Free-Space Ultrashort Pulsed Beam Propagation Beyond the Paraxial Approximation

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Applying the temporal and spatial Fourier transform method, we obtain a nonparaxial pulsed beam solution, which is based on the paraxial pulsed beam solution, where the nonparaxiality is evaluated by using a series of expansions. Specifically, the general lowest-order correction field is given in an integral form. Some special examples, such as the lowest-order correction to the paraxial approximation of a fundamental Gaussian pulsed Gaussian-like beam, whose waist plane has a parallel shift from the $z = 0$ plane, are presented.

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I. INTRODUCTION

It is well known that, for long light-pulse propagation, the scalar and the paraxial approximation of diffraction theory provides a successful description of the propagation of space-modulated light waves without considering their temporal structure [1–4]. However, for short light-pulse propagation, both the spatial modulation and the temporal modulation have to be considered simultaneously. For these short light-pulse beams, the paraxial approximation is no longer valid, and some additional terms must be taken into account. Many authors have used various approaches to study corrections to the paraxial approximation for a fundamental Gaussian beam and to that for a high-order Gaussian beam with a complex argument [5–16]. Recently, Fu et al. [17] proposed a correction to the pulsed beam, however, due to the complexity of the complex analytical signal (CAS) treatment, obtaining the specific correction terms for a Gaussian beam and for a Gaussian-like beam uniformly driven by a Gaussian pulse is difficult. In this paper, we use a different initial value from the one used in the previous CAS treatment, and we present the lowest-order correction to the paraxial approximation of a fundamental Gaussian pulsed Gaussian-like beam.

The paper is organized as follows: In Sec. II, by applying the spatial-temporal Fourier transform, we derive the nonparaxial correction to the paraxial pulsed beam solutions as a series of expansions. In Sec. III, the lowest-order correction to the paraxial approximation for an arbitrary freely propagating pulsed beam is obtained, and a concrete application of the lowest-order correction is given. The paper concludes in Sec. IV with a discussion.

II. TRANSFORM FROM A SOLUTION OF THE PARAXIAL WAVE EQUATION TO THE EXACT SOLUTION OF THE WAVE EQUATION

The free-space propagation of an electromagnetic pulse is governed by the wave equation

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E(r, z, t) = 0,$$

(1)

where $r = e_x x + e_y y$ is the transverse coordinate and $e_x$ and $e_y$ are the unit vectors in the $x$ and $y$ directions, respectively. If the local variables, i.e., $t' = t - z/c$ and

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\( z' = z \) are used, Eq. (1) can be rewritten as
\[
\left( \nabla_\perp^2 - \frac{2}{c} \frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial z'^2} \right) E(r, z', t) = 0,
\]
where \( \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the transverse Laplacian. Ignoring the evanescent waves and using a temporal-spatial Fourier transform, Fu et al. [17] obtain a spatial-temporal-frequency solution of Eq. (2):
\[
\tilde{\psi}(\mathbf{f}, z', \omega) = \sum_{n=2m=0}^{+\infty} \frac{i}{m!} \left[ -ikz'(2n-3)!! \right]^{m} (\lambda^2 f^2)^n \tilde{\psi}^{(0)}(\mathbf{f}, z', \omega),
\]
where \( \lambda = 2\pi/k(\omega) \) is the wavelength, \( \tilde{\psi}^{(0)}(\mathbf{f}, z', \omega) \) is the paraxial pulsed-beam solution in spatial-temporal-frequency solution. Taking the spatial inverse Fourier transform of Eq. (3) yields the ultrashort pulsed-beam solution in the temporal-frequency domain:
\[
\psi(r, z', \omega) = D \tilde{\psi}^{(0)}(r, z', \omega),
\]
where
\[
D = \prod_{n=2m=0}^{+\infty} \frac{(-1)^{mn}}{m!} \left[ -iz'(2n-3)!! \right]^{m} \nabla_{2mk}(1-2n)
\]
is an \( \omega \)-dependent operator, and \( \tilde{\psi}^{(0)}(r, z', \omega) \) is the paraxial pulsed-beam solution in the temporal-frequency domain with the temporal inverse Fourier transform of Eq. (4), one obtains the exact short pulsed-beam solution:
\[
E(r, z', t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} D \tilde{\psi}^{(0)}(r, z', \omega) \exp(-i\omega t') d\omega.
\]
According to Eqs. (4) and (5), in principle, one can obtain the exact solution for ultrashort pulsed-beam propagation in free space, which is a correction of the paraxial solution \( \psi^{(0)} \) in terms of \( D \), and the exact solution is always valid for an arbitrarily freely propagating pulsed beam when the evanescent waves can be ignored.

Now we will analyze the magnitudes of the \( (m, n) \)-order terms of the corrections. The maximum spatial frequency is denoted by \( f_{\text{max}} = |f|_{\text{max}} \); it is evident that \( \lambda^2 f^2_{\text{max}} < 1 \). According to Eq. (3), when \( m = M \) and \( n = N \) are sufficiently big, Eq. (3) can be approximated by
\[
\tilde{\psi}_{M,N}(\mathbf{f}, z', \omega) \approx \prod_{n=2m=0}^{M} \frac{(-1)^{mn}}{m!} \left[ -iz'(2n-3)!! \right]^{m} \nabla_{2mk}(1-2n) \tilde{\psi}^{(0)}(\mathbf{f}, z', \omega).
\]
From Eq. (5), the approximate short pulsed-beam solution is given by
\[
E_{M,N}(r, z', t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} D \tilde{\psi}^{(0)}(r, z', \omega) \exp(-i\omega t') d\omega,
\]
where
\[
D = \prod_{n=2m=0}^{M,N} \frac{(-1)^{mn}}{m!} \left[ -iz'(2n-3)!! \right]^{m} \nabla_{2mk}(1-2n+1).
\]
Comparing Eq. (7) with Eq. (5), one finds that the approximate solution \( E_{M,N}(r, z', t') \) approaches the exact solution \( E(r, z', t') \) when the parameters \( M \) and \( N \) are increased, particularly, when those parameters both become infinite. In practice, the discrepancy between Eq. (7) and (5) is negligible when
\[
Q_{N+1} = k z'^{(2N-1)!!}(\lambda^2 f^2_{\text{max}})^{(N+1)} \ll 1.
\]
\[
R_{n,M+1} = \frac{1}{(M_n + 1)!} \left[ k z'(2n-3)!!(\lambda^2 f^2_{\text{max}})^n \right]^{(M_n+1)} \ll 1.
\]
(8) (9)
are satisfied, where \( 2 \leq n \leq N \). Equations (8) and (9) can be used to choose appropriate value for \( M \) and \( N \) so as to obtain an appropriate approximate solution \( E_{M,N}(r, z', t') \) that is accurate enough, but not too complicated. For example, if \( f_{\text{max}} \leq 0.1 \lambda^{-1} \) and \( k z' \lambda^4 f^2_{\text{max}} \leq 0.5 \), namely, the propagation distance \( z' \) is not long, then \( Q_3 \approx 0.0003 \ll 1 \), \( R_{3,1} \approx 0.06 \), and \( R_{3,2} \approx 0.0001 \ll 1 \), so the parameters \( N = 2 \), \( M = 1 \) can be chosen to. For instance, for a Gaussian beam uniformly driven by a Gaussian pulse, if one takes \( f_{\text{max}} = 0.1 \lambda^{-1} \), namely, \( \theta = 0.1 \), where \( \theta \) is the divergence angle of the far field, and \( k z' \lambda^4 f^2_{\text{max}} = 0.5 \) as the spatial frequency width \( 1/(\pi w_0) \) and the propagation distance, respectively, where \( w_0 \) is the waist width, one obtains that \( w_0 \approx 3.2 \lambda \) and \( z' = 25z_R \), where \( z_R = k(\omega)w_0^2/2 \) is the Rayleigh range. This example shows that the order of correction increases when the divergence angle of the far field increases.

III. LOWEST-ORDER CORRECTION TO THE PARAXIAL APPROXIMATION OF AN ARBITRARY FREELY PROPAGATING PULSED BEAM

From Eqs. (8) and (9), one can deduce that the simplest approximate solution, \( E_{1,2}(r, z', t') \), beyond the paraxial approximation, which corresponds to \( M = 1, N = 2 \), is usually sufficient for a nonparaxial pulsed light beam when the frequencies of the frequency components are not high, evanescent wave components do not exist, and the propagation distance \( z' \) is not too long. From Eq. (7), one can directly obtain the lowest-order correction,
\[
E^{(2)}(r, z', t') = -\frac{iz' c^3}{16\pi} \int_{-\infty}^{+\infty} \tilde{\psi}^{(0)}(r, z', \omega) \exp(-i\omega t') d\omega.
\]
by letting $E_{1,2}(r,z',t') = E^{(0)}(r,z',t') + E^{(2)}(r,z',t')$. From the paraxial equation, Eq. (10), can be rewritten as

$$E^{(2)}(r,z',t') = \frac{i z' c}{4\pi} \frac{\partial^2}{\partial z'^2} \int_{-\infty}^{+\infty} \psi^{(0)}(r,z',\omega) \exp(-i\omega t') d\omega, \quad (11)$$

Compared with Eq. (10), Eq. (11) is more appropriate for obtaining the lowest-order correction to the paraxial approximate solution for an arbitrary pulsed beam on the $z'$ axis because Eq. (11) only involves the axial differential operator $\partial/\partial z'$. By choosing between Eq. (10) and Eq. (11), we can obtain the suitable lowest-order correction to the paraxial approximation for an arbitrary pulsed beam in a fixed $z'$ plane and on the $z'$ axis.

Let us illustrate the above method to give the lowest-order correction to the paraxial approximation for the concrete case of a Gaussian beam in the $z' = z_0$ plane, $g(r) = \exp(-r^2/\omega_0^2)$, where $r^2 = x^2 + y^2$, uniformly driven by a pulse $f(t') = \exp[-t'^2/T^2 - i(\omega_0 t' + \phi)]$, i.e., $E(r,z',t') = g(r)f(t')$. In the derivation of $E(r,z',t')$, the solution in Ref. [5] should be applied to avoid the difficulty arising from the CAS theory. For simplicity, we have only applied the first-order correction, which is in the temporal-frequency domain, so the zeroth-order solution is

$$\psi^{(0)}(r,z',\omega) = -\frac{i z' \sqrt{\pi} T}{q} \left[ 1 + \frac{(\omega - \omega_0) z'}{q \omega_0} \right] L_1 \left( -\frac{i \omega_0 r^2}{2qc} \right) \times \exp \left[ \frac{i k_0 r^2}{2q} - \frac{T^2 (\omega - \omega_0)^2}{4} - i \varphi \right], \quad (12)$$

where $q(\omega) = z' - i z_R(\omega)$ is the $q$-parameter of the Gaussian-like beam, and $L_1(u)$ is the first order Laguerre polynomial and is given by $L_1(u) = 1 - u$. However, deriving the analytical solution is very difficult if $z_R$ depends on $\omega$. Nevertheless, since it is possible to control $z_R$ and to make it experimentally independent of $\omega$, after substituting Eq. (12) into Eq. (10), we can write the lowest-order correction as

$$E^{(2)}(r,z',t') = A(r,z',t') \exp \left[ -\frac{T^2 \omega_0^2}{4} - i \varphi + \frac{i \omega_0 r^2}{2qc} \right], \quad (13)$$

where the complex function $A(r,z',t')$ is given by

$$A(r,z',t') = A_0 \left\{ -\frac{T^2}{2} + \left( \frac{T^2 \omega_0}{2} - i t' \right)^2 \times \left[ L_2 \left( -\frac{i \omega_0 r^2}{2qc} \right) + \frac{3i z'}{q} L_3 \left( -\frac{i \omega_0 r^2}{2qc} \right) - \frac{2i z'}{q \omega_0} \right] \times \left[ \frac{T^2 \omega_0}{2} - i t' \right] \left[ 2 + 2L_2 \left( \frac{i \omega_0 r^2}{2qc} \right) + 3L_3 \left( \frac{i \omega_0 r^2}{2qc} \right) \right] \right\}, \quad (14)$$

where $A_0 = -\sqrt{\pi} z_R T c \omega_0^2/2q^3$, and $L_3(u)$, and $L_3(u)$ are, respectively, the second and the third order Laguerre polynomials and are given by $L_2(u) = \frac{1}{2} u^2 - 2u + 1$, and $L_3(u) = \frac{1}{4} (-u^3 + 9u^2 - 18u + 6)$. In the extreme case of a half-cycle pulse, we take $a = 1$ mm, and $\omega_0 = 1.9$ ps$^{-1}$, with $T = 1.4$ ps, and $\alpha = 1$ mm. The solid, dashed, and dotted curves represent the solution given by Porras', our solution corrected to the lowest-order, and the exact paraxial solution (CAS solution).

**Fig. 1.** Spatial distribution of the (a) front, (b) center and (c) trailing edges of a pulsed beam in the $z = z_R = 63$ mm plane, where the instantaneous intensity is given by $(Re E)^2$ and $E(r,t) = \exp(-r^2/a^2) \exp(-t^2/T^2) \exp(-i\omega_0 t)$, with $\omega_0 = 1.9$ ps$^{-1}$, $T = 1.4$ ps, and $a = 1$ mm. The solid, dashed, and dotted curves represent the solution given by Porras', our solution corrected to the lowest-order, and the exact paraxial solution (CAS solution).
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Fig. 2. On-axis field distribution in the (a) $z' = z_R$ plane, in the (b) $z' = 5z_R$ plane. The solid, dashed, and dotted curves represent the solution given by Porras’, our solution corrected to the lowest-order and the exact paraxial solution (CAS solution).

IV. DISCUSSION AND CONCLUSION

Making use of the Fourier transform for the spatial variable, and starting from the nonparaxial pulsed beam propagation equation in the temporal-frequency domain, we derived the solution for nonparaxial pulsed-beam propagation as a correction to the paraxial solution, which is an infinite series containing the derivative operator $\hat{D}$. In particular, the nonparaxial pulsed Gaussian-like beam solution is given. A magnitude analysis shows that the higher-order correction terms have less effect than lower-order ones; hence the corrections can be made by a finite series with the derivative operator $\hat{D}$. The lowest-order correction $E^{(2)}(r, z', t')$, to the paraxial approximation of an arbitrary propagation of a pulsed beam in free space was obtained in a very simple form. The results obtained in this paper can be used to study propagation problems involving general nonparaxial pulsed laser beams (especially those nonparaxial pulsed beams whose zeroth-order fields are given in analytical form) in free space.

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