ENHANCEMENT OF NEAR CLOAKING FOR THE FULL MAXWELL EQUATIONS

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Abstract. Recently published methods for the quasi-static limit of the Helmholtz equation is extended to consider near cloaking for the full Maxwell equations. Effective near cloaking structures are described for the electromagnetic scattering problem at a fixed frequency. These structures are, prior to using the transformation optics, layered structures designed so that their first scattering coefficients vanish. As a result, any target inside the cloaking region has near-zero scattering cross section for a band of frequencies. Analytical results show that this construction significantly enhances the cloaking effect for the full Maxwell equations.

Key words. cloaking, transformation optics, Maxwell equations, scattering amplitude, scattering coefficients

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1. Introduction. The cloaking problem is to make a target invisible from far-field electromagnetic measurements [29, 19, 23, 12, 11, 20]. Many schemes for cloaking are currently under active investigation. These include exterior cloaking in which the cloaking region is outside the cloaking device [25, 1, 24, 10, 9, 2], active cloaking [14], and interior cloaking, which is the focus of this paper.

In interior cloaking, the difficulty is to construct electromagnetic material parameter distributions of a cloaking structure such that any target placed inside the structure is undetectable to waves. One approach is to use transformation optics [29, 12, 30, 32, 15], which takes advantage of the fact that the equations governing electromagnetism have transformation laws under change of variables. This allows one to design structures that steer waves around a hidden region, returning them to their original path on the far side. The cloaking method based on change of variables uses a singular transformation to boost the material properties so that it makes a cloaking region look like a point to outside measurements. However, this transformation induces the singularity of material constants in the transversal direction (also in the tangential direction in two dimensions), which causes difficulty both in theory and in applications. To overcome this weakness, so-called near cloaking is naturally considered, which is a regularization or an approximation of singular cloaking. In [18], instead of the singular transformation, the authors use a regular one to push forward the material constant in the conductivity equation describing the quasi-static limit of electromagnetism, in which a small ball is blown up to the cloaking region. In
[17], this regularization point of view is adopted for the Helmholtz equation. See also [21, 28]. More recently, Bao and Liu [8] considered near cloaking for the full Maxwell equations. They derived sharp estimates for the boundary effect due to a small inclusion with arbitrary material parameters enclosed by a thin high-conducting layer. Their results show that the near cloaking scheme can be applied to cloak targets from electromagnetic boundary measurements.

In the recently published papers [5, 6], it is shown that near cloaking, from measurements of the Dirichlet-to-Neumann map for the conductivity equation and of the scattering cross section for the Helmholtz equation, can be drastically enhanced by using multilayered structures. The structures are designed so that their generalized polarization tensors (GPTs) or scattering coefficients vanish (up to a certain order). GPTs are building blocks of the far-field behavior of solutions in the quasi-static limits (conductivity equations) [4] and the scattering coefficients are “Fourier coefficients” of the scattering amplitude [6, 7]. The multilayered structures combined with the usual change of variables (transformation optics) greatly reduce the visibility of an object. This fact is also confirmed by numerical experiments [3].

The purpose of this paper is to extend the results of [5, 6] to full Maxwell’s equations. It shows that near cloaking from cross section scattering measurements at a fixed frequency can be enhanced by using layered structures together with transformation optics. Of particular importance is the notion of scattering coefficients of an inclusion, which is extended in this paper to full Maxwell equations. As for the Helmholtz equation, the layered structures, prior to using the transformation optics, are designed so that their first scattering coefficients vanish. It is also shown that inside the cloaking region, any target has a near-zero scattering cross section for a band of frequencies. Analytical results prove that this construction significantly enhances the near cloaking effect for the full Maxwell equations. It is worth mentioning that even if the basic scheme of this work is parallel to that of [6], the analysis is much more complicated due to the vectorial nature of the Maxwell equations.

The paper is organized as follows. In section 2, some fundamental results on the scattering problem for the full Maxwell equations are recalled. Section 3 introduces the scattering coefficients of an electromagnetic inclusion and proves that the scattering coefficients are basically the spherical harmonic expansion coefficients of the far-field pattern. Section 4 is devoted to the construction of layered structures with vanishing scattering coefficients. Numerical examples of the scattering coefficient vanishing structures are presented. Section 5 shows that the near cloaking is enhanced if a scattering coefficient vanishing structure is used.

2. Multipole solutions to the Maxwell equations. In this section, a few fundamental results related to electromagnetic scattering, which will be essential in what follows, are recalled.

Consider the time-dependent Maxwell equations

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\mu \frac{\partial}{\partial t} \mathbf{H}, \\
\nabla \times \mathbf{H} &= \epsilon \frac{\partial}{\partial t} \mathbf{E},
\end{align*}
\]

where \(\mu\) is the magnetic permeability and \(\epsilon\) is the electric permittivity.

In the time-harmonic regime, one looks for the electromagnetic fields of the form

\[
\begin{align*}
\mathbf{H}(\mathbf{x}, t) &= \mathbf{H}(\mathbf{x}) e^{-i\omega t}, \\
\mathbf{E}(\mathbf{x}, t) &= \mathbf{E}(\mathbf{x}) e^{-i\omega t},
\end{align*}
\]

where \(\omega\) is the frequency. The couple \((\mathbf{E}, \mathbf{H})\) is a solution to the harmonic Maxwell equations.
One says that \((E, H)\) is radiating if it satisfies the Silver–Müller radiation condition:
\[
\lim_{|x| \to \infty} |x|(\sqrt{\mu H} \times \hat{x} - \sqrt{\epsilon}E) = 0,
\]
where \(\hat{x} = x/|x|\). In what follows, one sets the wave number \(k = \omega \sqrt{\mu/\epsilon}\).

For \(m = -n, \ldots , n\) and \(n = 1, 2, \ldots\), set \(Y^m_n\) to be the spherical harmonics defined on the unit sphere \(S\). For a wave number \(k > 0\), the function
\[
v_{n,m}(k; x) = h_n^{(1)}(k|x|)Y^m_n(\hat{x})
\]
satisfies the Helmholtz equation \(\Delta v + k^2 v = 0\) in \(\mathbb{R}^3 \setminus \{0\}\) and the Sommerfeld radiation condition:
\[
\lim_{|x| \to \infty} |x| \left( \frac{\partial v_{n,m}}{\partial |x|}(k; x) - ik v_{n,m}(k; x) \right) = 0.
\]
Here, \(h_n^{(1)}\) is the spherical Hankel function of the first kind and order \(n\), which satisfies the Sommerfeld radiation condition. Similarly, \(\tilde{v}_{n,m}(x)\) is defined as
\[
\tilde{v}_{n,m}(k; x) = j_n(k|x|)Y^m_n(\hat{x}),
\]
where \(j_n\) is the spherical Bessel function of the first kind. The function \(\tilde{v}_{n,m}\) satisfies the Helmholtz equation in all \(\mathbb{R}^3\).

In the same manner, one can make solutions to the Maxwell system with the vector version of spherical harmonics. Define the vector spherical harmonics as
\[
U_{n,m} = \frac{1}{\sqrt{n(n+1)}} \nabla_S Y^m_n(\hat{x}) \quad \text{and} \quad V_{n,m} = \hat{x} \times U_{n,m}
\]
for \(m = -n, \ldots , n\) and \(n = 1, 2, \ldots\). Here, \(\hat{x} \in S\) and \(\nabla_S\) denotes the surface gradient on the unit sphere \(S\). The vector spherical harmonics defined in (2.4) form a complete orthogonal basis for \(L^2_T(S)\), where \(L^2_T(S) = \{u \in (L^2(S))^3 \mid \nabla \cdot u = 0\}\) and \(\nabla\) is the outward unit normal to \(S\).

Multiplying the vector spherical harmonics to the Hankel function, one can make the so-called multipole solutions to the Maxwell system. To keep the analysis simple, one separates the solutions into transverse electric, \((E \cdot x) = 0\), and transverse magnetic, \((H \cdot x) = 0\). Define the exterior transverse electric multipoles to (2.1) as
\[
\left\{ \begin{array}{ll}
E^{TE}_{n,m}(k; x) &= -\sqrt{n(n+1)} h_n^{(1)}(k|x|) V_{n,m}(\hat{x}), \\
H^{TE}_{n,m}(k; x) &= -\frac{i}{\omega \epsilon} \nabla \times \left(-\sqrt{n(n+1)} h_n^{(1)}(k|x|) V_{n,m}(\hat{x})\right),
\end{array} \right.
\]
and the exterior transverse magnetic multipoles as
\[
\left\{ \begin{array}{ll}
E^{TM}_{n,m}(k; x) &= \frac{i}{\omega \epsilon} \nabla \times \left(-\sqrt{n(n+1)} h_n^{(1)}(k|x|) V_{n,m}(\hat{x})\right), \\
H^{TM}_{n,m}(k; x) &= -\sqrt{n(n+1)} h_n^{(1)}(k|x|) V_{n,m}(\hat{x}).
\end{array} \right.
\]
The exterior electric and magnetic multipoles satisfy the radiation condition. In the same manner, one defines the interior multipoles \((E_{n,m}^{TE}, H_{n,m}^{TE})\) and \((\tilde{E}_{n,m}^{TM}, \tilde{H}_{n,m}^{TM})\) with \(h_n^{(1)}\) replaced by \(j_n\), i.e.,

\[
\begin{align}
\tilde{E}_{n,m}^{TE}(k; \mathbf{x}) &= -\frac{\sqrt{n(n+1)}}{|\mathbf{x}|} j_n^{(1)}(k|\mathbf{x}|) \mathbf{V}_{n,m}(\mathbf{x}), \\
\tilde{H}_{n,m}^{TE}(k; \mathbf{x}) &= -\frac{i}{\omega \mu} \nabla \times \tilde{E}_{n,m}^{TE}(k; \mathbf{x})
\end{align}
\]

and

\[
\begin{align}
\tilde{E}_{n,m}^{TM}(k; \mathbf{x}) &= -\frac{\sqrt{n(n+1)}}{|\mathbf{x}|} j_n^{(1)}(k|\mathbf{x}|) \mathbf{V}_{n,m}(\mathbf{x}), \\
\tilde{H}_{n,m}^{TM}(k; \mathbf{x}) &= \frac{j}{\omega \epsilon} \nabla \times \tilde{H}_{n,m}^{TM}(k; \mathbf{x}).
\end{align}
\]

The wave number \(k\) will sometimes be omitted in the notation of the multipoles.

Note that one has

\[
\nabla \times \mathbf{E}_{n,m}^{TE}(k; \mathbf{x}) = \frac{\sqrt{n(n+1)}}{|\mathbf{x}|} \mathcal{H}_n(k|\mathbf{x}|) \mathbf{U}_{n,m}(\mathbf{x}) + \frac{n(n+1)}{|\mathbf{x}|} \hat{h}_n^{(1)}(k_0|\mathbf{x}|) Y_n(\mathbf{x}) \mathbf{\hat{x}},
\]

\[
\nabla \times \mathbf{\tilde{E}}_{n,m}^{TE}(k; \mathbf{x}) = \frac{\sqrt{n(n+1)}}{|\mathbf{x}|} \mathcal{J}_n(k|\mathbf{x}|) \mathbf{U}_{n,m}(\mathbf{x}) + \frac{n(n+1)}{|\mathbf{x}|} \hat{j}_n^{(1)}(k_0|\mathbf{x}|) Y_n(\mathbf{x}) \mathbf{\hat{x}},
\]

where \(\mathcal{H}_n(t) = h_n^{(1)}(t) + t(h_n^{(1)})'(t)\) and \(\mathcal{J}_n(t) = j_n(t) + t j_n'(t)\).

The solutions to the Maxwell system can be represented as separated variable sums of the multipole solutions; see [27, section 5.3]. With multipole solutions and the Helmholtz solutions in (2.2) and (2.3), it is also possible to expand the fundamental solution to the Helmholtz operator. For \(k > 0\), the fundamental solution \(\Gamma_k\) to the Helmholtz operator \((\Delta + k^2)\) in \(\mathbb{R}^3\) is

\[
\Gamma_k(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{4\pi |\mathbf{x}|},
\]

Let \(\mathbf{p}\) be a fixed vector in \(\mathbb{R}^3\). For \(|\mathbf{x}| > |\mathbf{y}|\), the following addition formula holds (see [26, section 9.3.3]):

\[
\Gamma_k(\mathbf{x} - \mathbf{y}) \mathbf{p} = -\sum_{n=1}^{\infty} \frac{ik}{n(n+1)} \frac{\epsilon}{\mu} \sum_{m=-n}^{n} \mathbf{E}_{n,m}^{TM}(k; \mathbf{x}) \mathbf{\overline{E}}_{n,m}^{TM}(k; \mathbf{y}) \cdot \mathbf{p}
\]

\[
+ \sum_{n=1}^{\infty} \frac{ik}{n(n+1)} \sum_{m=-n}^{n} \mathbf{E}_{n,m}^{TE}(k; \mathbf{x}) \mathbf{\overline{E}}_{n,m}^{TE}(k; \mathbf{y}) \cdot \mathbf{p}
\]

\[
- \frac{i}{k} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \nabla v_{n,m}(k; \mathbf{x}) \nabla \overline{v}_{n,m}(k; \mathbf{y}) \cdot \mathbf{p},
\]

with \(v_{n,m}\) and \(\overline{v}_{n,m}\) being defined by (2.2) and (2.3).

Plane wave solutions to the Maxwell equations have the expansion using the multipole solutions as well (see [16]). The incoming wave \(\mathbf{E}'(\mathbf{x}) = ik(\mathbf{q} \times \mathbf{p}) \times q e^{ikq \cdot \mathbf{x}}\),
where \( \mathbf{q} \in S \) is the direction of propagation and the vector \( \mathbf{p} \in \mathbb{R}^3 \) is the direction of polarization, is expressed as

\[
E'(x) = ik \sum_{p=1}^{\infty} \frac{4\pi p}{\sqrt{p(p+1)}} \sum_{q=-p}^{p} \left[ (V_{p,q}(q) \cdot \mathbf{c}) E^{TF}_{p,q}(x) - \frac{1}{i\omega \mu} (U_{p,q}(q) \cdot \mathbf{c}) E^{TM}_{p,q}(x) \right],
\]

where \( \mathbf{c} = (\mathbf{q} \times \mathbf{p}) \times \mathbf{q} \).

3. Scattering coefficients of an inclusion. This section introduces the notion of scattering coefficients of an inclusion associated to Maxwell equations. It extends the notions and results established in [6] for the Helmholtz equation.

Let \( D \) be a bounded domain in \( \mathbb{R}^3 \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \), and let \( (\epsilon_0, \mu_0) \) be the pair of electromagnetic parameters (permittivity and permeability) of \( \mathbb{R}^3 \setminus \overline{D} \), and \( (\epsilon_1, \mu_1) \) that of \( D \). Assume that \( \epsilon_0, \epsilon_1, \mu_0, \) and \( \mu_1 \) are positive constants.

Then the permittivity and permeability distributions are given by

\[
\epsilon = \epsilon_0 \chi(\mathbb{R}^3 \setminus \overline{D}) + \epsilon_1 \chi(D) \quad \text{and} \quad \mu = \mu_0 \chi(\mathbb{R}^3 \setminus \overline{D}) + \mu_1 \chi(D),
\]

where \( \chi \) denotes the characteristic function. In what follows, let \( k = \omega \sqrt{\epsilon_1 \mu_1} \) and \( k_0 = \omega \sqrt{\epsilon_0 \mu_0} \).

For a given solution \( (E', H') \) to the Maxwell equations

\[
\begin{cases}
\nabla \times E' = i\omega \mu_0 H' & \text{in } \mathbb{R}^3, \\
\nabla \times H' = -i\omega \epsilon_0 E' & \text{in } \mathbb{R}^3,
\end{cases}
\]

let \( (E, H) \) be the solution to the following Maxwell equations:

\[
\begin{cases}
\nabla \times E = i\omega \mu H & \text{in } \mathbb{R}^3, \\
\nabla \times H = -i\omega \epsilon E & \text{in } \mathbb{R}^3, \\
(E - E', H - H') & \text{satisfies the Silver–Müller radiation condition.}
\end{cases}
\]

It is worth emphasizing that along the interface \( \partial D \), the following transmission condition holds:

\[
[\nu \times E] = [\nu \times H] = 0.
\]

Here, \([\nu \times E]\) denotes the jump of \( \nu \times E \) along \( \partial D \), namely,

\[
[\nu \times E] = (\nu \times E)_{\partial D}^+ - (\nu \times E)_{\partial D}^-.
\]

Let \( \nabla_{\partial D} \cdot \) denote the surface divergence. Let the function space

\[
TH(\text{div}, \partial D) := \left\{ u \in L^2_T(\partial D) : \nabla_{\partial D} \cdot u \in L^2(\partial D) \right\}
\]

be equipped with the norm

\[
\|u\|_{TH(\text{div}, \partial D)} = \|u\|_{L^2(\partial D)} + \|\nabla_{\partial D} \cdot u\|_{L^2(\partial D)}.
\]

For a density \( \varphi \in TH(\text{div}, \partial D) \), one defines the single-layer potential associated with the fundamental solutions \( \Gamma_k \) given in (2.11) by

\[
S_D^k[\varphi](x) := \int_{\partial D} \Gamma_k(x - y) \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^3.
\]
For a scalar density contained in $L^2(\partial D)$, the single-layer potential is defined the same way. One also defines the boundary integral operators:

$$L_D^k[\varphi](x) := (\nu \times (k^2 S_D^k[\varphi] + \nabla S_D^k[\nabla_{\partial D} \cdot \varphi]))(x),$$

$$M_D^k[\varphi](x) := \text{p.v.} \int_{\partial D} \nu(x) \times ((\nabla x \times (\Gamma_k(x - y) \varphi(y))) d\sigma(y), \quad x \in \partial D.$$

In the same way, one defines $S_D^{k_0}$, $L_D^{k_0}$, and $M_D^{k_0}$ associated with $\Gamma_{k_0}$ instead of $\Gamma_k$.

Then the solution to (3.1) can be represented as the following:

$$E(x) = \begin{cases} E^i(x) + \mu_0 \nabla \times S_D^k[\varphi](x) + \nabla \times \nabla \times S_D^k[\psi](x), & x \in \mathbb{R}^3 \setminus \overline{\mathcal{T}}, \\ \mu_1 \nabla \times S_D^k[\varphi](x) + \nabla \times \nabla \times S_D^k[\psi](x), & x \in D, \end{cases}$$

and

$$H(x) = -\frac{i}{\omega \mu} (\nabla \times E)(x), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

where the pair $(\varphi, \psi) \in TH(\text{div}, \partial D) \times TH(\text{div}, \partial D)$ is the unique solution to

$$\begin{align*}
\begin{bmatrix}
\frac{\mu_1 + \mu_0}{2} I + \mu_1 M_D^k - \mu_0 M_D^k \\
L_D^k - L_D^{k_0}
\end{bmatrix}
\begin{bmatrix}
\varphi \\
\psi
\end{bmatrix}
&= \frac{k^2}{2\mu_0} \begin{bmatrix}
I \\
\frac{k^2}{\mu_1} - \frac{k_0^2}{\mu_0}
\end{bmatrix} \begin{bmatrix}
\varphi \\
\psi
\end{bmatrix} \\
&= \left[ \frac{E^i \times \nu}{i \omega H^i \times \nu} \right]_{\partial D}.
\end{align*}$$

The invertibility of the system of equations (3.4) on $TH(\text{div}, \partial D) \times TH(\text{div}, \partial D)$ was proved in [31]. Moreover, there exists a constant $C = C(\epsilon, \mu, \omega)$ such that

$$\|\varphi\|_{TH(\text{div}, \partial D)} + \|\psi\|_{TH(\text{div}, D)} \leq C \left( \|E^i \times \nu\|_{TH(\text{div}, \partial D)} + \|H^i \times \nu\|_{TH(\text{div}, \partial D)} \right).$$

From (2.12) (with $k_0$ in the place of $k$) and (3.3) it follows that, for sufficiently large $|x|$,\n
$$E^i(x) = \sum_{n=1}^{\infty} \frac{ik_0}{n(n+1)} \sum_{m=-n}^{n} \left( \alpha_{n,m} E_{n,m}^{TE}(k_0; x) + \beta_{n,m} E_{n,m}^{TM}(k_0; x) \right),$$

where

$$\alpha_{n,m} = -i\omega \epsilon_0 \mu_0 \int_{\partial D} \overline{E_{n,m}^{TE}(k_0; y)} \cdot \varphi(y) + k_0^2 \int_{\partial D} \overline{E_{n,m}^{TM}(k_0; y)} \cdot \psi(y),$$

$$\beta_{n,m} = -i\omega \epsilon_0 \mu_0 \int_{\partial D} \overline{E_{n,m}^{TM}(k_0; y)} \cdot \varphi(y) - \omega^2 \epsilon_0 \int_{\partial D} \overline{E_{n,m}^{TE}(k_0; y)} \cdot \psi(y).$$

**Definition 3.1.** Let $(\varphi_{p,q}^{TE}, \psi_{p,q}^{TE})$ be the solution to (3.4) when $E^i = \overline{E_{p,q}^{TE}}(k_0; y)$ and $H^i = \overline{H_{p,q}^{TE}}(k_0; y)$, and $(\varphi_{p,q}^{TM}, \psi_{p,q}^{TM})$ when $E^i = \overline{E_{p,q}^{TM}}(k_0; y)$ and $H^i = \overline{H_{p,q}^{TM}}(k_0; y)$. The scattering coefficients $(W_{(n,m)(p,q)}^{TE,E,TE}, W_{(n,m)(p,q)}^{TE,TM,TE}, W_{(n,m)(p,q)}^{TM,TE}, W_{(n,m)(p,q)}^{TM,TM})$ associated with the permittivity and the permeability distributions $\epsilon, \mu$ and the frequency $\omega$ (or...
\( k, k_0, D \) are defined to be

\[
W_{(n,m)}^{TE,TE} = -i\omega\varepsilon_0\mu_0 \int_{\partial D} E_{n,m}^{TE}(k_0; y) \cdot \varphi_{p,q}^{TE}(y) \, d\sigma(y) \\
+ k_0^2 \int_{\partial D} E_{n,m}^{TE}(k_0; y) \cdot \psi_{p,q}^{TE}(y) \, d\sigma(y),
\]

\[
W_{(n,m)}^{TE,TM} = -i\omega\varepsilon_0\mu_0 \int_{\partial D} E_{n,m}^{TM}(k_0; y) \cdot \varphi_{p,q}^{TM}(y) \, d\sigma(y) \\
+ k_0^2 \int_{\partial D} E_{n,m}^{TM}(k_0; y) \cdot \psi_{p,q}^{TM}(y) \, d\sigma(y),
\]

\[
W_{(n,m)}^{TM,TE} = -i\omega\varepsilon_0\mu_0 \int_{\partial D} E_{n,m}^{TE}(k_0; y) \cdot \varphi_{p,q}^{TE}(y) \, d\sigma(y) \\
- \omega^2 c_0^2 \int_{\partial D} E_{n,m}^{TM}(k_0; y) \cdot \psi_{p,q}^{TM}(y) \, d\sigma(y),
\]

\[
W_{(n,m)}^{TM,TM} = -i\omega\varepsilon_0\mu_0 \int_{\partial D} E_{n,m}^{TE}(k_0; y) \cdot \varphi_{p,q}^{TM}(y) \, d\sigma(y) \\
- \omega^2 c_0^2 \int_{\partial D} E_{n,m}^{TM}(k_0; y) \cdot \psi_{p,q}^{TM}(y) \, d\sigma(y).
\]

As it can be seen, the scattering coefficients appear naturally in the expansion of the scattering amplitude. One first obtains the following estimates of the scattering coefficients.

**Lemma 3.2.** There exists a constant \( C \) depending on \( (\varepsilon, \mu, \omega) \) such that

\[
(3.7) \quad \left| W_{(n,m)}^{TE,TE}(\varepsilon, \mu, \omega) \right| \leq \frac{Cn+p}{\eta^np^p}
\]

for all \( n, m, p, q \in \mathbb{N} \). The same estimates hold for \( W_{(n,m)}^{TE,TM}, W_{(n,m)}^{TM,TE}, \) and \( W_{(n,m)}^{TM,TM} \).

**Proof.** Let \( (\varphi, \psi) \) be the solution to \( (3.4) \) with \( E^i(y) = \tilde{E}_{p,q}^{TE}(k_0; y) \) and \( H^i = -\frac{1}{\mu_0} \nabla \times E^i \). Recall that the spherical Bessel function \( j_p \) behaves as

\[
j_p(t) = \frac{t^p}{1 \cdot 3 \cdot \cdots (2p + 1)} \left( 1 + O \left( \frac{1}{p} \right) \right) \quad \text{as } p \to \infty,
\]

uniformly on compact subsets of \( \mathbb{R} \). Using Stirling’s formula \( p! = \sqrt{2\pi p} \left( \frac{p}{e} \right)^p (1 + o(1)) \), one has

\[
(3.8) \quad j_p(t) = O \left( \frac{Cp^p}{p^p} \right) \quad \text{as } p \to \infty,
\]

uniformly on compact subsets of \( \mathbb{R} \) with a constant \( C \) independent of \( p \). Thus one has

\[
\left\| E^i \right\|_{TH(\text{div,}\partial D)} + \left\| H^i \right\|_{TH(\text{div,}\partial D)} \leq \frac{Cp^p}{p^p}
\]

for some constant \( C' \). It then follows from \( (3.5) \) that

\[
\left\| \varphi \right\|_{L^2(\partial D)} + \left\| \psi \right\|_{L^2(\partial D)} \leq \frac{Cp}{p^p}
\]
for another constant $C$. So one gets (3.7) from the definition of the scattering coefficients. \hfill \Box

Suppose that the incoming wave is of the form
\begin{equation}
E(x) = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \left( a_{p,q} E_{p,q}^{TE}(k_0; x) + b_{p,q} E_{p,q}^{TM}(k_0; x) \right)
\end{equation}
for some constants $a_{p,q}$ and $b_{p,q}$. Then the solution $(\varphi, \psi)$ to (3.4) is given by
\begin{align*}
\varphi &= \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \left( a_{p,q} \varphi_{p,q}^{TE} + b_{p,q} \varphi_{p,q}^{TM} \right), \\
\psi &= \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \left( a_{p,q} \psi_{p,q}^{TE} + b_{p,q} \psi_{p,q}^{TM} \right).
\end{align*}

By (3.6) and Definition 3.1, the solution $E$ to (3.1) can be represented as
\begin{equation}
(E - E^i)(x) = \sum_{n=1}^{\infty} \frac{i k_0}{n(n+1)} \sum_{m=-n}^{n} \left( \alpha_{n,m} E_{n,m}^{TE}(k_0; x) + \beta_{n,m} E_{n,m}^{TM}(k_0; x) \right), \quad |x| \to \infty,
\end{equation}
where
\begin{equation}
\begin{cases}
\alpha_{n,m} = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \left( a_{p,q} W_{(n,m),p,q}^{TE,TE} + b_{p,q} W_{(n,m),p,q}^{TE,TM} \right), \\
\beta_{n,m} = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \left( a_{p,q} W_{(n,m),p,q}^{TM,TE} + b_{p,q} W_{(n,m),p,q}^{TM,TM} \right).
\end{cases}
\end{equation}

Using (3.10), (3.11), and the behavior of the spherical Bessel functions, the far-field pattern of the scattered wave $(E - E^i)$ can be estimated. The far-field pattern (also called the scattering amplitude) $A_{\infty}[\epsilon, \mu, \omega]$ is defined by
\begin{equation}
E(x) - E^i(x) = \frac{e^{i k_0 |x|}}{k_0 |x|} A_{\infty}[\epsilon, \mu, \omega](\hat{x}) + o(|x|^{-1}) \quad \text{as } |x| \to \infty.
\end{equation}

Since the spherical Bessel function $h_n^{(1)}$ behaves like
\begin{align*}
\left( h_n^{(1)}(t) \right) &\sim \frac{1}{t} e^t e^{-i \frac{n+1}{2} \pi} \quad \text{as } t \to \infty, \\
\left( h_n^{(1)}(t) \right)' &\sim \frac{1}{t} e^t e^{-i \frac{n}{2} \pi} \quad \text{as } t \to \infty,
\end{align*}
one can easily see by using (2.9) that
\begin{align*}
E_{n,m}^{TE}(k_0; x) &\sim \frac{e^{i k_0 |x|}}{k_0 |x|} e^{-i \frac{n+1}{2} \pi} \left( - \sqrt{n(n+1)} \right) \psi_{n,m}(\hat{x}) \quad \text{as } |x| \to \infty, \\
E_{n,m}^{TM}(k_0; x) &\sim \frac{e^{i k_0 |x|}}{k_0 |x|} \sqrt{\frac{\mu_0}{\epsilon_0}} e^{-i \frac{n+1}{2} \pi} \left( - \sqrt{n(n+1)} \right) \varphi_{n,m}(\hat{x}) \quad \text{as } |x| \to \infty.
\end{align*}
The following result holds.
Proposition 3.3. If \( \mathbf{E}^i \) is given by (3.9), then the corresponding scattering amplitude can be expanded as

\[
(3.13) \quad A_\infty[\epsilon, \mu, \omega](\hat{x}) = \sum_{n=1}^{\infty} \frac{-i^n k_0}{\sqrt{n(n+1)}} \sum_{m=-n}^{n} \left( \alpha_{n,m} \mathbf{V}_{n,m}(\hat{x}) + \beta_{n,m} \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{U}_{n,m}(\hat{x}) \right),
\]

where \( \alpha_{n,m} \) and \( \beta_{n,m} \) are defined by (3.11).

It is worth emphasizing that since \( \{\mathbf{V}_{n,m}, \mathbf{U}_{n,m}\} \) forms an orthogonal basis of \( L^2_\theta(S) \), the conversion of the far field to the near field is achieved via formula (3.10).

Consider the case where the incident wave \( \mathbf{E}^i \) is given by a plane wave \( e^{ik \cdot x} \) with \( |k| = k_0 \) and \( k \cdot c = 0 \). It follows from (2.13) that

\[
e^{ik \cdot x} = \sum_{p=1}^{\infty} \frac{4\pi i p}{\sqrt{p(p+1)}} \sum_{q=-p}^{p} \left( (\mathbf{V}_{p,q}(\hat{k}) \cdot c) \tilde{E}_{p,q}^{TE}(k_0; x) - \frac{1}{i\omega \mu_0} (\mathbf{U}_{p,q}(\hat{k}) \cdot c) \tilde{E}_{p,q}^{TM}(k_0; x) \right),
\]

where \( \hat{k} = k/k_0 \in S \), and therefore

\[
a_{p,q} = \frac{4\pi i p}{\sqrt{p(p+1)}} (\mathbf{V}_{p,q}(\hat{k}) \cdot c) \quad \text{and} \quad b_{p,q} = -\frac{4\pi i p}{\sqrt{p(p+1)}} \frac{1}{i\omega \mu_0} (\mathbf{U}_{p,q}(\hat{k}) \cdot c).
\]

Hence, the scattering amplitude, denoted by \( A_\infty[\epsilon, \mu, \omega](\mathbf{c}, \hat{k}; \hat{x}) \), is given by (3.13) with the coefficients \( \alpha_{n,m} \) and \( \beta_{n,m} \):

\[
(3.14) \quad \begin{cases}
\alpha_{n,m} = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \frac{4\pi i p}{\sqrt{p(p+1)}} \left( (\mathbf{V}_{p,q}(\hat{k}) \cdot c) W_{n,m}^{TE}(k_0; p,q) - \frac{1}{i\omega \mu_0} (\mathbf{U}_{p,q}(\hat{k}) \cdot c) W_{n,m}^{TM}(k_0; p,q) \right), \\
\beta_{n,m} = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} \frac{4\pi i p}{\sqrt{p(p+1)}} \left( (\mathbf{V}_{p,q}(\hat{k}) \cdot c) W_{n,m}^{TM}(k_0; p,q) - \frac{1}{i\omega \mu_0} (\mathbf{U}_{p,q}(\hat{k}) \cdot c) W_{n,m}^{TM}(k_0; p,q) \right).
\end{cases}
\]

These formulas show that the scattering coefficients appear in the expansion of the scattering amplitude.

The low frequency behavior of the scattering coefficients is now investigated. Let \( \Gamma(x) := -1/(4\pi|x|) \) denote the fundamental solution corresponding to the case \( k = 0 \), and \( \mathcal{M}_D \) the associated boundary integral operator:

\[
\mathcal{M}_D[\varphi](x) := \text{p.v.} \int_{\partial D} \nu(x) \times \left( \nabla_x \times \left( \Gamma(x - y) \varphi(y) \right) \right) d\sigma(y), \quad \varphi \in TH(\text{div}, \partial D).
\]

Analogously to (3.4), one can prove that there is a unique solution \( (\varphi^{(0)}, \psi^{(0)}) \in TH(\text{div}, \partial D) \times TH(\text{div}, \partial D) \) to the following equations:

\[
(3.15) \quad \begin{bmatrix}
(\mu_1 - \mu_0) \left( \frac{\mu_1 + \mu_0}{2(\mu_1 - \mu_0)} I + \mathcal{M}_D \right) & 0 \\
0 & (\epsilon_1 - \epsilon_0) \left( \frac{\epsilon_1 + \epsilon_0}{2(\epsilon_1 - \epsilon_0)} I + \mathcal{M}_D \right)
\end{bmatrix}
\begin{bmatrix}
\varphi^{(0)} \\
\omega \psi^{(0)}
\end{bmatrix} = \begin{bmatrix}
\mathbf{E}^i \times \nu \\
i\mathbf{H}^i \times \nu
\end{bmatrix}.\]

In fact, since \( \partial D \) is \( C^{1,\alpha} \), \( \mathcal{M}_D \) is compact and one may apply the Fredholm alternative to prove unique solvability of above equation. Moreover, one has

\[
(3.16) \quad \|\varphi^{(0)}\|_{TH(\text{div}, \partial D)} + \omega \|\psi^{(0)}\|_{TH(\text{div}, \partial D)} \leq C(\|\mathbf{E}^i \times \nu\|_{TH(\text{div}, \partial D)} + \|\mathbf{H}^i \times \nu\|_{TH(\text{div}, \partial D)}),
\]
with a constant $C = C(\epsilon, \mu)$. Let $\rho$ be a small positive number and consider the boundary integral equation (3.4) with $k$, $k_0$, and $\omega$ replaced by $\rho k$, $\rho k_0$, and $\rho \omega$, respectively. Then one has

$$M^{\rho k}_D - M_D = O(\rho^2), \quad M^{\rho k_0}_D - M_D = O(\rho^2),$$

and

$$L^{\rho k}_D - L^{\rho k_0}_D = O(\rho^2).$$

Since

$$\left(\frac{k^2}{2\mu_1} + \frac{k_0^2}{2\mu_0}\right) I + \frac{k^2}{\mu_1} M^{\rho k}_D - \frac{k_0^2}{\mu_0} M^{\rho k_0}_D = \rho^2 \omega^2 \left[ I + (\epsilon_1 - \epsilon_0) M_D + O(\rho^2) \right],$$

if one expresses the solution $(\varphi, \psi)$ to (3.4) as $(\varphi^\rho, \psi^\rho)$ then it satisfies

$$\left( A + O(\rho) \right) \begin{bmatrix} \varphi^\rho \\ \rho \omega \psi^\rho \end{bmatrix} = \begin{bmatrix} \mathbf{E}^i \times \nu \\ i \mathbf{H}^i \times \nu \end{bmatrix} \bigg|_{\partial D},$$

where $A$ is the 2-by-2 matrix appeared on the left-hand side of (3.15). From the invertibility of $A$, it follows that there are constants $\rho_0$ and $C = C(\epsilon, \mu, \omega)$ independent of $\rho$ as long as $\rho \leq \rho_0$ such that

$$\|\varphi^\rho\|_{TH(\text{div}, \partial D)} + \rho \omega \|\psi^\rho\|_{TH(\text{div}, \partial D)} \leq C \left( \|\mathbf{E}^i \times \nu\|_{TH(\text{div}, \partial D)} + \|\mathbf{H}^i \times \nu\|_{TH(\text{div}, \partial D)} \right).$$

**Lemma 3.4.** There exists $\rho_0$ such that, for all $\rho \leq \rho_0$,

$$\left| W_{(n,m)(p,q)}^{TE,TE}(\epsilon, \mu, \rho \omega) \right| \leq \frac{C^{n+p}}{n!^{p+1}} \rho^{n+p+1}$$

for all $n, m, p, q \in \mathbb{N}$, where the constant $C$ depends on $(\epsilon, \mu, \omega)$ but is independent of $\rho$. The same estimate holds for $W_{(n,m)(p,q)}^{TE,TM}$, $W_{(n,m)(p,q)}^{TM,TE}$, and $W_{(n,m)(p,q)}^{TM,TM}$.

**Proof.** Let $(\varphi, \psi)$ be the solution to (3.4) with $\mathbf{E}^i(y) = \mathbf{E}_{p,q}^{TE}(\rho k_0; y)$ and $\mathbf{H}^i = -\frac{i}{\rho_0 \omega} \nabla \times \mathbf{E}^i$. Then, from (3.8), it follows that

$$\left| \mathbf{E}^{i,\rho} \right|_{TH(\text{div}, \partial D)} + \left| \mathbf{H}^{i,\rho} \right|_{TH(\text{div}, \partial D)} \leq \frac{C^p}{p^p} \rho^p,$$

where $C$ is independent of $\rho$, and hence

$$\left| \varphi^\rho \right|_{L^2(\partial D)} + \rho \left| \psi^\rho \right|_{L^2(\partial D)} \leq \frac{C^p}{p^p} \rho^p$$

for $\rho \leq \rho_0$ for some $\rho_0$. So one gets (3.18) from the definition of the scattering coefficients in Definition 3.1.

**4. S-vanishing structures.** The purpose of this section is to construct multi-layered structures whose scattering coefficients vanish, which are called $S$-vanishing structures. The multilayered structure is defined as follows: For positive numbers $r_1, \ldots, r_{L+1}$ with $2 = r_1 > r_2 > \cdots > r_{L+1} = 1$, let

$$A_j := \{x : r_{j+1} \leq |x| < r_j\}, \quad j = 1, \ldots, L,$$

$$A_0 := \mathbb{R}^3 \setminus B_2(0), \quad A_{L+1}(= D) := \{x : |x| < 1\},$$

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where $B_2(0)$ denotes the central ball of radius 2 and
\[ \Gamma_j = \{ |x| = r_j \}, \quad j = 1, \ldots, L + 1. \]

Let $(\mu_j, \epsilon_j)$ be the pair of permeability and permittivity parameters of $A_j$ for $j = 1, \ldots, L + 1$. Set $\mu_0 = 1$ and $\epsilon_0 = 1$. Then define
\[ \mu = \sum_{j=0}^{L+1} \mu_j \chi(A_j) \quad \text{and} \quad \epsilon = \sum_{j=0}^{L+1} \epsilon_j \chi(A_j), \]
which are permeability and permittivity distributions of the layered structure.

The scattering coefficients $(W^{TE,TE}_{(n,m)(p,q)}, W^{TE,TM}_{(n,m)(p,q)}, W^{TM,TE}_{(n,m)(p,q)}, W^{TM,TM}_{(n,m)(p,q)})$ are defined as before, namely, if $\mathbf{E}^i$ is given as in (3.9), the scattered field $\mathbf{E} - \mathbf{E}^i$ can be expanded as (3.10) and (3.11). The transmission condition on each interface $\Gamma_j$ is given by
\[ [\mathbf{x} \times \mathbf{E}] = [\mathbf{x} \times \mathbf{H}] = 0. \]

Assume that the core $A_{L+1}$ is perfectly conducting (PEC), namely,
\[ \mathbf{E} \times \mathbf{\nu} = 0 \quad \text{on} \quad \Gamma_{L+1} = \partial A_{L+1}. \]

Thanks to the symmetry of the layered (radial) structure, the scattering coefficients are much simpler than the general case. In fact, if the incident field is given by $\mathbf{E}^i = \tilde{\mathbf{E}}_{n,m}^{TE}$, then the solution $\mathbf{E}$ to (3.1) takes the form
\[ \mathbf{E}(\mathbf{x}) = \tilde{a}_j \tilde{\mathbf{E}}_{n,m}^{TE}(\mathbf{x}) + a_j \mathbf{E}_{n,m}^{TE}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, \ldots, L, \]
with $\tilde{a}_0 = 1$. From (2.9) and (2.10), the interface condition (4.2) amounts to
\[
\begin{bmatrix}
  j_n(k_j r_j) & h_n^{(1)}(k_j r_j) \\
  \frac{1}{\mu_j} j_n(k_j r_j) & \frac{1}{\mu_j} h_n(k_j r_j)
\end{bmatrix}
\begin{bmatrix}
  \tilde{a}_j \\
  a_j
\end{bmatrix} =
\begin{bmatrix}
  j_{n-1}(k_{j-1} r_j) & h_{n-1}^{(1)}(k_{j-1} r_j) \\
  \frac{1}{\mu_{j-1}} j_{n-1}(k_{j-1} r_j) & \frac{1}{\mu_{j-1}} h_{n-1}(k_{j-1} r_j)
\end{bmatrix}
\begin{bmatrix}
  \tilde{a}_{j-1} \\
  a_{j-1}
\end{bmatrix}, \quad j = 1, \ldots, L,
\]
where $\mathcal{H}_n(t) = h_n^{(1)}(t) + t(h_n^{(1)})'(t)$ and $\mathcal{J}_n(t) = j_n(t) + t j_n'(t)$, and the PEC boundary condition on $\Gamma_{L+1}$ is
\[ \begin{bmatrix}
  j_n(k_L) & h_n^{(1)}(k_L) \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  \tilde{a}_L \\
  a_L
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

Since the matrices in (4.5) are invertible, one can see that there are $a_j$ and $\tilde{a}_j$, $j = 0, 1, \ldots, L$, satisfying (4.5) and (4.6). Similarly, one can see that if the incident field is given by $\mathbf{E}^i = \tilde{\mathbf{E}}_{n,m}^{TM}(\mathbf{x})$, then the solution $\mathbf{E}$ takes the form
\[ \mathbf{E}(\mathbf{x}) = \tilde{b}_j \tilde{\mathbf{E}}_{n,m}^{TM}(\mathbf{x}) + b_j \mathbf{E}_{n,m}^{TM}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, 1, \ldots, L, \]
for some constants $b_j$ and $\tilde{b}_j$ ($\tilde{b}_0 = 1$). One can now see from (4.4) and (4.7) that the scattering coefficients satisfy
\[
\begin{align*}
W^{TE,TE}_{(n,m)(p,q)} &= W^{TM,TE}_{(n,m)(p,q)} = 0 \quad \text{for all} \quad (m,n) \text{ and } (p,q), \\
W^{TE,TM}_{(n,m)(p,q)} &= W^{TM,TM}_{(n,m)(p,q)} = 0 \quad \text{if} \quad (m,n) \neq (p,q),
\end{align*}
\]
and, since (4.4) and (4.7) hold independently of $m$, one has
\[
W_{n,m}^{TE,TE} = W_{n,m}^{TM,TE},
\]
\[
W_{n,0}^{TM, TM} = W_{n,0}^{TM, TE}
\]
for $-n \leq m \leq n$.

Moreover, if one writes
\[
W_n^{TE} := W_{(n,0)}^{TE} \quad \text{and} \quad W_n^{TM} := W_{(n,0)}^{TM},
\]
then one has
\[
W_n^{TE} = -\frac{in(n+1)}{k_0}a_0 \quad \text{and} \quad W_n^{TM} = -\frac{in(n+1)}{k_0}b_0.
\]

Suppose now that $\tilde{E}_n^{TE}$ is the incident field and the solution $E$ is given by
\[
E(x) = \tilde{a}_j \tilde{E}_{n,0}^{TE}(x) + a_j E_{n,0}^{TE}(x), \quad x \in A_j, \quad j = 0, \ldots, L,
\]
with $\tilde{a}_0 = 1$, where the coefficients $\tilde{a}_j$'s and $a_j$'s are determined by (4.5) and (4.6).

From (4.5) it follows that
\[
\begin{bmatrix}
\tilde{a}_j \\
a_j
\end{bmatrix} = \begin{bmatrix}
j_n(k_j r_j) \\
\frac{1}{\mu_j^2} J_n(k_j r_j)
\end{bmatrix}^{-1} \begin{bmatrix}
j_n(k_{j-1} r_j) \\
\frac{1}{\mu_{j-1}^2} J_n(k_{j-1} r_j)
\end{bmatrix} \begin{bmatrix}
\tilde{a}_{j-1} \\
a_{j-1}
\end{bmatrix}
\]
for $j = 1, \ldots, L$. Substituting these relations into (4.6) yields
\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = P_n^{TE} [\varepsilon, \mu, \omega] \begin{bmatrix}
\tilde{a}_0 \\
a_0
\end{bmatrix},
\]
where
\[
P_n^{TE} [\varepsilon, \mu, \omega] := \begin{bmatrix}
P_{n,1}^{TE} & P_{n,2}^{TE} \\
0 & 0
\end{bmatrix} = (-i\omega)^L \left( \prod_{j=1}^{L} \frac{\mu_j^2}{\mu_j + k_L} \right) \begin{bmatrix}
j_n(k_L) & h_n^{(1)}(k_L) \\
0 & 0
\end{bmatrix}
\]
\[
\times \prod_{j=1}^{L} \begin{bmatrix}
\frac{1}{\mu_j} H_n(k_j r_j) & -h_n^{(1)}(k_j r_j) \\
\frac{1}{\mu_j} J_n(k_j r_j) & j_n(k_j r_j)
\end{bmatrix} \begin{bmatrix}
j_n(k_{j-1} r_j) & h_n^{(1)}(k_{j-1} r_j) \\
\frac{1}{\mu_{j-1}} J_n(k_{j-1} r_j) & \frac{1}{\mu_{j-1}} H_n(k_{j-1} r_j)
\end{bmatrix}.
\]

Then (4.9) yields
\[
W_n^{TE} = -\frac{in(n+1)}{k_0}a_0 = -\frac{in(n+1)}{k_0} \frac{P_n^{TE}}{P_{n,2}^{TE}}.
\]

Similarly, for $W_n^{TM}$, one looks for another solution $E$ of the form
\[
E(x) = \tilde{b}_j \tilde{E}_{n,0}^{TM}(x) + b_j E_{n,0}^{TM}(x), \quad x \in A_j, \quad j = 0, \ldots, L,
\]
with $\tilde{b}_0 = 1$. The transmission conditions become
\[
\begin{bmatrix}
j_n(k_j r_j) \\
\frac{1}{\varepsilon_j} J_n(k_j r_j)
\end{bmatrix} \begin{bmatrix}
\tilde{b}_j \\
b_j
\end{bmatrix} = \begin{bmatrix}
j_n(k_{j-1} r_j) \\
\frac{1}{\varepsilon_{j-1}} J_n(k_{j-1} r_j)
\end{bmatrix} \begin{bmatrix}
\tilde{b}_{j-1} \\
b_{j-1}
\end{bmatrix}, \quad j = 1, \ldots, N + 1,
\]
and the PEC boundary condition on the innermost layer is

\[
\left[ \mathcal{J}_n(k_L) \quad \mathcal{H}_n(k_L) \right] \begin{bmatrix} b_L \\ b_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

From (4.12) and (4.13), one obtains

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P_{n}^{TM}[\varepsilon, \mu, \omega] \begin{bmatrix} b_0 \\ b_0 \end{bmatrix},
\]

where

\[
P_{n}^{TM}[\varepsilon, \mu, \omega] := \begin{bmatrix} p_{n,1}^{TM} & p_{n,2}^{TM} \\ 0 & 0 \end{bmatrix} = (i\omega)^L \left( \prod_{j=1}^{L} \frac{1}{\varepsilon_j} \right) \begin{bmatrix} \mathcal{J}_n(k_L) \\ \mathcal{H}_n(k_L) \end{bmatrix}
\]

\[
\times \prod_{j=1}^{L} \begin{bmatrix} h_j^{(1)}(k_j r_j) & -\frac{1}{r_j} \mathcal{J}_n(k_j r_j) \\ -j_n(k_j r_j) & \frac{1}{r_j} \mathcal{H}_n(k_j r_j) \end{bmatrix} \begin{bmatrix} \mathcal{J}_n(k_j r_j) \\ \mathcal{H}_n(k_j r_j) \end{bmatrix}.
\]

From the definition of \( W_{n}^{TE} \) and (4.14),

\[
W_{n}^{TE} = -\frac{in(n + 1) b_0}{k_0} b_0 = -\frac{in(n + 1) P_{n,1}^{TM}}{P_{n,2}^{TM}}.
\]

It is worth emphasizing that \( p_{n,2}^{TM} \neq 0 \) and \( P_{n,2}^{TM} \neq 0 \). In fact, if \( p_{n,2}^{TE} = 0 \), then (4.9) can be fulfilled with \( \bar{a}_0 = 0 \) and \( a_0 = 1 \). This means that there exists \((\mu, \varepsilon)\) on \( \mathbb{R}^3 \setminus \overline{\mathcal{D}} \) such that the following problem has a solution:

\[
\begin{cases}
\nabla \times \mathbf{E} = i\omega \mu \mathbf{H} \quad &\text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}, \\
\nabla \times \mathbf{H} = -i\omega \varepsilon \mathbf{E} \quad &\text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}, \\
(\mathbf{x} \times \mathbf{E})^+ = 0 \quad &\text{on } \partial \mathcal{D}, \\
\mathbf{E}^+ = \mathbf{E}_{n,0}^{TE}(\mathbf{x}) \quad &\text{for } |\mathbf{x}| > 2.
\end{cases}
\]

Applying the following Green's theorem on \( \Omega = \{ \mathbf{x} \mid 1 < |\mathbf{x}| < R \} \),

\[
\int_{\Omega} (\mathbf{E} \cdot \Delta \mathbf{F} + \text{curl} \mathbf{E} \cdot \text{curl} \mathbf{F} + \text{div} \mathbf{E} \cdot \text{div} \mathbf{F}) \, d\mathbf{x}
= \int_{\partial \Omega} (\nu \times \mathbf{E} \cdot \text{curl} \mathbf{F} + \nu \cdot \mathbf{F} \cdot \text{div} \mathbf{E}) \, d\mathbf{\sigma}(\mathbf{x})
\]

with \( \mathbf{F} = \mathbf{E}_{n,0}^{TE}(\mathbf{x}) \) and the PEC boundary condition on \( \{ |\mathbf{x}| = 1 \} \), it follows that

\[
\int_{|\mathbf{x}| = R} (\nu \times \mathbf{E} \cdot \mathbf{\Pi}) \, d\mathbf{\sigma}(\mathbf{x}) = ik_0 \int_{\Omega} (|\mathbf{H}|^2 - |\mathbf{E}|^2) \, d\mathbf{x}.
\]

In particular, the left-hand side is real valued. Hence,

\[
\int_{|\mathbf{x}| = R} |\mathbf{H} \times \nu - \mathbf{E}|^2 \, d\mathbf{\sigma}(\mathbf{x}) = \int_{|\mathbf{x}| = R} (|\mathbf{H} \times \nu|^2 + |\mathbf{E}|^2 - 2\Re\{ (\nu \times \mathbf{E} \cdot \mathbf{\Pi}) \}) \, d\mathbf{\sigma}(\mathbf{x})
= \int_{|\mathbf{x}| = R} (|\mathbf{H} \times \nu|^2 + |\mathbf{E}|^2) \, d\mathbf{\sigma}(\mathbf{x}).
\]
From the radiation condition, the left-hand side goes to zero as $R \to \infty$, and it contradicts the behavior of the Hankel functions. One can show that $p_{n,2}^{TM} \neq 0$ in a similar way.

To construct the S-vanishing structure at a fixed frequency $\omega$, one looks for $(\mu, \epsilon)$ such that

$$W_n^{TE}[\epsilon, \mu, \omega] = 0, \quad W_n^{TM}[\epsilon, \mu, \omega] = 0, \quad n = 1, \ldots, N,$$

for some $N$. More ambitiously one may look for a structure $(\mu, \epsilon)$ for a fixed $\omega$ such that

$$W_n^{TE}[\mu, \epsilon, \rho \omega] = 0, \quad W_n^{TM}[\mu, \epsilon, \rho \omega] = 0$$

for all $1 \leq n \leq N$ and $\rho \leq \rho_0$ for some $\rho_0$. Such a structure may not exist. So instead one looks for a structure such that

$$W_n^{TE}[\mu, \epsilon, \rho \omega] = o(\rho^{2N+1}), \quad W_n^{TM}[\mu, \epsilon, \rho \omega] = o(\rho^{2N+1})$$

for all $1 \leq n \leq N$ and $\rho \leq \rho_0$ for some $\rho_0$. Such a structure is called an S-vanishing structure of order $N$ at low frequencies. In the following section, the scattering coefficients are expanded at low frequencies and conditions for the magnetic permeability and the electric permittivity to be an S-vanishing structure are derived.

Suppose that $(\mu, \epsilon)$ is an S-vanishing structure of order $N$ at low frequencies. Let the incident wave $E'$ be given by a plane wave $e^{i\rho k \cdot \hat{n}}$ with $|k| = k_0$ and $k \cdot \hat{n} = 0$. From (3.14), the corresponding scattering amplitude, $A_{\infty}[\mu, \epsilon, \rho \omega](\hat{c}, \hat{k}; \hat{x})$, is given by (3.13) with the following $\alpha_{n,m}$ and $\beta_{n,m}$:

$$\begin{align*}
\alpha_{n,m} &= \frac{4\pi i^n}{\sqrt{n(n+1)}} (V_{n,m}(\hat{k}) \cdot \hat{c}) W_n^{TE}[\mu, \epsilon, \rho \omega], \\
\beta_{n,m} &= \frac{4\pi i^n}{\sqrt{n(n+1)}} \frac{1}{i\omega \mu_0} (U_{n,m}(\hat{k}) \cdot \hat{c}) W_n^{TM}[\mu, \epsilon, \rho \omega].
\end{align*}$$

Applying (3.18) and (4.17),

$$A_{\infty}[\mu, \epsilon, \rho \omega](\hat{c}, \hat{k}; \hat{x}) = o(\rho^{2N+1})$$

uniformly in $(\hat{k}, \hat{x})$ if $\rho \leq \rho_0$. Thus, using such a structure, the visibility of scattering amplitude is greatly reduced.

4.1. Asymptotic expansion of the scattering coefficients. The spherical Bessel functions of first and second kinds have the series expansions

$$j_n(t) = \sum_{l=0}^{\infty} \frac{(-1)^l t^{n+2l}}{2^l l!(1 \cdot 3 \cdots (2n + 2l + 1))}$$

and

$$y_n(t) = -\frac{(2n)!}{2^{n+1} n!} \sum_{l=0}^{\infty} \frac{(-1)^l (2l-n-1)}{2^l l!(2n+1)(2n+3) \cdots (2n+2l-1)}.$$
one has
\begin{equation}
    j_n(t) = \frac{t^n}{(2n+1)!!} (1 + o(t)) \quad \text{for } t \ll 1
\end{equation}

and
\begin{equation}
    y_n(t) = - ((2n-1)!!) t^{-n+1} (1 + o(t)) \quad \text{for } t \ll 1.
\end{equation}

One now computes $P_n^{TE}[\varepsilon, \mu, t]$ for small $t$. For $n \geq 1$,
\begin{equation*}
P_n^{TE}[\varepsilon, \mu, t] = (-it)^L \left( \prod_{j=1}^{L} \frac{iQ(n)}{\mu_j (z_j r_j)^n} t^{-n+1} + o(t^{-n+1}) \right)
\end{equation*}
\begin{equation*}
\times \prod_{j=1}^{L} \left[ \begin{array}{ccc}
\frac{iQ(n)}{\mu_j (z_j r_j)^n} t^{-n+1} + o(t^{-n+1}) & 0 & -iQ(n) t^{-n+1} \\
0 & \frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & 0 \\
\frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & \frac{Q(n)}{\mu_j (z_j r_j)^n} t^{-n+1} + o(t^{-n+1}) & 0 \\
\end{array} \right],
\end{equation*}
where $z_j = \sqrt{\varepsilon_j \mu_j}$ and $Q(n) = (2n-1)!!$. One then has
\begin{equation*}
P_n^{TE}[\varepsilon, \mu, t] = \left[ \begin{array}{ccc}
\frac{iQ(n)}{\mu_j (z_j r_j)^n} t^{-n+1} + o(t^{-n+1}) & 0 & -iQ(n) t^{-n+1} \\
0 & \frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & 0 \\
\frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & \frac{Q(n)}{\mu_j (z_j r_j)^n} t^{-n+1} + o(t^{-n+1}) & 0 \\
\end{array} \right]
\end{equation*}

Similarly, for the transverse magnetic case, one has
\begin{equation*}
P_n^{TM}[\varepsilon, \mu, t] = \left[ \begin{array}{ccc}
\frac{(n+1)z_j^2 t^n + o(t^n)}{2n+1} & 0 & -iQ(n) t^n \\
0 & \frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & 0 \\
\frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) & \frac{Q(n)}{\mu_j (z_j r_j)^n} t^n + o(t^n) & 0 \\
\end{array} \right].
\end{equation*}

Using the behavior of spherical Bessel functions for small arguments, one can see that $p_{n,1}^{TE}$ and $p_{n,2}^{TE}$ admit the following expansions:
\begin{equation}
p_{n,1}^{TE}[\mu, \varepsilon, t] = t^n \left( \sum_{l=0}^{N-n} f_{n,l}^{TE}(\mu, \varepsilon) t^{2l} + o(t^{2N-2n}) \right)
\end{equation}
and
\begin{equation}
p_{n,2}^{TE}[\mu, \varepsilon, t] = t^{-n-1} \left( \sum_{l=0}^{N-n} g_{n,l}^{TE}(\mu, \varepsilon) t^{2l} + o(t^{2N-2n}) \right).
\end{equation}
Similarly, $p_{n,1}^{TM}$ and $p_{n,2}^{TM}$ have the following expansions:

\begin{equation}
(4.23) \quad p_{n,1}^{TM}[\mu, \varepsilon, t] = t^n \left( \sum_{l=0}^{N-n} f_{n,l}^{TM}(\mu, \varepsilon) t^{2l} + o(t^{2N-2n}) \right)
\end{equation}

and

\begin{equation}
(4.24) \quad p_{n,2}^{TM}[\mu, \varepsilon, t] = t^{-n-1} \left( \sum_{l=0}^{N-n} g_{n,l}^{TM}(\mu, \varepsilon) t^{2l} + o(t^{2N-2n}) \right)
\end{equation}

for $t = \rho \omega$ and some functions $f_{n,l}^{TE}, g_{n,l}^{TE}, f_{n,l}^{TM}$, and $g_{n,l}^{TM}$ independent of $t$.

**Lemma 4.1.** For any pair of $(\mu, \varepsilon)$, one has

\begin{equation}
(4.25) \quad g_{n,0}^{TE}(\mu, \varepsilon) \neq 0
\end{equation}

and

\begin{equation}
(4.26) \quad g_{n,0}^{TM}(\mu, \varepsilon) \neq 0.
\end{equation}

**Proof.** Assume that there exists a pair of $(\mu, \varepsilon)$ such that $g_{n,0}^{TE}(\mu, \varepsilon) = 0$. Since $p_{n,2}^{TM}[\mu, \varepsilon, \rho \omega] = o(\rho^{-n-1})$, the solution given by (4.4) with $a_0 = 1$ and $\tilde{a}_0 = 0$ satisfies

\begin{align*}
\nabla \times \left( \frac{1}{\mu} \nabla \times E \right) - \rho^2 \omega^2 \varepsilon E &= 0 \quad \text{in} \mathbb{R}^3 \setminus D, \\
\nabla \cdot E &= 0 \quad \text{in} \mathbb{R}^3 \setminus D, \\
(\nu \times E)_{\mid_+} &= o(\rho^{-(n+1)}) \quad \text{on} \partial D, \\
E(x) &= h_n^{(1)}(\rho k_0 |x|) V_{n,0}(\hat{x}) \quad \text{for} |x| > 2.
\end{align*}

Let $V(x) = \lim_{\rho \to 0} \rho^{n+1} E(x)$. Using (4.20) one knows that the limit $V$ satisfies

\begin{align*}
\nabla \times \left( \frac{1}{\mu} \nabla \times V \right) &= 0 \quad \text{in} \mathbb{R}^3 \setminus \overline{D}, \\
\nabla \cdot V &= 0 \quad \text{in} \mathbb{R}^3 \setminus \overline{D}, \\
(\nu \times V)_{\mid_+} &= 0 \quad \text{on} \partial D, \\
V(x) &= -((2n-1)!!) V_{n,0}(\hat{x}) \quad \text{for} |x| > 2.
\end{align*}

Since $V_{n,0}(\hat{x}) = O(|x|^{-1})$, one gets $V(x) = 0$ by Green’s formula, which is a contradiction. Thus $g_{n,0}^{TE}(\mu, \varepsilon) \neq 0$. In a similar way, (4.26) can be proved. \qed

From Lemma 4.1, one obtains the following theorem.

**Proposition 4.2.** One has

\begin{equation}
W_{n,1}^{TE}[\mu, \varepsilon, t] = t^{2n+1} \sum_{l=0}^{N-n} W_{n,l}^{TE}(\mu, \varepsilon) t^{2l} + o(t^{2N+1})
\end{equation}

and

\begin{equation}
W_{n,1}^{TM}[\mu, \varepsilon, t] = t^{2n+1} \sum_{l=0}^{N-n} W_{n,l}^{TM}(\mu, \varepsilon) t^{2l} + o(t^{2N+1}),
\end{equation}

where $t = \rho \omega$ and the coefficients $W_{n,l}^{TE}(\mu, \varepsilon)$ and $W_{n,l}^{TM}(\mu, \varepsilon)$ are independent of $t$.

Hence, if one has $(\mu, \varepsilon)$ such that

\begin{equation}
(4.27) \quad W_{n,l}^{TE}[\mu, \varepsilon] = W_{n,l}^{TM}[\mu, \varepsilon] = 0 \quad \text{for all} \ 1 \leq n \leq N, \ 0 \leq l \leq (N-n),
\end{equation}

$(\mu, \varepsilon)$ satisfies (4.17); in other words, it is an $S$-vanishing structure of order $N$ at low frequencies. It is quite challenging to construct $(\mu, \varepsilon)$ analytically satisfying (4.27). The next section presents some numerical examples of such structures.
4.2. Numerical examples. This section provides numerical examples of S-vanishing structures of order $N$ at low frequencies based on (4.27). To do this, the gradient descent method for the suitable energy functional is used, as in [5] and [6], to compute the enhanced near-cloaking structures for the conductivity problem and the Helmholtz problem. As in [6], one symbolically computes the scattering coefficients. In the place of spherical Bessel functions and spherical Hankel functions, one puts its low frequency asymptotic expansions in (4.10) and (4.15) and symbolically computes $W_{nE}^{TE}$ and $W_{nM}^{TM}$ to have $W_{nE}^{TE}[\mu, \varepsilon]$ and $W_{nM}^{TM}[\mu, \varepsilon]$.

The following example is an S-vanishing structure of order $N = 2$ made of 6 multilayers. The radii of the concentric disks are $r_j = 2 - \frac{j-1}{6}$ for $j = 1, \ldots, 7$. From Proposition 4.2, the nonzero leading terms of $W_{nE}^{TE}[\mu, \varepsilon, t]$ and $W_{nM}^{TM}[\mu, \varepsilon, t]$ up to $t^5$ are

- $[t^3, t^5]$ terms in $W_{1}^{TE}[\mu, \varepsilon, t]$, i.e., $W_{1,0}^{TE}, W_{1,1}^{TE}$,
- $[t^3, t^5]$ terms in $W_{1}^{TM}[\mu, \varepsilon, t]$, i.e., $W_{1,0}^{TM}, W_{1,1}^{TM}$,
- $[t^5]$ term in $W_{2}^{TE}[\mu, \varepsilon, t]$, i.e., $W_{2,0}^{TE}$.
- $[t^5]$ term in $W_{2}^{TM}[\mu, \varepsilon, t]$, i.e., $W_{2,0}^{TM}$.

Consider the mapping

$$
(\mu, \varepsilon) \rightarrow (W_{1,0}^{TE}, W_{1,1}^{TE}, W_{1,0}^{TM}, W_{1,1}^{TM}, W_{2,0}^{TE}, W_{2,0}^{TM}),
$$

where, $\mu = (\mu_1, \ldots, \mu_6)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_6)$. One looks for $(\mu, \varepsilon)$ which has the right-hand side of (4.28) as small as possible. Since (4.28) is a nonlinear equation, one solves it iteratively. Initially, one sets $\mu = \mu^{(0)}$ and $\varepsilon = \varepsilon^{(0)}$. One iteratively modifies $(\mu^{(i)}, \varepsilon^{(i)})$

$$
[\mu^{(i+1)}, \varepsilon^{(i+1)}]^T = [\mu^{(i)}, \varepsilon^{(i)}]^T - A_i^T b^{(i)},
$$

where $A_i^T$ is the pseudoinverse of

$$
A_i := \left. \frac{\partial(W_{1,0}^{TE}, W_{1,1}^{TE}, \ldots, W_{2,0}^{TM})}{\partial(\mu, \varepsilon)} \right|_{(\mu, \varepsilon) = (\mu^{(i)}, \varepsilon^{(i)})}
$$

and

$$
b^{(i)} = \begin{bmatrix}
W_{1,0}^{TE} \\
W_{1,1}^{TE} \\
\vdots \\
W_{2,0}^{TM}
\end{bmatrix}_{(\mu, \varepsilon) = (\mu^{(i)}, \varepsilon^{(i)})}.
$$

Example 1. Figures 4.1 and 4.2 show computational results of a 6-layer S-vanishing structure of order $N = 2$. One sets $r = (2, \frac{13}{6}, \ldots, \frac{7}{6})$, $\mu^{(0)} = (3, 6, 3, 6, 3, 6)$, and $\varepsilon^{(0)} = (3, 6, 3, 6, 3, 6)$ and modifies them following (4.29) with the constraints that $\mu$ and $\varepsilon$ belong to the interval between 0.1 and 10. The obtained material parameters are $\mu = (0.1000, 1.1113, 0.2977, 2.0436, 0.1000, 1.8260)$ and $\varepsilon = (0.4356, 1.1461, 0.2899, 1.8199, 0.1000, 3.1233)$, respectively. Differently from the no-layer structure with PEC condition at $|x| = 1$, the obtained multilayer structure has the nearly zero coefficients of $W_{nE}^{TE}[\mu, \varepsilon, t]$ and $W_{nM}^{TM}[\mu, \varepsilon, t]$ up to $t^5$. 
5. Enhancement of near cloaking. In this section one constructs a cloaking structure based on the following lemma in [8, 22].
LEMMA 5.1. Let $F$ be an orientation-preserving diffeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^3$ such that $F(\mathbf{x})$ is the identity for $|\mathbf{x}|$ large enough. If $(\mathbf{E}, \mathbf{H})$ is a solution to
\begin{align}
\nabla \times \mathbf{E} &= i \omega \mu \mathbf{H} \quad \text{in } \mathbb{R}^3, \\
\nabla \times \mathbf{H} &= -i \omega \epsilon \mathbf{E} \quad \text{in } \mathbb{R}^3,
\end{align}
(5.1)
then $(\mathbf{E}, \mathbf{H})$ is radiating,
\begin{align}
\nabla \times \mathbf{E} &= i \omega (F_\ast \mu) \mathbf{H} \quad \text{in } \mathbb{R}^3, \\
\nabla \times \mathbf{H} &= -i \omega (F_\ast \epsilon) \mathbf{E} \quad \text{in } \mathbb{R}^3,
\end{align}
where $(\mathbf{E}'(\mathbf{y}), \mathbf{H}'(\mathbf{y})) = ((DF)^{-T} \mathbf{E}(F^{-1}(\mathbf{y})), (DF)^{-T} \mathbf{H}(F^{-1}(\mathbf{y})))$ satisfies
\begin{align}
(F_\ast \mu)(\mathbf{y}) &= \frac{DF(\mathbf{x})\mu(\mathbf{x})DF^T(\mathbf{x})}{\det(DF(\mathbf{x}))} \quad \text{and} \quad (F_\ast \epsilon)(\mathbf{y}) = \frac{DF(\mathbf{x})\epsilon(\mathbf{x})DF^T(\mathbf{x})}{\det(DF(\mathbf{x}))},
\end{align}
where $\mathbf{x} = F^{-1}(\mathbf{y})$ and $DF$ is the Jacobian matrix of $F$. Hence,
\begin{align}
A[\mu, \epsilon, \omega] = A[F_\ast \mu, F_\ast \epsilon, \omega].
\end{align}

To compute the scattering amplitude which corresponds to the material parameters before the transformation, one considers the following scaling function, for small parameter $\rho$:
\begin{align}
\Psi_\frac{1}{\rho}(\mathbf{x}) = \frac{1}{\rho} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.
\end{align}
Then one has the following relation between the scattering amplitudes which correspond to two sets of differently scaled material parameters and frequency:
\begin{align}
A_\infty \left[ \mu \circ \Psi_\frac{1}{\rho}, \epsilon \circ \Psi_\frac{1}{\rho}, \omega \right] = A_\infty[\mu, \epsilon, \rho \omega].
\end{align}

To see this, consider $(\mathbf{E}, \mathbf{H})$, which satisfies
\begin{align}
\begin{cases}
(\nabla \times \mathbf{E})(\mathbf{x}) = i \omega (\mu \circ \Psi_\frac{1}{\rho})(\mathbf{x}) \mathbf{H}(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B}_\rho, \\
(\nabla \times \mathbf{H})(\mathbf{x}) = -i \omega (\epsilon \circ \Psi_\frac{1}{\rho})(\mathbf{x}) \mathbf{E}(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B}_\rho, \\
\hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0 & \text{on } \partial B_\rho, \\
(\mathbf{E} - \mathbf{E}' \circ \Psi_\frac{1}{\rho}, \mathbf{H} - \mathbf{H}' \circ \Psi_\frac{1}{\rho}) \text{ is radiating},
\end{cases}
\end{align}
with the incident wave $\mathbf{E}^i(\mathbf{x}) = e^{i \mathbf{k} \cdot \mathbf{x}} \hat{\mathbf{c}}$ and $\mathbf{H}^i = -\frac{1}{i \omega \epsilon} \nabla \times \mathbf{E}^i$ with $\mathbf{k} \cdot \hat{\mathbf{c}} = 0$ and $|\mathbf{k}| = k_0$. Here $B_\rho$ is the ball of radius $\rho$ centered at the origin. Set $\mathbf{y} = \frac{1}{\rho} \mathbf{x}$ and define
\begin{align}
(\mathbf{E}(\mathbf{y}), \mathbf{H}(\mathbf{y})) := \left( \left( \mathbf{E} \circ \Psi_\frac{1}{\rho} \right)(\mathbf{y}), \left( \mathbf{H} \circ \Psi_\frac{1}{\rho} \right)(\mathbf{y}) \right) = \left( \left( \mathbf{E} \circ \Psi_\rho \right)(\mathbf{y}), \left( \mathbf{H} \circ \Psi_\rho \right)(\mathbf{y}) \right)
\end{align}
and
\begin{align}
(\mathbf{E}^i(\mathbf{y}), \mathbf{H}^i(\mathbf{y})) := \left( \left( \mathbf{E}^i \circ \Psi_\rho \right)(\mathbf{y}), \left( \mathbf{H}^i \circ \Psi_\rho \right)(\mathbf{y}) \right).
\end{align}
Then one has
\[
\begin{align*}
\begin{cases}
\left( \nabla_y \times \tilde{E} \right)(y) = i \rho \omega y \tilde{H}(y) & \text{for } y \in \mathbb{R}^3 \setminus \overline{B_1}, \\
\left( \nabla_y \times \tilde{H} \right)(y) = -i \rho \omega \epsilon(y) \tilde{E}(y) & \text{for } y \in \mathbb{R}^3 \setminus \overline{B_1}, \\
\hat{y} \times \tilde{E}(y) = 0 & \text{on } \partial B_1,
\end{cases}
\end{align*}
\]

Recall that the scattered wave can be represented using the scattering amplitude as follows:
\[
(\mathbf{E} - \mathbf{E}^i)(x) \sim \frac{e^{i k_0 |x|}}{k_0 |x|} A_{\infty} \left[ \mu \circ \Psi_{\hat{\rho}}, \epsilon \circ \Psi_{\hat{\rho}}, \omega \right](c, \hat{k}; \hat{x}) \quad \text{as } |x| \to \infty
\]

and
\[
(\tilde{E} - \tilde{E}^i)(y) \sim \frac{e^{i k_0 |y|}}{k_0 |y|} A_{\infty} \left[ \mu, \epsilon, \omega \right](c, \hat{k}; \hat{x}) \quad \text{as } |y| \to \infty.
\]

Since the left-hand sides of the previous equations are coincident, one has (5.2).

Suppose that \((\mu, \epsilon)\) is an S-vanishing structure of order \(N\) at low frequencies as in section 4. From (4.18) and (5.2), one has
\[
A_{\infty} \left[ \mu \circ \Psi_{\hat{\rho}}, \epsilon \circ \Psi_{\hat{\rho}}, \omega \right](c, \hat{k}; \hat{x}) = o(\rho^{2N+1}).
\]

Then one defines the diffeomorphism \(F_\rho\) as
\[
F_\rho(x) := \begin{cases} 
\frac{x}{2(1 - \rho)} + \frac{1}{4(1 - \rho)} |x| \frac{x}{|x|} & \text{for } |x| \geq 2, \\
\frac{1}{2} \left( 1 + \frac{1}{2\rho} |x| \right) \frac{x}{|x|} & \text{for } 2 \rho \leq |x| \leq 2, \\
\frac{x}{\rho} & \text{for } |x| \leq \rho.
\end{cases}
\]

One then gets from (5.3) and Lemma 5.1 the main result of this paper.

**Theorem 5.2.** If \((\mu, \epsilon)\) is an S-vanishing structure of order \(N\) at low frequencies, then there exists \(\rho_0\) such that
\[
A_{\infty} \left[ ((F_\rho)_\ast (\mu \circ \Psi_{\hat{\rho}})), (F_\rho)_\ast (\epsilon \circ \Psi_{\hat{\rho}}), \omega \right](c, \hat{k}; \hat{x}) = o(\rho^{2N+1})
\]

for all \(\rho \leq \rho_0\), uniformly in \((\hat{k}, \hat{x})\).

Note that the cloaking structure \(((F_\rho)_\ast (\mu \circ \Psi_{\hat{\rho}})), (F_\rho)_\ast (\epsilon \circ \Psi_{\hat{\rho}}))\) in Theorem 5.2 satisfies the PEC boundary condition on \(|x| = 1\).

**6. Conclusion.** In this paper, near-cloaking examples for the Maxwell equation have been shown. Based on an extension of the method of [5, 6] to electromagnetic scattering problems, a cloaking device that achieves an enhanced cloaking effect has been designed. Any target placed inside the cloaking device has an approximately zero scattering amplitude. Such a cloaking device has been obtained by the blow-up using the transformation optics of a multicoated inclusion with PEC boundary condition. The cloaking device has anisotropic permittivity and permeability parameters.
REFERENCES


