constant matrix representing the known range of $B(x, t)$, and $\hat{f}(x, t)$ the unknown parts including uncertainties in the plant and disturbances, which are expressed as

$$\hat{f}(x, t) = f(x, t) + (B(x, t) - \hat{B})u + d(t).$$

(3)

Regarding the determination of $\hat{B}$, one will find more-detailed discussions in (Hsia and Gao, 1990; Youcef-Toumi and Reddy, 1992).

Desired performances can be specified with the response of a stable linear time-invariant reference model as

$$\dot{x}_m = A_m x_m + B_m r$$

(4)

where $x_m \in \mathbb{R}^n$ denotes the state vector of the reference model, $A_m$ the system matrix, $B_m$ the command distribution matrix, and $r \in \mathbb{R}^r$ the command vector. Then the control objective is to make the state of the plant track the response of the reference model, equation (4). If the error is defined as $e = x_m - x$, then error dynamics becomes, from equation (3) and equation (4),

$$\dot{e} = A_m e + [-\hat{f}(x, t) + A_m x + B_m r - \hat{B} u].$$

(5)

If a control $u$ can be found, such that

$$-\hat{f}(x, t) + A_m x + B_m r - \hat{B} u = K e$$

(6)

where $K \in \mathbb{R}^{n \times n}$ denotes an error feedback gain matrix, then the error dynamics becomes

$$\dot{e} = (A_m + K) e$$

(7)

showing that the error vanishes as time goes on. Incidentally, the case of $K = 0$ may be accepted as a special case, since with $A_m$ alone the error dynamics can be made stable.

Equation (6), however, cannot be always satisfied if the number of controls is smaller than the number of the states, in which case a least-square approximate solution is used to determine the control $u$ as

$$u = \hat{B}^+[-\hat{f}(x, t) + A_m x + B_m r - K e]$$

(8)

where $\hat{B}^+$ denotes a pseudo-inverse of $\hat{B}$. Incidentally, equation (8) is used as the basic algorithm of TDC. The error dynamics in this case is obtained, by substituting equation (8) into equation (5), as the following:

$$\dot{e} = (A_m + K) e +$$

$$[I - \hat{B} \hat{B}^+] [-\hat{f}(x, t) + A_m x + B_m r - K e].$$

(9)

This equation obviously satisfies equation (7), resulting in the same error dynamics, provided that

$$[I - \hat{B} \hat{B}^+] [-\hat{f}(x, t) + A_m x + B_m r - K e] = 0.$$

(10)
The conditions on which equation (10) is always satisfied are discussed in (Youcef-Toumi and Ito, 1990). Especially, it is noteworthy that this equation is always satisfied when the phase variables are used as state variables.

When implementing the algorithm in equation (8), it is required to estimate \( \hat{f}(x,t) \), the total effect of plant uncertainties. An especially efficient estimation method results from the following ideas: First, make use of the fact that \( f(x,t) \) may be assumed mostly as a continuous function, from which it follows that, for sufficiently small \( L \),

\[
\hat{f}(x,t) \approx \hat{f}(x,t - L) \tag{11}
\]

Secondly, use equation (2) together with equation (11). Then, one obtains the following estimation for the total effect of uncertainties:

\[
\hat{f}(x,t) = x - \hat{Bu} \approx \hat{x}(t - L) - \hat{Bu}(t - L). \tag{12}
\]

Substituting this estimation into equation (8) leads to the following TDC control law:

\[
u = \hat{B}^+[-\hat{x}(t-L)+\hat{Bu}(t-L)]+A_mz+Bu(t-L)+K(x_m-x). \tag{13}
\]

3. OBSERVER DESIGN

As is clearly shown in equation (13), the TDC control law requires the estimation of states and their derivatives. In practice, this requirement sets non-trivial limitations on the application of TDC to real plants. As a solution to this problem, an observer design specifically for the TDC controller is proposed.

3.1 Derivation of observer equation

In a case where there are no uncertainties in the plant, perhaps a Luenberger-type observer would suffice (Luenberger, 1966). In the presence of substantial uncertainties in the plant, however, observer design becomes more complex, since it is necessary to estimate the uncertainties in addition to states. Hence, in this case, the structure of the observer often becomes nonlinear and the stability of the observer may not be easily analyzed (Mielczarski, 1987).

Nevertheless, the observer design for TDC becomes especially simple because of the fact: TDC enables the plant dynamics to immediately follow the reference model. More specifically, it is the reference model (linear and certain) — instead of the plant model (nonlinear and uncertain) — that is used to reconstruct the states.

Thus in the system with uncertainties expressed in equation (2), the states are reconstructed by using the following linear observer:

\[
\dot{z} = A_mz + B_mr + F(\dot{y} - y) = A_mz + B_mr + FC(z - x) \tag{14}
\]

where \( z \in \mathbb{R}^m \) denotes an observer state vector, \( F \) an \((n \times m)\) constant observer gain matrix, and \( \dot{y} \in \mathbb{R}^m \) an observer output vector.

When this observer (together with TDC) is connected to the plant, the control input \( u \) is to be obtained, by using the reconstructed states \( z \), instead of the states \( x \). In addition, the uncertainties at time \( t \) are to be estimated with the reconstructed state \( z \) at \( t - L \). Thus, the control input \( u \) and the estimation using time delay are determined as

\[
\dot{u} = B^+[-\dot{f}(z,t)+A_mz+Bu(t-L)] \tag{15}
\]

\[
\dot{f}(z,t) \approx \dot{z}(t - L) - \hat{Bu}(t - L). \tag{16}
\]

The overall system can be illustrated with the block diagram in Fig. 1.

**Fig. 1.** Block diagram of TDC system with proposed observer

3.2 Overall stability of observer and controller

As to the stability of resulting system consisting of the plant, TDC, and the proposed observer, the following theorem will be presented as a sufficient condition for the stability of the overall system.

Thus if the proposed observer is designed so that the observer gain matrix \( F \) and time delay \( L \) may meet this condition, then the resulting system is made stable.

**Theorem 1** The overall system with TDC and observer is internally stable, if \( \dot{f}(x,t) \) is a continuous function of time, and the eigenvalues, \( \lambda \),

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1 Note that this is an approximate estimation, which becomes more accurate as \( L \) decreases.
of the following characteristic equation (17) lie in the left half of the s-plane.

$$\det \left\{ (\lambda I - A_s) - \hat{B} \hat{B}^T (e^{-L\lambda} \lambda I - A_s) \right\} (\lambda I - A_s)^{-1} (\lambda I - A_m) = 0$$ (17)

where

$$A_s = (A_m + K), \quad A_m = (A_m + FC).$$ (18)

The proof of this theorem, based on the results of Youcef-Toumi and Reddy (1992), is listed in Appendix 1. In addition to the overall stability, the convergence of error, $e = x_m - x$, may be stated in the following theorem, the proof of which is also found in Appendix 2.

**Theorem 2**

1. The overall system is internally stable, and
2. $\dot{e}(x, t)$ is a continuous function of time with a constant steady state value,

then the error between reference model and plant, $(e = x_m - x)$, converges to zero as time approaches infinity.

4. SIMULATION

To demonstrate the effectiveness of the proposed observer, the observer design has been carried out for the following second-order system:

$$\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x 
\end{align*}$$ (19)

where

$$\alpha \equiv 2 \times z_2 \times \sin z_1 / (2 \sin z_1)$$
$$\beta \equiv \cos z_1 / (2 \sin z_1)$$
$$b \equiv 1 / (2 \sin z_1)$$

with $z_1$ and $z_2$ representing the states of the system. In this example, $\alpha$ and $\beta$ are assumed to be completely unknown, whereas the range of $b$ is known. The reference model was selected as a second-order system described by

$$\begin{align*}
\dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \omega_n \end{bmatrix} r 
\end{align*}$$ (20)

where its natural frequency $\omega_n$ and damping ratio $\zeta$ are set to be 10 rad/sec and 1, respectively. The input command $r$ was chosen to be 1 rad. By setting the observer gain matrix to be $F = [F_1 \quad F_2]^T$, the proposed observer can be designed as

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ \omega_n \end{bmatrix} r + F(y - \hat{y}).$$ (21)

From equation (19), equation (20) and equation (15), TDC control law becomes:

$$u(t) = u(t-L) + \frac{1}{T} \left[ -\dot{z}_3 (t-L) - \omega_n^2 z_1 - 2\zeta \omega_n z_2 + \omega_n^2 r \right]$$

where $\dot{b}$ denotes the nominal value of $b$, which is set to be 1.5.

The performance of the observer was verified in two respects: first, whether it can satisfactorily reconstruct the states and their derivatives; second, how the use of the observer affects overall system performance as compared to the use of numerical differentiation. To this end, the proposed observer with the gain matrix of $F = [230 \quad 7600]^T$ was chosen. More specifically, the observer gains were selected so that the eigenvalues of $A_m + FC$ may have the values of $-67$, $-180$, which are about seven times larger than the eigenvalues of $A_m$, which are $-10$, $-10$.

In this simulation, the plant dynamics in equation (19) was computed by using the fourth-order Runge-Kutta method, with the time step of 0.0001 second. In conjunction with plant dynamics, the observer dynamics in equation (21) was also computed by the same Runge-Kutta method with the time step of 0.001 second; and the reconstructed states and state derivatives were used to obtain the control input $u$ in equation (22) with the sampling time of 0.01 second. In the second simulation, a certain amount of sensor noise was applied to the closed-loop system, and noise-corrupted response was obtained for the two cases: when numerical differentiation was used, and when the proposed observer was used. For sensor noise, zero-mean Gaussian noises with standard deviations of $\sigma = 0.002$ and 0.02, respectively — corresponding to 0.2% and 2% of the magnitude of the command signal — were applied. As predicted, the responses in Fig. 3 show that state derivatives estimated by numerical differentiation are very sensitive to sensor noise. When using the proposed observer, however, the reconstructed states and their derivatives are not so sensitive to sensor noise; the error due to sensor noise is less than 1/10 of the error with numerical differentiation.

As shown in Fig. 2, the proposed observer reconstructed states and their derivatives well, with the plant satisfactorily following the desired model.

In the second simulation, a certain amount of sensor noise was applied to the closed-loop system, and noise-corrupted response was obtained for the two cases: when numerical differentiation was used, and when the proposed observer was used. For sensor noise, zero-mean Gaussian noises with standard deviations of $\sigma = 0.002$ and 0.02, respectively — corresponding to 0.2% and 2% of the magnitude of the command signal — were applied. As predicted, the responses in Fig. 3 show that state derivatives estimated by numerical differentiation are very sensitive to sensor noise. When using the proposed observer, however, the reconstructed states and their derivatives are not so sensitive to sensor noise; the error due to sensor noise is less than 1/10 of the error with numerical differentiation.

From these simulation results, it can be seen that proposed observer can effectively reconstruct states and their derivatives, and at the same time is quite insensitive to sensor noise as compared to
5. EXPERIMENTS

In order to assure the validity of the proposed observer in a real system, a DC servo motor system was selected to make experiments. The experimental setup is shown in Fig. 4. The following three points have been examined:

- the robustness of the observer (with TDC controller) in the presence of parameter variations,
- the performance under external disturbance, and
- the sensitivity to sensor noise.

On the first two points, checks were made to see if the proposed observer with TDC preserves the performances that TDC with a numerical differentiator has shown (Youcef-Toumi and Ito, 1990). For the purpose of comparison, a PD controller was also used on the plant. On the third point, the focus is on the effectiveness of the use of the observer as compared to the case with TDC with a numerical differentiator.

The dynamics of the DC servo motor can be characterized by a second-order equation:

$$M\ddot{\theta} + B\dot{\theta} + D(\theta, \dot{\theta}, t) = \tau$$

where $\theta$ denotes the motor shaft rotating angle, $M$ the effective moment of inertia of motor, $B$ the effective viscous friction coefficient, $D(\theta, \dot{\theta}, t)$ an unknown dynamics including various internal nonlinearities and external disturbances, and $\tau$ the input torque to the motor.

Defining the following second-order system as the reference model,

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \tau \quad (24)$$

the following control law can be derived:

$$\tau(t) = \tau(t-L) + \dot{M}[-\dot{\theta}(t-L) + \omega_n^2(\tau-\theta) - 2\zeta\omega_n\dot{\theta}] \quad (25)$$

where $\hat{\theta}$, $\dot{\theta}$ are estimated by the following observer:

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} z + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \tau + F(y-\hat{y}). \quad (26)$$

To be more specific, the observer dynamics was computed by the Euler method, with the sampling time of 0.0008 second; the TDC control, (25), was
Fig. 5. Experimental results without uncertainty for PD and TDC control system.

Fig. 6. Experimental results with inertia variations: (a) PD control case; (b) TDC control case

Fig. 7. Experimental results with external spring disturbance: (a) PD control case; (b) TDC control case.

Fig. 8. Experimental results when numerical differentiator and the proposed observer are compared.

In the first experiment, neither external disturbance nor parameter variation was applied to the motor system. As shown in Fig. 5, both the two closed-loop systems with TDC and PD followed the desired response well; by comparison, the system with PD control shows the faster response, yet having a slight steady-state error.

In the second experiment, the robustness of the control systems to parameter variations was tested by increasing the value of $M$ to 0.1 $kg - cm^2$ (1.7 times to the nominal value) and 0.7 $kg - cm^2$ (12 times to the nominal value). This range of variation could be observed in the joint motors of a robot with a small amount of gear reduction, when the end effector loads (or unloads) payloads.

As shown in (a) and (b) of Fig. 6, the system with TDC (and the proposed observer) is hardly affected by the wide range of parameter variations. In contrast, the PD control system shows larger overshoot due to the change of closed-loop pole location, as the value of $M$ increases.

In the third experiment, robustness to external disturbances was tested, where, as external disturbances, springs were attached to the load. More specifically, two springs with different values of stiffness were used: (1) a soft spring with $K = 1.2kgf - cm$ applying disturbance torque of $-K(0.5 - \sin \theta)$, which amounts up to about 30% of motor stall torque; (2) a hard spring with $K = 1.8kgf - cm$ applying disturbance of $-K \sin \theta$, which amounts up to about 45% of stall torque.

As shown in Fig. 7(b), regardless of the magnitude of spring disturbances, the responses of TDC control system are almost identical to the desired response, demonstrating its ability to reject external disturbances. In contrast, the PD control system, as shown in Fig. 7(a), has a steady-state error, which increases as the magnitude of the spring disturbance increases.
Incidentally, adding some integral action to the controller (thus the PID controller) could reduce the steady-state error. Nevertheless, it was very difficult to find an integral gain that ensures stability under such a wide range of parameter variations. Hence the case with PID was not included.

In order to see the effectiveness of the proposed observer, the response using the observer was compared with the response using numerical differentiation. As shown in (a), (b), and (c) of Fig. 8, it can be seen that the observer works well and the states and their derivative reconstructed by observer are somewhat less noisy compared to those achieved by numerical differentiation.

6. CONCLUSION

In this paper an observer design method for TDC has been proposed, and the overall stability of observer and controller has been analyzed. It was shown that the proposed observer reconstructs states and their derivatives very well in the presence of plant uncertainties, while preserving the performance of TDC alone. Thus the TDC algorithm may be expanded to systems where all the states and their derivatives are not measurable. In addition, since the proposed observer does not require an accurate model of plant dynamics with uncertainties, its structure is simple and easy to implement.

The effectiveness of the proposed observer was evaluated through simulations and experiments. Through the simulations, for a second-order nonlinear plant with uncertainties, it was demonstrated that the proposed observer reconstructed the states and their derivatives very well, and the good control performance of TDC was not degraded by the observer. It was also demonstrated that the control system using the observer is less sensitive to sensor noise than the control system using numerical differentiation.

The simulation results have also been confirmed by experiment. It was demonstrated that the designed control system is very robust to external disturbance and motor inertia variation. It turned out that the proposed observer worked well, and the overall system of observer and controller was somewhat less sensitive to sensor noise.

7. REFERENCES


APPENDIX 1: PROOF OF THEOREM 1

If \( u \) in equation (15) together with equation (16) are substituted into equation (2), the overall system dynamics becomes

\[
\dot{x} = f(x, t) + \hat{B}\hat{B}^+[-x(t-L) + \hat{B}u(t-L) + A_m x + B_m r - K e]
\]

\[
+ \hat{B}\hat{B}^+[-\dot{x}(t-L) + \dot{x}(t-L) + (A_m + K)(z - x)].
\]

(27)

Since the overall system is assumed to satisfy the constraint, equation (10), equation (27) may be expressed as

\[
x = (A_m + K)x + B_m r - Kx + f(x, t) - f(x, t-L)
\]

\[
+ \hat{B}\hat{B}^+[-\dot{x}(t-L) + \dot{x}(t-L) + (A_m + K)(z - x)].
\]

(28)

Taking the Laplace transform of equation (14) with \( z(0) = 0 \) leads to

\[
z(s) = -G(s)z(s) + (I - L'sI - A_c)X(s) + B_m R(s).
\]

(29)

Taking the Laplace transform of equation (28) with \( x(0) = 0 \) and \( z(0) = 0 \), into which equation (29) together with equation (18) is substituted, then equation (28) becomes

\[
\begin{align*}
(sI - A_c) - \hat{B}\hat{B}^+(e^{-L'sI} - A_c).

(sI - A_c)^{-1}(sI - A_m) \& X(s) =

[I - \hat{B}\hat{B}^+(e^{-L'sI} - A_c)](sI - A_c)^{-1}B_m R(s)

- KX_m(s) + (1 - e^{-L'sI})\hat{F}(s)
\end{align*}
\]

(30)

where \( \hat{F}(s) \) denotes the Laplace transform of \( \hat{f}(x, t) \). Meanwhile, taking the Laplace transform of the reference model, equation (4), with \( x_m(0) = 0 \) leads to

\[
x_m(s) = (sI - A_m)^{-1}B_m R(s).
\]

(31)

Substitute equation (31) into equation (30), and rearrange it. Then equation (30) results in

\[
\Phi(s)X(s) = \Phi(s)(sI - A_m)^{-1}B_m R(s)
\]

\[
+ (1 - e^{-L'sI})\hat{F}(s)
\]

(32)

where

\[
\Phi(s) = \{(sI - A_c) - \hat{B}\hat{B}^+(e^{-L'sI} - A_c).
\]

\[
(sI - A_c)^{-1}(sI - A_m)\}
\]

(33)

From equations (32) and (33), the characteristic equation of the overall system is

\[
\text{det}\left\{(sI - A_c) - \hat{B}\hat{B}^+(e^{-L'sI} - A_c), (sI - A_m)\right\} = 0.
\]

(34)

Accordingly, if \( \hat{f}(x, t) \) is a continuous function, then the overall system is internally stable, provided that the roots of equation (34), \( \lambda \), lie in the left half plane. \( Q.E.D. \)

APPENDIX 2: PROOF OF THEOREM 2

From the final value theorem, the steady state error is determined as

\[
\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s).
\]

(35)

By using \( e = x_m - x \) and the following reference model,

\[
x_m(s) = A_m x_m(s) + B_m R(s)
\]

equation (30) may be expressed as

\[
\begin{align*}
\{(sI - A_c) - \hat{B}\hat{B}^+(e^{-L'sI} - A_c).

(sI - A_c)^{-1}(sI - A_m)\& E(s) =

-I - \hat{B}\hat{B}^+(e^{-L'sI} - A_c)(sI - A_m)^{-1}B_m R(s)

- (1 - e^{-L'sI})\hat{F}(s).
\end{align*}
\]

(37)

Using equation (31) together with the assumption of internal stability leads equation (37) to

\[
\begin{align*}
\{(sI - A_c) - \hat{B}\hat{B}^+(e^{-L'sI} - A_c).

(sI - A_c)^{-1}(sI - A_m)\& E(s) =

- (1 - e^{-L'sI})\hat{F}(s).
\end{align*}
\]

(38)

Then, from equation (38) the steady state error is determined as the following:

\[
\lim_{s \to 0} sE(s) = -\Gamma^{-1}\lim_{s \to 0}\{(1 - e^{-L'sI})s\hat{F}(s)\}
\]

(39)

where nonsingular matrix, \( \Gamma \), is defined as

\[
\Gamma = -A_c + \hat{B}\hat{B}^+A_cA_c^{-1}A_m.
\]

(40)

Provided that \( \hat{f}(x, t) \) has a constant steady state value, the following holds:

\[
\Gamma^{-1}\lim_{s \to 0}\{(1 - e^{-L'sI})s\hat{F}(s)\} = 0
\]

(41)

which immediately leads to the zero steady state error. \( Q.E.D. \)