Dissipative kinetic Alfvén solitary waves resulting from viscosity

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Nonlinear small-amplitude kinetic Alfvén solitary waves (KASWs) are investigated with their “anomalous” kinetic viscosity effect on electrons. It is found that the structure of a hump-type KASW solution develops into a shock-type (or double layer) KASW solution for large amplitude KASWs when viscosity exists. For small amplitude KASWs, the Korteweg-de Vries (KdV) equation with an approximate pseudopotential was solved, and it is found that the hump-type KASWs develop into oscillating shock-type (kink-type) KASWs. It is also found that the oscillating scale of this structure is related to the propagation velocity and plasma beta, while the damping scale is inversely proportional to the viscosity. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4834495]

I. INTRODUCTION

Since Hasegawa and Mima1 first investigated the kinetic Alfvén solitary wave (KASW) in low β plasmas, numerous studies on KASWs have been conducted under various plasma conditions.2–21 For example, Chen et al.19 examined the kinetic Alfvén wave instability driven by a field-aligned current in high β plasmas. Hollweg and Kaghashvili20 examined the density fluctuations of a linearly polarized Alfvén wave in a shear flow. Ofman and Davila7 investigated the KASW solution in Ref. 21 was a pure double layer. The present paper is organized as follows. In Sec. II, the basic equations are introduced and the Sagdeev pseudopotential, which is analyzed in detail using a graphical method, is derived. Because an exact analytical solution is difficult to obtain for the Sagdeev pseudopotential due to the electron viscosity term, the Korteweg-de Vries (KdV) equation is formulated corresponding to the Sagdeev pseudopotential in Sec. III, and then it is solved in Sec. IV for approximate solutions of the small amplitude KASWs with suitable approximations. Finally, conclusions are provided in Sec. V.

II. BASIC EQUATIONS

The focus of this research is on the obliquely propagating Alfvénic mode in magnetized plasma with low β plasma that satisfies \( Q \ll \beta \ll 1 \), where \( Q = m_e/m_i \) with \( m_e \) and \( m_i \) representing the electron and ion masses, respectively. When the plasma is strongly magnetized and compressional magnetic perturbation is not present, the transversal electric fields can be expressed in terms of two potentials:22

\[
E_z = -\frac{\partial \phi}{\partial z} \quad \text{and} \quad E_x = -\frac{\partial \psi}{\partial x},
\]

where \( z \) and \( x \) are the co-ordinates parallel and perpendicular to the magnetic field, respectively. These potentials produce pure shear perturbations along the magnetic field. For electrons, the continuity equation can be written as follows:

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} (n_e v_{ez}) = 0, \tag{1}
\]

and the massless momentum equation with viscosity can be written as follows:

\[
\frac{\beta}{2Q} \frac{\partial \psi}{\partial z} - \frac{\beta}{2Q} \frac{\partial}{\partial z} \ln(n_e) + \mu \frac{\partial^2 v_{ez}}{\partial z^2} = 0. \tag{2}
\]
The basic equations for ions can also be written as follows:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_{ix}) + \frac{\partial}{\partial z} (n_i v_{iz}) = 0$$  \hspace{1cm} (3)

and

$$\frac{\partial v_{iz}}{\partial t} + v_{ix} \frac{\partial v_{iz}}{\partial x} + v_{iz} \frac{\partial v_{iz}}{\partial z} = -\frac{1}{\rho} \left( \frac{\partial \psi}{\partial z} \right)$$  \hspace{1cm} (4)

The ion motion has polarization drift in the $x$ direction only, as follows:

$$v_{ix} = -\frac{\partial^2 \phi}{\partial x \partial t}.$$  \hspace{1cm} (5)

Here, the following normalized variables were used:

$$n_{i(e)} = \frac{n_i}{n_0} \rightarrow n_{i(e)'}, \quad \frac{x}{r_s} \rightarrow x', \quad \frac{z}{\omega_p/c} \rightarrow z', \quad \Omega_t \rightarrow \Omega_t',$$

$$v_{ix}/C_s = v_{ix}', \quad v_{iz}/v_A = v_{iz}', \quad e\phi/T_e \rightarrow \psi',$$

$$e\psi/T_e \rightarrow \psi', \quad \text{and} \quad \frac{\omega_{pi}^2}{\Omega^2} \rightarrow \mu \rightarrow \mu'.$$

where $r_s = (T_e/m_i)^{1/2}/\Omega$ is the ion gyroradius, $\Omega = eB_0/m_i$ is the ion gyrofrequency, $C_s = \sqrt{T_e/m_i}$ is the ion sound speed, $v_A = B_0/\sqrt{\mu_0 n_e}$ is the Alfvén speed, and $\omega_{pi}$ is the ion plasma frequency. Here, $\mu$ is the kinematic viscosity of the electron fluid. Note that, for simplicity, all primes have been deleted from the normalized equations above. A finite viscosity for the electrons was assumed; this viscosity might arise when background magnetic turbulence exists. Biskamp has demonstrated that when there is a small-scale magnetic turbulence, the “anomalous” kinematic viscosity is given by the following equation:

$$\mu = \frac{\tau_B}{2} (B_z^2),$$  \hspace{1cm} (6)

where $\tau_B$ is the magnetic correlation time and $\langle B_z^2 \rangle$ is the small-scale turbulent magnetic field intensity. This formula is applicable for low $\beta$ plasma when small-scale and large-scale perturbations have well separated spectra.

Using Faraday’s law and Ampere’s law, the following is obtained:

$$\frac{\partial^4}{\partial x^2 \partial z^2} (\phi - \psi) = \frac{\partial^2 n_e}{\partial t^2}.$$  \hspace{1cm} (7)

Again, in this equation, the current in the $x$ direction in Ampere’s law is neglected.

Now, the above equations are transformed into a moving frame with $\xi = l_x x + l_z z - M t$, where $l_x = \frac{\tau}{2} \sin \theta$, $l_z = \cos \theta$, and $M$ is the Mach number normalized by the Alfvén velocity ($v_A$). Then, the following equation is obtained:

$$\frac{\partial}{\partial t} = -M \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = l_x \frac{d}{d\xi}, \quad \frac{\partial}{\partial z} = l_z \frac{d}{d\xi}.$$  \hspace{1cm} (8)

After some algebra and by assuming a neutral charge condition, the following is obtained:

$$v_{iz} = \frac{\beta}{2M} l_z n + Q \mu \int n \frac{d^2}{d\xi^2} \left( \frac{1}{n} \right) d\xi,$$  \hspace{1cm} (9)

where $n = n_i = n_e$.

Because the ion velocity along the parallel direction is not important and $v_{iz}$ is consequently set to 0, the following is finally obtained:

$$\frac{d}{d\xi} \left( \frac{1}{n} \frac{dn}{d\xi} \right) = \left( 1 - \frac{1}{n} \left( \frac{M^2 n - l_z^2}{l_z^2} \right) \right) - 2QM\mu \frac{d^3}{d\xi^3} \left( \frac{1}{n} \right).$$  \hspace{1cm} (10)

which is the same result as that presented by Woo et al. If the focus is limited to a situation without viscosity ($\mu = 0$), then Eq. (10) is the same as that derived by Hasegawa. 1 In Fig. 1, the numerical solution of Eq. (10) is presented for several different viscosity values. As the viscosity changes from $\mu = 0$ to $\mu = 0.5$, the structure of the large amplitude hump-type KASW develops into a shock-type (or double layer) KASW. It should be noted that the oscillating part that was noted in Wu’s result 13 is not present in this research.

### III. DERIVATION OF THE KORTEweg-De VRIErs (KDV) EQUATION

It is difficult to calculate the precise pseudopotential function corresponding to Eq. (10) due to the viscosity term; therefore, a quasi-exact solution is sought using the KdV formalism in this section. The stretched co-ordinates are introduced using the reductive perturbation method 24 as follows:

$$\xi = \epsilon^{1/2} (z - V_0 t), \quad \eta = \epsilon^{1/2} x, \quad \mu = \epsilon^{1/2} \mu_0, \text{ and } \tau = \epsilon^{3/2} t,$$  \hspace{1cm} (11)

where $\epsilon$ is the small expansion parameter and $V_0$ is the phase velocity of a solitary wave. In this new co-ordinate system, the physical parameters are expanded as follows:

![FIG. 1. The viscosity effect on the kinetic Alfvén solitary wave for $\mu = 0$ (dashed line), 0.005 (solid line), and 0.5 (dotted line).](image-url)
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\[ n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \cdots, \]
\[ n_i = 1 + \varepsilon n_i^{(1)} + \varepsilon^2 n_i^{(2)} + \cdots, \]
\[ v_{ci}(x) = v_{ci}(x) + \varepsilon^2 v_{ci}(x)^{(2)} + \cdots, \]
\[ \phi = \phi^{(1)} + \varepsilon \phi^{(2)}, \quad \text{and} \]
\[ \psi = \psi^{(1)} + \varepsilon^2 \psi^{(2)} + \cdots. \]

After substituting the stretching co-ordinates into Eq. (11) and the perturbation expansions of Eq. (12) into Eqs. (1)–(5) and (7), the first order \((O(\varepsilon))\) equations can be obtained as described in the Appendix. From the first order \((O(\varepsilon))\) equations, the following is obtained:

\[ n_e^{(1)} = \psi^{(1)} \] (13)

and

\[ n_i^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{\beta}{2 V_0^2} \psi^{(1)}. \] (14)

Furthermore, from Eq. (5), the first order of the \(x\)-component of the ion velocity can be expressed as follows:

\[ v_{ix}^{(1)} = V_0 \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \eta}. \] (15)

From the next higher order \((O(\varepsilon^2))\) equations (see Appendix), the following can be obtained:

\[ V_0 \frac{\partial^2 n_i^{(2)}}{\partial \xi^2} - \frac{\partial^2 \phi^{(2)}}{\partial \eta^2 \partial \xi} = 2 V_0 \frac{\partial^2 n^{(1)}}{\partial \xi^2} - \frac{\partial^2 \psi^{(1)}}{\partial \eta^2 \partial \xi} \] (16)

The dispersion relation for the velocity \((V_0)\) can be obtained as follows:

\[ V_0^2 - \frac{\beta}{2} + \frac{\beta}{2} = 0. \] (17)

After some complicated algebra from the next higher order \((O(\varepsilon^2))\) equations using the relation presented in Eq. (17), the following can be obtained:

\[ A \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + B \frac{\partial^2 \phi^{(1)}}{\partial \eta^2 \partial \xi} + C \frac{\partial^2 \phi^{(1)}}{\partial \eta^2 \partial \xi} + D \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \eta} + E \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \xi} + F \frac{\partial^2 \phi^{(1)}}{\partial \eta \partial \eta} = 0, \] (18)

where

\[ A = 4 V_0^2 + 2 V_0^2 \beta + \beta^2, \quad B = -4 Q V_0^\gamma \mu_0, \quad C = 2 V_0^2 (2 V_0^2 + \beta), \quad D = 4 V_0^2, \quad \text{and} \quad E = 2 V_0^2 (2 V_0^2 - 6 V_0^2 + \beta). \]

In Eq. (18), the third term is the result of the coupling between the polarization current and electric field. This term is the source contribution because the kinetic Alfvén wave is propagating in an oblique direction with changes in the \(x\) direction.

Because the focus of this research remains on a stationary solution that propagates in an arbitrary direction, Eq. (18) is transformed into a moving frame as follows:

\[ \zeta = M \tau - l_o \eta - l_o \xi. \] (19)

Then, Eq. (18) can be rewritten as follows:

\[ -A l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + B l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \zeta \partial \zeta} + C l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \eta^2 \partial \zeta} + D l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + E l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \eta} + F l_o^2 \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} = 0. \] (20)

Equation (14) can be transformed as follows:

\[ n_i^{(1)} \equiv n_i^{(1)} = \frac{1}{V_0^2} \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} + \frac{\beta}{V_0^2} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}. \] (21)

Thus, Eq. (20) is reduced to a simple form as follows:

\[ \frac{d^2 n_i^{(1)}}{d \zeta^2} = a n_i^{(1)} + b n_i^{(1)} + c \frac{dn_i^{(1)}}{d \zeta}, \] (22)

where

\[ a = \frac{EM}{2 V_0^2}, \quad b = \frac{(A-C) V_0^2}{2 V_0^2} = -\frac{1}{\beta}, \quad \text{and} \quad c = \frac{b}{V_0^2}. \]

**IV. RESULTS AND DISCUSSION**

While Cveticanin\(^{25}\) obtained an exact analytical solution of the energy-displacement function in Eq. (22) for \(b = 0\), the highly nonlinear term \((n_i^{(1)})^2\) increases the difficulty of solving this equation without assumptions. Therefore, the pseudopotential is modified slightly, as will be demonstrated in this section, in order to appreciate the behavior of the solution.

First, if the viscosity effect is neglected, the following is obtained from Eq. (22):

\[ \frac{1}{2} \left( \frac{dn_i^{(1)}}{d \zeta} \right)^2 + V(n) = 0, \] (23)

where \(n_i^{(1)} = n\) is the conventional notation and the pseudopotential \(V(n) = -\frac{\alpha}{2} n^2 - \frac{\mu_0}{2} n^3\). From the shape of the potential, it is evident that the coefficient \(a\) must always be positive, as shown in Fig. 2(a). Hence, the condition of the velocity of a solitary wave is as follows:

\[ 0 < V_0 < 1/\sqrt{2}. \] (24)

The analytic solution of Eq. (23) without viscosity is well known as follows:

\[ n(\zeta) = \frac{3a}{2b} \left[ \tanh \left( \sqrt{\frac{a}{2}} \zeta \right) - 1 \right], \] (25)
as shown in Fig. 2(b).

Now, Eq. (22) will be solved including the viscosity effect. By multiplying both sides of Eq. (22) by \(dn_i^{(1)}/d \zeta\), the following is obtained:

\[ \frac{dR}{d \zeta} = c \left( \frac{dn_i^{(1)}}{d \zeta} \right)^2, \] (26)
where the total pseudo-particle energy \( R(ζ) \) is defined as

\[
R(ζ) = \frac{1}{2} \left( \frac{dn}{dζ} \right)^2 - \frac{a}{2} n^2 - \frac{b}{3} n^3.
\]  

(27)

Now, the boundary condition can be obtained as follows:

\[
R(ζ \to \infty) = 0 \quad \text{and} \quad n(ζ \to \infty) = 0.
\]  

(28)

Therefore, this condition can be used as the initial condition with

\[
R(n = 0) = 0.
\]  

(29)

Thus, the equation between the total pseudo-particle energy and density is found as follows:

\[
\left( \frac{dR}{dn} \right)^2 = c^2 [2R - 2V(n)].
\]  

(30)

Now, \( V(n) \) is approximated in quadratic form and denoted by \( V(ζ) \). This approximated solution should be asymptotically the same as the original potential with the following three requirements:

\[
V\left( n = \frac{-a}{b} \right) = V\left( n = \frac{-a}{b} \right)
\]

and

\[
V'\left( n = \frac{-a}{b} \right) = V'\left( n = \frac{-a}{b} \right) = 0
\]

and

\[
V''\left( n = \frac{-a}{b} \right) = V''\left( n = \frac{-a}{b} \right)
\]

where \( n = -a/b \) is the local minimum of \( V(n) \). The solution is \( V(n) = \frac{4}{9} n^2 + \frac{2}{3} n + \frac{2}{3b} \). In Fig. 3(a), \( \tilde{V}(n) \) is compared with \( V(n) \) for \( a = 1 \) and \( b = -1.5 \). While the pseudo-particle experiences the potential \( V(n) \), \( \tilde{V}(n) \) is introduced for the purpose of mathematical tractability only, as demonstrated below. Equation (30) can be rewritten as follows:

\[
\left( \frac{dR}{dn} \right)^2 \approx c^2 [2R - 2\tilde{V}(n)] = c^2 \left[ 2R - a \left( n + \frac{a}{b} \right)^2 + \frac{a^3}{3b^2} \right].
\]  

(31)

Next, the following is set

\[
R' = R + \frac{a^3}{6b^2} \quad \text{and} \quad n' = n + \frac{a}{b}.
\]  

(32)

Then, the equation of motion becomes

\[
\left( \frac{dR'}{dn'} \right)^2 = c^2 (2R' - an'^2).
\]  

(33)

After some algebra to set \( R' = \frac{1}{2} \left[ \frac{c^2}{\nu^2} + an'^2 \right] \) and \( \nu = \frac{c}{b} \), Eq. (33) is rewritten as follows:

\[
\frac{dn'}{\nu^2} = \frac{v^2}{c^2 a + c^2 \nu - v^2}
\]  

(34)

Therefore, Eq. (34) gives the following:

\[
\ln \left| c^2 an'^2 - c^2 \nu^2 + \nu^2 \right| = \frac{2c}{\sqrt{-4a + c^2}} \arctanh \left( \frac{-c^2 + 2\nu}{\sqrt{-4c^2a + c^4}} \right) + \Gamma,
\]  

(35)

where \( \Gamma \) is an integration constant.
Because $v = p/n'$, Eq. (35) provides the following:

$$\ln|c^2an'^2 - c^2pn' + p^2| = \frac{2c}{\sqrt{-4a + c^2}} \arctanh\left(\frac{-c^2 + 2p/n'}{\sqrt{-4c^2a + c^4}}\right) + \Gamma. \quad (36)$$

Because $p = \pm c\sqrt{2R' - an'^2}$, Eq. (36) can be rewritten as follows:

$$\ln|-pn' + 2R'| = \frac{2c}{\sqrt{4a - c^2}} \arctan\left(\frac{n'^2 - 2p}{n'^2c\sqrt{4a - c^2}}\right) = \Gamma, \quad (37)$$

where the relation of $\arctan(-ix) = -i \arctan(x)$ is used and $c^2$ in the logarithm is included in $\Gamma$.

Because the arctangent function in Eq. (37) is discontinuous at $n' = 0$ (dashed vertical line in Fig. 3(a)) for an arbitrary value of $p$, two cases are considered

$$\frac{dn'}{d\zeta} > 0 \quad \text{and} \quad p = c\sqrt{2R' - an'^2} \quad (38)$$

and

$$\frac{dn'}{d\zeta} < 0 \quad \text{and} \quad p = -c\sqrt{2R' - an'^2}. \quad (39)$$

For $n' > 0$ and $n' < 0$, the following is clear

$$\lim_{n' \to 0} \ln\left|-pn' + 2R'\right| - \frac{2c}{\sqrt{4a - c^2}} \arctan\left(\frac{c^2n'^2 - 2p}{cn'^2\sqrt{4a - c^2}}\right) = \ln|2R'| + \frac{c\pi}{\sqrt{4a - c^2}} \quad (40)$$

and

$$\lim_{n' \to 0} \ln\left|-pn' + 2R'\right| - \frac{2c}{\sqrt{4a - c^2}} \arctan\left(\frac{c^2n'^2 - 2p}{cn'^2\sqrt{4a - c^2}}\right) = \ln|2R'| - \frac{c\pi}{\sqrt{4a - c^2}}. \quad (41)$$

Thus, at this $n' = 0$ point, the integration constant $\Gamma$ is continuous and the total pseudo-particle energy ($R'$) is also continuous across the point. For example, if the coefficients are set to $a = 1$, $b = -1.5$, and $c = -0.05$, the first integration constant ($\Gamma^0$), which indicates the loss of energy through the viscosity after the first bounce, can be calculated as shown in Fig. 3(b). Then, after the first bounce, the integration constant is as follows:

$$R' = \frac{2}{\sqrt{3}}, \quad p = 0, \quad \text{and} \quad n' = \frac{\sqrt{3}}{3}. \quad (42)$$

Thus, $\Gamma^0 = -1.91079$; and after one bounce, $\Gamma^1 = -2.06792$ and $\Gamma^2 = -2.225$. Now, it is clear that after the $m$th crossing, i.e., the minimum point, the integration constant can be obtained simply as follows:

$$\Gamma^m = \Gamma^0 + \frac{2mc\pi}{\sqrt{4a - c^2}}. \quad (43)$$

As $m \to \infty$, $n \to -a/b$ and $R \to -a^2/(b)^2$. Thus, the particle tends to sit in the potential minimum. When the density is oscillating near $-a/b$, this is a simple harmonic oscillation and the change in the density is very small. With $n'^{(1)} = -a/b$ in the quadratic term, Eq. (27) can be rewritten as follows:

$$\frac{d^2n}{d\zeta^2} \approx a_n + \frac{a^2}{b} + \frac{c}{d\zeta}. \quad (44)$$

Then, this linear equation can be easily solved and the following is obtained:

$$n \approx \frac{a}{b} + n_0 e^{-a\zeta}, \quad (45)$$

where $\Omega = \frac{1}{\tau} (ic \pm \sqrt{4a - c^2}) = \Omega_r + i\Omega_i$. Because the viscosity term is included in coefficient $c$, the oscillation with damping in Eq. (45) is related to the viscosity. As seen in Fig. 4, there is a slight difference between the exact pseudopotential and the approximated pseudopotential in the early phases for the density oscillation, but the two oscillations equalize after a sufficient amount of time. By comparing Fig. 2(b) with Fig. 4, it can also be seen that the hump-type KASW develops into a shock-type KASW due to the viscosity in the small amplitude limit. For this double layer, the spatial oscillating scale is $L_0 = \frac{4\pi}{\sqrt{4a - c^2}} \approx \frac{2\pi}{\sqrt{a}}$, and the damping scale is $L_d = \frac{3}{2}$. 

V. CONCLUSION

In this paper, the viscosity effect of electrons on a nonlinear KASW was investigated for low $\beta$ plasma obliquely propagating with respect to an external magnetic field. It was found that the structure of the hump-type KASW solution developed into a shock-type (or double layer) solution for a large amplitude KASW when viscosity existed. For a small amplitude KASW, the KdV equation, which had the form of a nonlinear damped harmonic oscillation, was derived; this equation was solved using an approximated pseudopotential. The primary difference between Figs. 1 and 4 is the existence of the damping phenomenon: the large amplitude KASW did not have an oscillatory phenomenon; however, the small amplitude KASW had a damping oscillation due to its electron viscosity. The present results imply that dissipative KASWs could produce a local shock-like structure.
(i.e., a double layer) in the presence of viscosity that might arise due to magnetic fluctuations. This structure could accelerate particles locally, particularly in space plasmas where KASWs have been frequently observed.

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APPENDIX: FIRST AND SECOND ORDER EQUATIONS

After substituting the stretching co-ordinates into Eq. (11) and the perturbation expansions of Eq. (12) into Eqs. (1)–(5) and (7), the first order \(O(\varepsilon)\) equations can be obtained as follows:

\[
-V_0 \frac{\partial n^{(1)}_e}{\partial \xi} + \frac{\beta}{2Q} \frac{\partial \psi^{(1)}}{\partial \xi} = 0, \tag{A1}
\]

\[
- 2Q \frac{\partial n^{(1)}_e}{\partial \xi} + \frac{\partial n^{(1)}_e}{\partial \eta} = 0, \tag{A2}
\]

\[
-V_0 \frac{\partial n^{(1)}_e}{\partial \xi} + \frac{\partial n^{(1)}_e}{\partial \eta} + \frac{\partial n^{(1)}_e}{\partial \xi} = 0, \tag{A3}
\]

and

\[
-V_0 \frac{\partial \psi^{(1)}}{\partial \xi} = \frac{\beta}{2} \frac{\partial \psi^{(1)}}{\partial \xi}. \tag{A4}
\]

The next higher order \(O(\varepsilon^2)\) equations are as follows:

\[
-V_0 \frac{\partial n^{(2)}_e}{\partial \xi} + \frac{\partial n^{(2)}_e}{\partial \xi} = \frac{\partial n^{(1)}_e}{\partial \eta} + \frac{\partial n^{(1)}_e}{\partial \xi}, \tag{A5}
\]

\[
\frac{dn^{(2)}_e}{d\xi} - \frac{\partial \psi^{(2)}}{\partial \xi} = \frac{2Qn^{(1)}_e}{\beta} \frac{\partial \psi^{(1)}}{\partial \xi}, \tag{A6}
\]

\[
V_0 \frac{\partial n^{(1)}_e}{\partial \xi} - \frac{\partial \psi^{(1)}}{\partial \eta} - \frac{\partial n^{(1)}_e}{\partial \xi} = \frac{\partial n^{(1)}_e}{\partial \eta} + \frac{\partial \psi^{(1)}}{\partial \xi} - \frac{\partial \psi^{(1)}}{\partial \xi} = 0, \tag{A7}
\]

\[
\frac{\partial \psi^{(2)}}{\partial \xi} = \frac{1}{V_0} \frac{\partial n^{(1)}_e}{\partial \eta} + \frac{1}{V_0} \frac{\partial n^{(1)}_e}{\partial \xi} + \frac{1}{V_0} \frac{\partial n^{(1)}_e}{\partial \tau}, \tag{A8}
\]

and

\[
\frac{\partial \psi^{(2)}}{\partial \eta} - \frac{\partial \psi^{(1)}}{\partial \eta} = - \frac{\partial \psi^{(1)}}{\partial \tau}. \tag{A9}
\]