On the computation of the near-tip stress intensities for three-dimensional wedges via two-state \textit{M}-integral

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Abstract

Singular stress fields around three-dimensional wedges are examined, and the near-tip intensity is calculated via the two-state \textit{M}-integral with the aid of the domain integral representation. Two numerical examples demonstrate the effectiveness and accuracy of the present scheme for computing the stress intensities of singular stresses near the generic three-dimensional wedges. © 2003 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Singular stress fields around generic wedges have been an important area of research in the past half-century. For a crack, which is a special case of the generic wedge, it is well known that the stress intensity factor governs the near-tip stress field, and so it is contemplated as an essential fracture parameter for describing the crack behaviors including crack growth under static or fatigue loading. The existence of such a fracture parameter representing the near-tip singular stress field is not limited to cracks, but it is the case for generic wedges as well, as singular stress fields are formed near the wedge vertices. Recently, Labossiere and Dunn (2001) conducted a series of very elaborate testing to show that the intensities of the singular stresses around the three-dimensional wedge on the interface corner of the two joining materials are

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in an excellent correlation with the initiation of the failure at the wedge vertices of the specimens. Hence, of paramount importance is an efficient and accurate calculation of these near-tip intensities.

The computation of the near-tip intensities of the singular fields may be rather straightforward for the two-dimensional wedges, and there are many schemes available (see Im and Kim, 2000 and the papers cited therein, for examples). Among others, the application of the two-state conservation integral (Im and Kim, 2000; Kim et al., 2001) is known to be a robust method of computation. The computation of the intensities for three-dimensional wedges is much more complicated in comparison to those of the aforementioned two-dimensional ones. There are many researchers that have discussed only the order of stress singularities on three-dimensional wedge vertices, for examples, Koguchi and Muramoto (2000), Picu and Gupta (1997), Ghahremani and Shih (1992), Ghahremani (1991), Barsoum and Chen (1991), Somaratna and Ting (1986), Bazant and Estenssoro (1979), Benthem (1977, 1980) and others cited in these papers. However, no systematic computational schemes have been reported regarding the calculation of the near-tip intensities of the singular stress fields around the three-dimensional wedges, to the best of the authors’ knowledge.

The purpose of the present paper is to report on a numerical scheme for finding the near-tip intensities around three-dimensional wedges with the aid of the two-state $M$-integral. A brief review is first given of the eigenfunction expansion of the solution for three-dimensional elastic wedges. This is followed by a summary regarding the two-state conservation law with emphasis on the three-dimensional two-state $M$-integral. The two-state $M$-integral is then applied for calculating the near-tip intensity by utilizing the complementarity relationship for the eigenvalues of the three-dimensional wedges. That is, the path or surface independence property of the two-state $M$-integral for a pair of the complementary eigenvalues is exploited to equate its value calculated on the vanishing near-field surface around the vertex to the value from finite element analysis on the far-field surface. This procedure enables us to calculate the intensity of the singular stress field around the wedge vertex in an efficient manner.

For numerical examples, we firstly choose the intersection between a crack front line and a free surface, which is the three-dimensional crack-tip corner or the vertex of a quarter infinite crack (see Benthem, 1977, 1980). Next we take the corner of three-dimensional bimaterial interface, which was recently discussed by Labossiere and Dunn (2001). It is shown that our present solution via two-state $M$-integral is in good agreement with their direct finite element solution. The two examples demonstrate the effectiveness and accuracy of the proposed scheme.

2. Eigenfunction expansion of the solution for three-dimensional elastic wedge problems

Consider a conical region $V$, which has its vertex $O$, the lateral boundary $S_L$, and the far-field boundary $S_F$, as shown in Fig. 1. The surface $S_F$ may be subjected to a traction or displacement boundary condition, but the lateral surface $S_L$ is free from traction, or subjected to a fixed rigid fixture constraining the displacements to be zero.
This represents generic three-dimensional wedges or notches, and the typical examples include a three-dimensional crack-tip corner created at the intersection vertex between a crack front line and a free surface, and a three-dimensional bimaterial corner, which is formed by two intersecting free edges.

For the purpose of analysis for stress singularities at the vertex \( O \), we introduce the eigenfunction expansion (see Bazant, 1974; Benthem, 1977; Ghahremani and Shih, 1992) for the elastic solution, which is given in the form of the separable displacement field:

\[
\begin{align*}
  u &= \frac{1}{2\mu} \text{Re} \left[ \sum_{\delta_n} \beta_n r^{\delta_n+1} \tilde{u}_n(\theta, \phi; \delta_n) \right] = \frac{1}{2\mu} \text{Re} \left[ \sum_{\lambda_n} \beta_n r^{\lambda_n} \tilde{u}_n(\theta, \phi; \lambda_n) \right], \\
  v &= \frac{1}{2\mu} \text{Re} \left[ \sum_{\delta_n} \beta_n r^{\delta_n+1} \tilde{v}_n(\theta, \phi; \delta_n) \right] = \frac{1}{2\mu} \text{Re} \left[ \sum_{\lambda_n} \beta_n r^{\lambda_n} \tilde{v}_n(\theta, \phi; \lambda_n) \right], \\
  w &= \frac{1}{2\mu} \text{Re} \left[ \sum_{\delta_n} \beta_n r^{\delta_n+1} \tilde{w}_n(\theta, \phi; \delta_n) \right] = \frac{1}{2\mu} \text{Re} \left[ \sum_{\lambda_n} \beta_n r^{\lambda_n} \tilde{w}_n(\theta, \phi; \lambda_n) \right].
\end{align*}
\]

(1)

where \( r, \theta \) and \( \phi \) are the spherical coordinates with the origin at the vertex \( O \), and \( (u, v, w) \) are the components of displacement in \((r, \theta, \phi)\) directions, respectively. Values of \( \lambda_n(=\delta_n + 1) \), which are employed for convenience of the expression instead of \( \delta_n + 1 \), are called the eigenvalues and its corresponding eigenfunctions are \( \tilde{u}_n(\theta, \phi; \lambda_n) \), \( \tilde{v}_n(\theta, \phi; \lambda_n) \) and \( \tilde{w}_n(\theta, \phi; \lambda_n) \). This displacement field is required to satisfy the equilibrium equation within the conical region \( V \), and the proper homogeneous boundary conditions on the lateral surface \( S_L \). Note that the expression above is the generalization of the series expansion in terms of the spherical harmonics (Gurtin, 1972), just as the two-dimensional analog (Im and Kim, 2000) is the generalization of the series solution in terms of the cylindrical harmonics.
When the displacement fields are given by Eq. (1), the stress fields are expressed in the following form, proportional to $r^{\delta_n-1}$ or $r^{\delta_n}$:

$$\sigma_{ij} = \text{Re} \left[ \sum_{\delta_n} \beta_n r^{\delta_n} \tilde{\sigma}_{ij}(\theta, \phi, \delta_n) \right].$$

(2)

Hence, the stress singularity occurs at the origin when $\text{Re}(\lambda_n) < 1$. On the other hand, the strain energy is bounded at the origin and this requires $\text{Re}(\lambda_n) > -1/2$ (boundedness of strain energy requires $\text{Re}(\lambda_n) > 0$ for two-dimensional wedge vertices). However, $\text{Re}(\lambda_n) < 0$ implies that the displacement fields are unbounded at $r = 0$, which is unrealistic except for a concentrated load applied at the vertex. Therefore, we are primarily interested in eigenvalues $\lambda_n(=\delta_n + 1)$ in the range $0 < \text{Re}(\lambda_n) < 1$.

Let the stress and strain components in the spherical coordinates be represented by 1-D arrays as follows:

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = (\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{\theta\phi}, \sigma_{r\phi}, \sigma_{r\theta}),$$

(3a)

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}, 2\varepsilon_{\theta\phi}, 2\varepsilon_{r\phi}, 2\varepsilon_{r\theta}).$$

(3b)

Material constitutive relationship is then given as

$$\sigma_i = C_{ij}\varepsilon_j,$$

(4)

where $i,j = 1–6$, and $C_{ij}$ is the material stiffness which satisfies $C_{ij} = C_{ji}$.

Consider now a small subdomain $V_N$ of the entire region $V$, as shown in Fig. 1 again. The subdomain $V_N$ is the region enclosed by the vertex, the lateral surface and the surface $S_N$, which is the concentric surface $r = r_N$. We choose the sufficiently small $r_N$ so that the near-field expression (1) is valid in the subdomain $V_N$.

Suppose we cut through the surface $S_N$ to obtain $V_N$. Then the eigenfunction expansion for the displacement field (1) should satisfy the equilibrium equation inside the cone, the near-field boundary conditions on $S_L$ and the appropriate traction condition on $S_N$. Therefore the principle of minimum potential energy is written as

$$\delta U - \int_{S_N} (t_r\delta u + t_\theta\delta v + t_\phi\delta w)r^2 \sin \theta \, d\theta \, d\phi = 0,$$

(5)

where

$$U = \int_{V_N} Wr^2 \sin \theta \, dr \, d\theta \, d\phi$$

is the total strain energy in the volume $V_N$ of the cone, and $W$ the strain energy density, which is a function of the strains. The strain components are functions of the displacements and their first derivatives. The expression (5) of the principle of minimum potential energy is rewritten for the eigenstates as (see Somaratna and Ting, 1986;
\[ \int_{V_N} \left\{ \psi_{ru} - \frac{\partial}{\partial r} (\psi_{ru_r}) \right\} \delta u + \psi_{ru_r} \delta u_r + \psi_{ru_\theta} \delta u_\theta \\
+ \left\{ \psi_{rw} - \frac{\partial}{\partial r} (\psi_{rw_r}) \right\} \delta w + \psi_{rw_r} \delta w_r + \psi_{rw_\theta} \delta w_\theta \\
+ \left\{ \psi_{sw} - \frac{\partial}{\partial r} (\psi_{sw_r}) \right\} \delta w + \psi_{sw_r} \delta w_r + \psi_{sw_\theta} \delta w_\theta \right\} dr d\theta d\phi = 0 \] (6)

and

\[ \sigma_r = t_r, \quad \sigma_\theta = t_\theta, \quad \sigma_\phi = t_\phi \] on \( S_N, \)

where \( \psi = W r^2 \sin \theta \) and the subscripts \( r, \theta \) and \( \phi \) in \( u, v \) and \( w \) denote the partial differentiation or the derivative of displacement like \( u_r = \partial u / \partial r \). Furthermore, the comma after \( \psi \) indicates the partial differentiation with respect to the subject variable, for example, \( \psi_{ur} = \partial \psi / \partial u_r \) and \( \psi_{ur_r} = \partial \psi / \partial (\partial u / \partial r) \), etc. Since we assume that the traction on \( S_N \) is denoted by \( (t_r, t_\theta, t_\phi) \), we come to the conclusion that Eq. (6) is a variational statement that would ensure the satisfaction of the equations of equilibrium in \( V_N \) and the boundary conditions on \( S_L \) only.

The variational statement of Eq. (6) can be evaluated using the finite element method by discretizing the concentric spherical surface or the \( \theta-\phi \) surface into finite elements. The nodal variables are chosen to be the spherical components of the eigenfunctions \( \tilde{u}_n(\theta, \phi; \delta_n), \tilde{v}_n(\theta, \phi; \delta_n) \) and \( \tilde{w}_n(\theta, \phi; \delta_n) \) in Eq. (1). Details for discretizing Eq. (6) are described in Somaratna and Ting (1986) and Ghahremani (1991). Thus finite element formulation of Eq. (6) leads to the following quadratic eigenvalue problem:

\[ (K + \lambda D + \lambda^2 M)p = 0, \] (7)

where \( K, D \) and \( M \) are nonsymmetric and square matrices and \( p \) is the eigenvector of the nodal displacements (see Somaratna and Ting, 1986 or Ghahremani, 1991 for details).

The eigenvalue problem for \( \lambda \) is quadratic, i.e., of the form \( (K + \lambda D + \lambda^2 M)p = 0 \). To calculate eigenvalues \( \lambda \) by the Muller method (Gerald and Wheatley, 1984), it is necessary to compute \( ||K + \lambda D + \lambda^2 M|| \) repeatedly for various values of \( \lambda \). While real and complex eigenvalues, and multiple eigenvalues as well are obtained by the Muller method, it is not straightforward to obtain eigenvalues and the corresponding eigenvectors simultaneously by the Muller method. Therefore we employ the inverse iterative method, which is very effectively applied for calculating an eigenvector (Bathe, 1996) together with the eigenvalue. We utilize the common scheme to convert the quadratic problem to a linear problem (Ghahremani, 1991).

Finite element mesh consists of two-dimensional 8-node isoparametric quadrilateral elements. Numerical integration is performed using Gaussian quadrature rule on a grid of \( n_G \times n_G \) integration points. In a general anisotropic material the material stiffness matrix \( C_{ij} \) is functions of \( \theta \) and \( \phi \), and it will display a strong dependence on these terms. Note that examination of the expressions involved in Eq. (6) or Eq. (7) shows...
that terms like $1/\sin \theta$ also are present. Although terms like $1/\sin \theta$ occur because the strains are obtained in the spherical coordinate system, the integrand of Eq. (6) is not singular as Eq. (6) is derived from the principle of minimum potential energy. However, this leads to some difficulties numerically in evaluating the integrals accurately especially in the region near the pole of the coordinate system where $\theta \cong 0$. Two approaches have been adopted to avoid the numerical difficulties in evaluating the integral of Eq. (6) at the poles: one finds the optimal number of integration points and the optimal mesh size by numerical testing (Somaratna and Ting, 1986) and, the other calculates the element stiffness matrices close to $\theta = \pi/2$ using a local coordinate system and then transforms the computed element stiffness matrices to the global coordinate frame (Ghahremani, 1991). Our implementation utilizes the former approach.

3. Two-state $M$-integral and its application for three-dimensional wedges

In this section we briefly summarize the two-state $M$-integral, which is utilized for computing the near-tip stress intensities of three-dimensional wedges. Firstly, the $M$-integral and the two-state $M$-integral are discussed, and then a domain (or volume) representation for three-dimensional bodies is derived to compute the two-state $M$-integral accurately on the far-field using the finite element solutions. The two-dimensional analogue was described in Im and Kim (2000).

The $M$-integral is written as (Knowles and Sternberg, 1978):

$$ M = \int_{S} \left\{ W_{x_{i}n_{i}} - t_{i}u_{i,k}x_{k} + \frac{m-n}{m} t_{i}u_{i} \right\} \, dS, \quad (8) $$

where “$S$” is a closed surface. Note that $W$ and $t_{i}$ indicate the strain energy density and the traction components, given as $W = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$ and $t_{i} = \sigma_{ij} n_{j}$. Furthermore $u_{i}$ is the displacement component, and $m$ the degree of homogeneity of the strain energy density, that is, 2 for the linear elastic problem, and $n$ the degree of freedom of the spatial dimension, e.g., $n$ is equal to 2 for two-dimensional domains or to 3 for three-dimensional bodies. Thus the $M$-integral for a three-dimensional linear elastic body is rewritten as

$$ M = \int_{S} \left\{ W_{x_{i}n_{i}} - t_{i}u_{i,j}x_{j} - \frac{1}{2} t_{i}u_{i} \right\} \, dS \quad (i,j = 1,2,3). $$

Suppose two independent elastic states, “$A$” and “$B$”. We consider another elastic state “$C$” obtained by superposing the two equilibrium states “$A$” and “$B$”. Then the above $M$-integral is written as

$$ M^{C} = M^{A} + M^{B} + M^{(A,B)} \quad (9) $$

where the superscripts “$A$”, “$B$” and “$C$” indicate the aforementioned elastic states, and $M^{(A,B)}$ is the two-state $M$-integral (Chen and Shield, 1977), given as

$$ M^{(A,B)} = \int_{S} \left[ C_{ijkl} \epsilon_{ij}^{A} \epsilon_{kl}^{B} n_{p} x_{p} - (t_{i}^{A} u_{i,p}^{B} + t_{i}^{B} u_{i,p}^{A}) x_{p} + \frac{m-n}{m} (t_{i}^{A} u_{i}^{B} + t_{i}^{B} u_{i}^{A}) \right] \, dS. $$
The integral $M^{(A,B)}$ results from the mutual interaction between the two elastic states “A” and “B”. This integral is referred to as the two-state $M$-integral in this context. For three-dimensional bodies under linear elastic deformations, we take $m=2$ and $n=3$ to reach the following expression for $M^{(A,B)}$.

$$M^{(A,B)} = \int_S \left[ C_{ijkl} e_{ij}^{A} e_{kl}^{B} n_{p} r_{p} - (t_{i}^{A} u_{i,p}^{B} + t_{i}^{B} u_{i,p}^{A}) r_{p} - \frac{1}{2} (t_{i}^{A} u_{i}^{B} + t_{i}^{B} u_{i}^{A}) \right] dS.$$  

Note that $M^{(A,B)}$ is the conservation integral for two equilibrium states since it identically vanishes for the domains with no singularities.

To explain the application of $M^{(A,B)}$ for generic three-dimensional wedges, we reconsider the conical domain as in Fig. 2, where each of the two surfaces $S_I$ and $S_{II}$, having the outward normal vectors, cuts through the lateral surface $S_L$ in an arbitrary manner. Recalling that the $M$-integral is dependent upon the origin of the coordinate system $(x_1, x_2, x_3)$, we take its origin at the wedge vertex. We take the closed surface $S_{II} - S_I + S_L$ where $-S_I$ means the reverse orientation of the surface $S_I$, that is, the same area but with the opposite normal vectors. With no singularities inside the region bounded by these surfaces, we can show the path independence of the $M$-integral as

$$M(S_I) = M(S_{II}),$$  

where $M(S_L) = 0$ and $M(-S_I) = -M(S_I)$ have been used. Furthermore, the path or surface independence of the two-state $M$-integral $M^{(A,B)}$ is apparent from the above and Eq. (9). That is, we have

$$M^{(A,B)}(S_I) = M^{(A,B)}(S_{II}).$$  

A convenient method of computing the two-state integral $M^{(A,B)}$ on the far field is a regular displacement based FEM in conjunction with the volume integral for...
three-dimensional domains (Li et al., 1985). Now utilizing the domain integral and going through some manipulation, we can reach the following expressions:

\[ M^{(A,B)} = -\int_{V_II-V_I} C_{ijkl} e_j \varepsilon_k \varepsilon_l + \left( \sigma_{ij} e_i \varepsilon_j + \sigma_{ij} e_j \varepsilon_i \right) - \frac{1}{2} \left( \sigma_{ij} e_i + \sigma_{ij} e_j \right) q_{ij} \, dV \]  

(13)

for three-dimensional bodies, where \( V_I \) and \( V_{II} \) represent the domains bounded by \( S_I \) and \( S_{II} \), and \( S_{II} \) and \( S_L \), respectively, and \( V_{II} - V_I \) indicates the region bounded by \( S_I \), \( S_{II} \) and \( S_L \) in Fig. 2. The function \( q(x_1,x_2,x_3) \) is a weight function that is defined as 1 on \( S_I \) and as 0 on \( S_{II} \) with smooth variation between \( S_I \) and \( S_{II} \). Note that expressions (11) and (12) indicate that \( M \) and \( M^{(A,B)} \) are conserved for an arbitrary banded volume \( V_{II} - V_I \).

We are now at the stage of applying the aforementioned two-state \( M \)-integral for finding the free constant \( \beta_I \) in the eigenfunction series Eq. (1). Let \( \beta_n \) denote the free constant for the singular stress, which is the first term in Eqs. (1) and (2). In the numerical examples to follow in the next section, we will focus on this term as it represents the near-tip intensity of the singular stress field under a proper normalization of the eigenfunction. However, the present scheme is equally applicable for finding any higher-order eigenfunction term as well, and so we explain the scheme for an arbitrary free constant \( \beta_I \).

The key idea of calculating \( \beta_I \) is to utilize the path or surface independence property of \( M^{(A,B)} \), as given in Eq. (12). Firstly, a convenient auxiliary state “B” is chosen, and the elastic field of the wedge under consideration is assigned to “A”. \( M^{(A,B)}(S_I) \) is then calculated semi-analytically on the \( \theta-\phi \) domain of the spherical coordinates. We need numerical integration to evaluate the resulting integral on this domain. Next, from finite element analysis we obtain \( M^{(A,B)}(S_{II}) \) on the right-hand side of Eq. (12) with the aid of the volume integral expression (13). Then Eq. (12) yields \( \beta_I \) and this value must be invariant with respect to the choice of the auxiliary elastic state as \( M^{(A,B)} \) is a bilinear functional of the two elastic states “A” and “B”.

The present procedure now boils down to the choice of a convenient auxiliary state “B”. For this we define a complementary eigenfield for a given eigenstate. Let the complementary eigenvalue \( \delta^c_I \) of an arbitrary eigenvalue \( \delta_I \) be defined in the \( M \)-integral sense as follows:

\[ \delta_I + \delta^c_I = -3 \quad \text{or} \quad \lambda_I + \lambda^c_I = -1. \]  

(14)

This may be compared with its two-dimensional analogue (see Im and Kim, 2000; Lee et al., 2001; Jeon and Im, 2001), given as

\[ \delta_I + \delta^c_I = -2 \quad \text{or} \quad \lambda_I + \lambda^c_I = 0. \]

As will be verified later, \( \delta^c_I \) constitutes another eigenvalue as long as \( \delta_I \) belongs to the eigenvalues for a given problem. Benthem (1980) discussed the fact that \( -\delta_I - 3 \) is also an eigenvalue if \( \delta_I \) is the vertex singularity of a quarter infinite crack. In general this seems to be true for every eigenvalue and for generic three-dimensional wedges wherein the \( M \)-integral is conserved, just as in two-dimensional wedges. Rigorous proof is not available yet for the three-dimensional case. In this paper, we will numerically verify it for a quarter infinite crack and a bimaterial interface corner.
Suppose we are interested in finding a free constant $\beta_l$. Then, for the auxiliary state we take the elastic state of the complementary eigenvalue $\delta_l^c$ (or $\lambda_l^c = \delta_l^c + 1$), which is written as

$$u^c = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c+1} \mathbf{\tilde{w}}^c(\theta, \phi, \delta_l^c)] = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c} w^c_l(\theta, \phi, \delta_l^c)],$$

$$v^c = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c+1} \mathbf{\tilde{v}}^c(\theta, \phi, \delta_l^c)] = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c} v^c_l(\theta, \phi, \delta_l^c)],$$

$$w^c = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c+1} \mathbf{\tilde{w}}^c(\theta, \phi, \delta_l^c)] = \frac{1}{2\mu} \text{Re}[\beta_l^c r^{\delta_l^c} w^c_l(\theta, \phi, \delta_l^c)],$$

(15)

where the intensity $\beta_l^c$ of the complementary eigenfield is prescribed arbitrarily. To calculate $M^{(A,B)}(S_1)$ in Eq. (12) we substitute the elastic field (1) for the elastic state “A”, and the complementary elastic field (15) for the elastic state “B”. Then, the following expression is obtained for the two-state integral $M^{(A,B)}(S_1)$ after some algebra:

$$M^{(A,B)}(S_1) = \sum_{\delta_n} \int_{S} \text{Re}[\beta_n \beta_l^c r^{\delta_n+\delta_l^c+3} F(\delta_n, \delta_l^c)] d\theta d\phi,$$

(16)

where $F(\delta_n, \delta_l^c)$ is given in Appendix A. Note that we take the summation sign outside the integral symbol by exploiting firstly the path or surface independence property and secondly the fact that each individual eigenfunction term in the series expansion (1) is a separate elastic state satisfying the governing equations and the near-field boundary conditions on $S_L$ in Fig. 1.

For Eq. (12) with Eq. (16) to be valid, the surface $S_1$ of $M^{(A,B)}(S_1)$ on the left-hand side of Eq. (12) should be located sufficiently close to the vertex $O$ because expression (1), which has been substituted to obtain Eq. (16), is legitimate near the vertex. However, in actuality the path or surface independence property of the two-state $M$-integral makes $M^{(A,B)}(S_1)$ invariant with respect to the radial coordinate $r$. This is apparent if $S_1$ is chosen to be a concentric surface with the radial distance, say $\hat{r}$ from the vertex. If $\delta_n + \delta_l^c > -3$, we take $\hat{r}$ to go to zero so that the $M$-integral contribution from $\delta_n$ may be shown to be zero. On the other hand, we choose $\hat{r}$ to be an infinitely large value in order to show that the $M$-integral contribution disappears for $\delta_n + \delta_l^c < -3$ as well. The only non-vanishing contribution originates from the case $\delta_n + \delta_l^c = -3$ or $\lambda_n + \lambda_l^c = -1$ (see Appendix A for further details), and all the other terms disappear, so that $M^{(A,B)}(S_1)$ has no $\hat{r}$ dependence and that expression (16) involves the integration merely on the $\theta$–$\phi$ domain. This implies that the only nonzero contribution to $M^{(A,B)}(S_1)$ occurs from $\delta_n = -3 - \delta_l^c = \delta_l$, that is, from the complementary pair of eigenvalues. Im and Kim (2000), Lee et al. (2001) and Jeon and Im (2001) showed that this property holds for the two-state $J$ and $M$-integral in two-dimensional cases. The right-hand side $M^{(A,B)}(S_{II})$ is now calculated from finite element analysis with the aid of the domain integral representation (13). Then Eq. (12) will yield the free constant $\beta_l$, and so the near-tip stress intensity $\beta_s$ of the singular field if $\delta_s$ is chosen for $\delta_l$. 
4. Numerical examples

In this section, we take two numerical examples to illustrate the computational scheme discussed in the foregoing section. The first example is the three-dimensional wedge vertex wherein a crack-tip line meets with a free surface. The vertex singularity for this quarter infinite crack was discussed by a host of researchers, for example, Benthem (1977, 1980), Bazant and Estenssoro (1979), etc. However, no solutions have been reported for the near-tip stress intensity. The second example is the three-dimensional interface corner, which is formed by two joining free edges. This problem has been selected for comparison to the existing solution from Labossiere and Dunn (2001).

4.1. Three-dimensional crack corners

Consider the three-dimensional crack of a thick plate as shown in Fig. 3. The material is isotropic with \( E = 200 \text{ GPa} \) and \( v = 0.3 \) for steel. We choose the origin to be at the vertex of the three-dimensional crack corner (vertex). The length is \( h \) both in the \( x \)- and \( z \)-direction, \( 2L \) in the \( y \)-direction, and the crack length is \( h/2 \). The crack surface is \( x > 0 \) and \( z \geq 0 \) in the \( x-z \) plane and ligament plane is \( x \leq 0 \) and \( z \geq 0 \) in the \( x-z \) plane.

At a point sufficiently close to a three-dimensional crack corner the asymptotic solutions should be characterized by the solutions of a quarter infinite crack in a half-space.
Fig. 4. Quarter-infinite crack in the half-space.

Fig. 5. Domain and boundary conditions for computing the eigenvalues of a quarter-infinite crack.

(Nakamura and Parks, 1988, 1989). The eigenvalues in the present case are obtained for the vertex of a quarter infinite crack as shown in Fig. 4. The domain is considered in the half-space, \( z \geq 0 \) and the crack front is along the \( z \)-axis. To compute the eigenvalues for a quarter infinite crack, we consider the surface on the unit sphere onto the \( \theta-\phi \) plane as shown in Fig. 5 and discretize this domain for finite element analysis utilizing the 8-node plane element. We consider the two modes: one is symmetric and the other antisymmetric with respect to the \( x-z \) plane. To compute the eigenvalues corresponding to each mode, we take the half-model and impose symmetric or antisymmetric boundary conditions on the plane of symmetry \( \phi = \pi \) and traction-free conditions on other boundaries as shown in Fig. 5. The domain was subdivided into \( n_\theta \times n_\phi \) elements: \( n_\theta \) is the number of elements in the \( \theta \)-direction and \( n_\phi \) the number of elements in the \( \phi \)-direction. The symmetric boundary conditions are \( w = 0, \sigma_{\theta \phi} = 0, \sigma_{r \phi} = 0 \) and the antisymmetric boundary conditions are \( u = 0, v = 0, \sigma_{\phi \phi} = 0 \) where \( u, v \) and \( w \) are the displacements in \( r, \theta \) and \( \phi \) directions, respectively.
Because we use the finite element method to compute the eigenvalues, the values of eigenvalue depend upon the mesh size and the number of integration points. Moreover, in the previous section we stated that the expression of the eigenproblem, Eq. (6) or Eq. (7) involves the terms like $1/\sin \theta$ in the integrand. These terms may cause the numerical difficulty in evaluating the integrals accurately in the region near the pole of the coordinate system ($\theta \approx 0$). Therefore, we should first determine the mesh size and the number of integration points to be used for the present problem. For this purpose, we examine the influence of the mesh size and of the number of integration points with Gaussian quadrature rule upon the values of the stress singularity for Poisson’s ratio, $\nu = 0.0$. When $\nu = 0.0$, the plane strain solutions satisfy the three-dimensional field equations and the boundary conditions (Benthem, 1977; Ghahremani, 1991). Thus the order of stress singularity, $\delta_s$ should be exactly $-0.5$ for $\nu = 0.0$ irrespective of the symmetric deformation or the antisymmetric deformation. We plot the stress singularity obtained by varying the mesh size and the number of integration points in Fig. 6 for the symmetric deformation and in Fig. 7 for the antisymmetric deformation. In Figs. 6 and 7, we use regular mesh, and obtain stress singularity with error less than 0.5% for both the symmetric deformation and the antisymmetric deformation with $8 \times 16$ mesh and $5 \times 5$ Gaussian integration points. Furthermore, we find that the influence of mesh size upon the stress singularity is stronger than that of the number of integration points in Figs. 6 and 7 (see also Koguchi and Muramoto, 2000). We choose the number of the Gaussian integration points to be $5 \times 5$, and utilize the $8 \times 16$ mesh in all the numerical computation to follow. If we use a more refined mesh, we can obtain the more accurate value. It is noticed that the eigenvalues have already converged to the exact value for the $8 \times 16$ mesh (in the case of $\nu = 0.0$).

For varying Poisson’s ratios, the eigenvalues of a quarter infinite crack are computed and tabulated in Table 1 for the symmetric deformation and in Table 2 for the antisymmetric deformation. They show that the eigenvalue $-\delta_l - 3$ is an eigenvalue when $\delta_l$ is an eigenvalue of the present problem so that the eigenvalues satisfy
Fig. 7. Stress singularity of a quarter-infinite crack for antisymmetric deformation (in case of $\nu = 0.0$).

Table 1
Complementary pairs of eigenvalues for the symmetric deformation of the quarter-infinite crack ($\delta_n + \delta'_n = -3$)

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\nu = 0.0$</th>
<th>$\nu = 1.5$</th>
<th>$\nu = 3.0$</th>
<th>$\nu = 4.0$</th>
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<tbody>
<tr>
<td>$-4.4807$</td>
<td>$-4.4405$</td>
<td>$-4.3763$</td>
<td>$-4.8612 \pm i0.0241$</td>
<td></td>
</tr>
<tr>
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<td>$-4.0232$</td>
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<tr>
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<td>$-3.4987$</td>
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<td>$-3.6808$</td>
<td>$-3.9864$</td>
<td></td>
</tr>
<tr>
<td>$-3.4137$</td>
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<td>$-3.2146$</td>
<td>$-3.1309$</td>
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<td>$1.4405$</td>
<td>$1.3763$</td>
<td>$1.8612 \pm i0.0241$</td>
<td></td>
</tr>
</tbody>
</table>

the complementarity relationship, $\delta_l + \delta'_l = -3$ in the three-dimensional $M$-integral sense. To verify our computation, the stress singularities have been compared with the results computed by others (Benthem, 1977; Bazant and Estenssoro, 1979; Somaratna and Ting, 1986), which were verified within 1% error by Ghahremani (1991).
Table 2
Complementary pairs of eigenvalues for the antisymmetric deformation of the quarter-infinite crack ($\delta_n + \delta_n^c = -3$)

<table>
<thead>
<tr>
<th>$\nu = 0.0$</th>
<th>$\nu = 1.5$</th>
<th>$\nu = 3.0$</th>
<th>$\nu = 4.0$</th>
</tr>
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<tbody>
<tr>
<td>$-4.7018$</td>
<td>$-4.6944$</td>
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<td>$-4.7108 \pm i0.5760$</td>
</tr>
<tr>
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<td>$-4.5375$</td>
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</tr>
<tr>
<td>$-4.2355$</td>
<td>$-4.2901$</td>
<td>$-4.6913 \pm i0.5510$</td>
<td>$-4.5312 \pm i0.3304$</td>
</tr>
<tr>
<td>$-3.9900$</td>
<td>$-3.9941$</td>
<td>$-3.9975$</td>
<td>$-4.4832 \pm i0.3772$</td>
</tr>
<tr>
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<td>$-3.6014$</td>
<td>$-3.5055$</td>
<td>$-4.2431 \pm i0.1257$</td>
</tr>
<tr>
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<td>$-3.4333$</td>
<td>$-3.5025$</td>
</tr>
<tr>
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<td>$-3.0008$</td>
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<td>$-2.0014$</td>
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<td>$2.3728 \pm i0.2352$</td>
<td>$1.7108 \pm i0.5760$</td>
</tr>
</tbody>
</table>

comparison has shown that the present results are in good agreement with those from the previous works.

Recall that a quarter infinite crack as shown in Fig. 4 has a singularity line of the crack front as well as the vertex of the crack front line intersecting with the free surfaces. That is, the asymptotic solutions of Eq. (6) contain the inverse square root singularity along the crack front line. To deal with the edge singularity we utilize the singular element of which the mid-side nodes are located at the quarter point to result in the $\rho^{-1/2}$ singularity near the crack front line of a quarter infinite crack where $\rho = \sqrt{x^2 + y^2}$.

We take the origin at the vertex where the crack front line is intersecting with the free surface. To apply the two-state $M$-integral to the three-dimensional crack corner, we cannot directly utilize the integral path (or surface) for the two-state $M$-integral as shown in Fig. 2 because a singularity line exists along the crack front line. The integral surface for the two-state $M$-integral should be modified as shown in Fig. 8. We take the closed surface $S = S_{\text{II}} - S_t + S_c^+ + S_c^- - S_I + S_L$ with no singularities inside, when $S_t$ indicates the outer surface of a circular tube enclosing the crack front line between the semi-hemispherical surface $S_I$ and $S_{\text{II}}$, with infinitesimally small radius, $S_c^+$ and $S_c^-$ the crack faces, $S_L$ the free surface at $z = 0$. Then the surface-independence of the
two-state $M$-integral is expressed as

$$M(S_I) = M(S_{II}) - M(S_t). \hspace{1cm} (17)$$

The second term $M(S_t)$ on the right-hand side of Eq. (17) is the contribution from the crack front singular field and is calculated on the surface along the crack front line as shown in Fig. 9. The path or surface independence of the two-state $M$-integral is apparent from the above expression and is rewritten as

$$M^{(A,B)}(S_I) = M^{(A,B)}(S_{II}) - M^{(A,B)}(S_t), \hspace{1cm} (18)$$
where the second term $M^{(A,B)}(S_I)$ is written as

$$M^{(A,B)}(S_I) = \lim_{\rho \to 0} \int_{S_I} \left\{ W^{(A,B)} \rho - t^A_i \frac{\partial u^B_i}{\partial \rho} \rho - t^B_i \frac{\partial u^A_i}{\partial z} \rho - t^A_i \frac{\partial u^B_i}{\partial z} \rho - t^B_i \frac{\partial u^A_i}{\partial z} \right\} \rho \, dz \, d\varphi,$$

(19)

where $(\rho = \sqrt{x^2 + y^2}, \varphi, z)$ are cylindrical coordinates and $W^{(A,B)} = C_{ijkl} u^A_i u^B_j$. Note that the tube $S_I$ does not include the singular vertex point $(r = 0)$, and that the two states “A” and “B” have the inverse square root singularity along the crack front line $z > 0$. Taking into account the fact that $\frac{\partial u^B_i}{\partial z}$ and $\frac{\partial u^A_i}{\partial z}$ are not singular along the crack front line $z > 0$, we see that the above integral goes to zero at the rate of $O(\rho)$ as $\rho$ approaches zero. Therefore, the path or surface independence of the two-state $M$-integral of Eq. (18) for the three-dimensional crack corner is rewritten as

$$M^{(A,B)}(S_I) = M^{(A,B)}(S_{II}).$$

We compute the two-state $M$-integral $M^{(A,B)}(S_I)$ with the aid of Eq. (16) on the $\theta$–$\phi$ surface, and next $M^{(A,B)}(S_{II})$ from the volume representation of Eq. (13) via FEM using the auxiliary fields (15) on the far field.

We investigate the influence of mesh size and the number of integration points upon the values of the two-state $M$-integral. In evaluating Eq. (16), it is found that the number of integration points does not matter if it is greater than $3 \times 3$ integration. We compute the two-state $M$-integral of Eq. (16) varying the number of elements in the $\theta$–$\phi$ plane and plot the result in Fig. 10. It is found that $M^{(A,B)}(S_I)$ of Eq. (16) yields a converging solution, as the total number of elements increases.

The finite element mesh with 4800 twenty-node solid elements, together with the loading conditions and the boundary conditions for each of the symmetric and antisymmetric modes, are shown in Fig. 11. The finite element mesh near the crack vertex is refined due to the presence of the three-dimensional singularity while the singular elements into which the $\rho^{-1/2}$ edge singularity has been embedded are adopted along the crack front line. Finite element analysis has been carried out using the package code ABAQUS. The surface independence of the two-state $M$-integral obtained utilizing the finite element solution for the given problem and expression (15) for the auxiliary field on far field is shown in Fig. 12 for the symmetric and the antisymmetric deformations in the case of Poisson’s ratio $\nu = 0.3$.

The free constant is obtained using the two-state $M$-integral and finite element analysis as described in the previous chapter. Note that the value of the free constant $\beta_s$ is dependent upon the way the eigenfunctions ($\tilde{u}, \tilde{v}, \tilde{w}$) are normalized. For this problem, we choose to normalize the eigenfunctions such that the maximum magnitude of the three spherical components $\tilde{u}, \tilde{v}$ and $\tilde{w}$ on $\theta = \pi/2$ and $\phi = \pi$, which is the ligament direction on the free surface ($z = 0$), may become 1. To confirm the solution convergence, we plot the free constant $\beta_{s,I}$ for the symmetric mode versus the number of finite elements employed in Fig. 13. The free constants or the near-tip intensities $\beta_{s,I}$ and $\beta_{s,II}$ for the two deformation modes are tabulated in Table 3. In Table 3 they
Fig. 10. The two-state $M$-integral near the vertex for varying the number of element for symmetric deformation with $\nu = 0.3$ $(M^{(A,B)}(S_{0}))$ indicates the value of the two-state $M$-integral with the most refined mesh.

decrease as Poisson’s ratio increases both for the symmetric deformation and for the antisymmetric deformation.

Fig. 14 shows the stresses obtained from the finite element solutions and the eigenfunction solutions with the free constants obtained above along the ligament direction. The normal stress $\sigma_{xx}$ is plotted for the symmetric deformation and the skew stress $\sigma_{xy}$ for antisymmetric deformation. Between the finite element solutions and the eigenfunction solutions in Fig. 14, only a small deviation is noticed in the region $r/a < 0.01$ for the dominant stress components, where $a$ is the crack length.

4.2. Three-dimensional bimaterial interface corner

For our next example, we choose a three-dimensional wedge or three-dimensional bimaterial interface corner (vertex). Koguchi (1997) computed the stress singularities of three-dimensional bimaterial interface corners using BEM. Later Koguchi and Muramoto (2000) used the finite element method and compared the results with those obtained from BEM. Labossiere and Dunn (2001) computed the stress singularity for three-dimensional bimaterial corners as shown in Fig. 15 using the finite element method. Furthermore, the near-tip stress intensity for this three-dimensional problem was calculated in their work. It involves a full-field finite element analysis with an enormously large number of degrees of freedom, ($\sim 100,000$), even for such a simple geometry, to cover the singular zone with a fine mesh. In order to be able to compare our results with theirs, we choose the same three-dimensional interface corner as designed by Labossiere and Dunn (2001) with the width $h = 12.5$ mm, and the
Fig. 11. Finite element mesh and loading conditions and boundary conditions of the half-model for symmetric and antisymmetric modes \((u_x, u_y, u_z)\) are displacement components in Cartesian coordinates and \(a = 1 \text{ MPa}\). (a) for symmetric mode. (b) for antisymmetric mode.

length \(L = 63.5\) mm in \(z\)-axis and the loading point distance \(l = 76.2\) mm as shown in Fig. 15. The structure consists of 6061-T6 aluminum and cast West System 105-205 epoxy. Each material is isotropic with \(E = 70.0\) GPa and \(v = 0.33\) for the aluminum, and \(E = 2.98\) GPa and \(v = 0.38\) for the epoxy. The structure is the four-point bending specimen with the square cross section with \(h \times h\) dimensions, where \(h = 12.5\) mm as shown in Fig. 15 (see Labossiere and Dunn, 2001 for details). They chose this specimen geometry and load configuration for several reasons: fracture initiation occurs at the three-dimensional bimaterial interface corner under the given loading; four-point flexure loading is relatively easy to perform by introducing only one length scale \(h\); there is only one stress singularity at the three-dimensional bimaterial interface corner (Labossiere and Dunn, 2001).

To compute the stress singularity first, we consider the surface on the unit sphere as shown in Fig. 16, and consider the domain with \(0 \leq \theta \leq \pi\) and \(0 \leq \phi \leq \pi/2\) on the \(\theta-\phi\) plane. We divide the domain with eight-node quadrilateral elements and apply the finite element technique according to the procedure in Section 2. We used 16 \(\times\) 8 mesh with eight-node elements and 5 \(\times\) 5 Gaussian integration points per element. Table 4 shows that the eigenvalue \(-\delta_i - 3\) is an eigenvalue when \(\delta_i\) is an eigenvalue so that
the eigenvalues appear as complementary pairs in the three-dimensional $M$-integral sense for this bimaterial corner. We obtain the stress singularity of $\delta_s = -0.3586$ while the stress singularity obtained by Labossiere and Dunn (2001) is $\delta_s = -0.351$. Its corresponding eigenvectors are also in good agreement with each other, as shown in Figs. 17 and 18.

Labossiere and Dunn (2001) proposed the asymptotic stress and a stress intensity with the stress singularity $\sigma_s$ for the three-dimensional interface corner.
Table 3
The intensities obtained from the two-state $M$-integral

<table>
<thead>
<tr>
<th>Poisson’s ratio</th>
<th>$\beta_{s,1}$ (symmetric deformation)</th>
<th>$\beta_{s,II}$ (antisymmetric deformation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>10.820</td>
<td>7.2275</td>
</tr>
<tr>
<td>0.15</td>
<td>7.7390</td>
<td>3.6336</td>
</tr>
<tr>
<td>0.3</td>
<td>4.9911</td>
<td>1.8833</td>
</tr>
<tr>
<td>0.4</td>
<td>4.8412</td>
<td>0.9230</td>
</tr>
</tbody>
</table>

Fig. 14. Comparison of the eigenfunction solutions with the finite elements results along the ligament surface ($\theta = \pi/2, \phi = \pi$).

geometry as

$$\sigma_{ij} \sim H^{3D} r^{\delta_i} \tilde{\sigma}_{ij}(\theta, \phi) \quad \text{as } r \to 0,$$

(22a)

$$H^{3D} = \sigma_0^{3D} h^{-\delta_3} Y^{3D} \left( \frac{E_1}{E_2}, v_1, v_2 \right),$$

(22b)

where $r$ is the radius from the bimaterial interface corner, $h$ the width as shown in Fig. 15, and $Y^{3D}(E_1/E_2, v_1, v_2)$ is a nondimensional function of the elastic mismatch. The above stress intensity $H^{3D}$ is equivalent to the free constant $\beta_{s}$ of Section 4.1 except that their scalings may be different from each other due to different normalizations between Eqs. (2) and (22a). Labossiere and Dunn (2001) normalized the asymptotic solution such that $\tilde{\sigma}_{i0}(\pi/2, \pi/4) = 1$ in equation (22a). They obtained $Y^{3D}$ from fitting, using the least-squares approach, the asymptotic displacements fields to the full-field displacements from the finite element solution in the vicinity of the three-dimensional interface corner along the specific rays emanating from the interface corner or vertex. Note that $\sigma_0^{3D}$ is the nominal bending stress that would exist at the bottom edge of a homogeneous
beam of dimension $h \times h$ under four-point loading in Fig. 15, and is written as

$$
\sigma_0^{3D} = \frac{3P(L-l)}{\sqrt{2}h^3},
$$

where $P$ is the applied loading.

The finite element mesh with 3200 twenty-node solid elements and with relatively fine discretization near the three-dimensional bimaterial interface corner has been employed. We compute the free constant $\beta_s$ utilizing the two-state $M$-integral and the finite element analysis using ABAQUS, as discussed in previous section. To compare the results of Labossiere and Dunn (2001) with the present results, we compute the nondimensional function $Y^{3D}$ in Eq. (22b) as tabulated in Table 5. When $\beta_s$ or $H^{3D}$ is computed, the nondimensional function $Y^{3D}$ is obtained using the stress singularity

---

**Fig. 15.** Geometric configuration of three-dimensional interface corner specimen.

**Fig. 16.** Surface for eigenvalue analysis of three-dimensional interface corner specimen.
Table 4
Complementary pairs of eigenvalues of the three-dimensional bimaterial interface corner ($\delta_n + \delta_\xi = -3$)

<table>
<thead>
<tr>
<th>Eigenvalue</th>
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<td></td>
</tr>
<tr>
<td>$-0.99978$</td>
<td></td>
</tr>
<tr>
<td>$-0.99972$</td>
<td></td>
</tr>
<tr>
<td>$-0.35861$</td>
<td></td>
</tr>
<tr>
<td>$0.00000$</td>
<td></td>
</tr>
<tr>
<td>$0.00005$</td>
<td></td>
</tr>
<tr>
<td>$0.00016$</td>
<td></td>
</tr>
<tr>
<td>$0.35770 \pm i0.55264$</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 17. Eigenfunctions as a function of $\theta$ for the three-dimensional aluminum/epoxy corner along $\phi = \pi/4$.

Table 5
Nondimensional function $Y^{3D}$ for the three-dimensional bimaterial interface corner

<table>
<thead>
<tr>
<th></th>
<th>Labossiere and Dunn’s results</th>
<th>The present results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stress singularity</td>
<td>$-0.351$</td>
<td>$-0.3586$</td>
</tr>
<tr>
<td>$Y^{3D}$</td>
<td>0.303</td>
<td>0.302</td>
</tr>
</tbody>
</table>
Fig. 18. Eigenfunctions as a function of $\phi$ for the three-dimensional aluminum/epoxy corner along the interface $\theta = \pi/2$.

$\delta_s = -0.3586$ while Labossiere and Dunn (2001) employed their result $\delta_s = -0.351$ to compute $Y^{3D}$. The two results of the nondimensional function $Y^{3D}$ are in good agreement. Using the free constant $\beta_n$, we obtain stress from the asymptotic solution of Eq. (2), and compare it with the result from the finite element analysis along the line $\theta = \pi/2$ and $\phi = \pi/4$ in Fig. 19. The finite element solution agrees well with the one term expansion (the singular term only) in the region $\rho = \sqrt{x^2 + y^2} < 0.9$ mm with the width $h = 12.5$ mm.

5. Conclusions

We have proposed a general and systematic computational scheme for computing the singular stress states near the three-dimensional vertices with the aid of the two-state $M$-integral and the eigenfunction expansion. We verify numerically that the eigenvalues of the given three-dimensional problem satisfy the complementarity relationship, $\delta_n + \delta_n^c = -3$, in the three-dimensional $M$-integral sense. This relationship and the surface independence of the two-state $M$-integral are applied for extracting the near-tip intensity of the singular stress fields for three-dimensional vertices. The two numerical examples demonstrate that the present scheme is effective and accurate for computing the intensities of singular stresses near the generic three-dimensional wedges.

Acknowledgements

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Fig. 19. Stresses versus distance from the vertex of the interface corner along \( \theta = \pi/2 \) and \( \phi = \pi/4 \) for the three-dimensional aluminum/epoxy interface corner with \( h = 12.5 \) mm; (a) \( \sigma_{xx} \), (b) \( \sigma_{yy} \).

Appendix A

Eq. (16) is written as

\[
M^{(A,B)}(S_1) = \sum_{\delta_x} \int_S \text{Re}[\beta_n \beta_1 r^{\delta_n + \delta_1^c + 3} F(\delta_n, \delta_1^c) + \beta_n \beta_1 r^{\delta_n + \delta_1^c + 3} F(\delta_n, \delta_1^c)] \, d\theta \, d\phi,
\]

where \( F(\delta_n, \delta_1^c) = \{I_1(\delta_n, \delta_1^c) + I_2(\delta_n, \delta_1^c) + I_3(\delta_n, \delta_1^c)\} \sin \theta \) with

\[
I_1(\delta_n, \delta_1^c) = \left( \frac{2\mu + k}{4\mu} \right) \left\{ (2 - \lambda_n \lambda_1^c - (\lambda_n + \lambda_1^c)/2) \tilde{u}_n \tilde{v}_1^c \cot^2 \theta + \tilde{v}_n \tilde{n}_1 \tilde{v}_1^c \right\}
\]
\[ +w_{n,\phi} \tilde{w}_n^c \sin^2 \theta + \tilde{u}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{w}_n^c + (\tilde{u}_{n,\phi} + \tilde{w}_{n,\phi}) \cot \theta \]

\[ + (\tilde{u}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi}) / \sin \theta + (\tilde{v}_{n,\phi} \tilde{w}_n^c + \tilde{v}_{n,\phi}) \cot \theta / \sin \theta \}

\[ I_2(\delta_n, \delta^c_1) = \frac{k}{4 \mu} \left\{ \left( \tilde{u}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{v}_n^c \right) / \sin \theta + \left( \tilde{v}_{n,\phi} \tilde{w}_n^c + \tilde{v}_{n,\phi} \tilde{v}_n^c \right) / \sin \theta \right\}, \]

\[ I_3(\delta_n, \delta^c_1) = \frac{1}{4} \left\{ \left( 2 - \iota_n \lambda^c_{1}\right) + \frac{5}{2} \left( \iota_n + \lambda^c_{1}\right) \left( \tilde{u}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{v}_n^c \right) + \tilde{u}_{n,\phi} \tilde{w}_n^c \right\}

\[ -\frac{3}{2} \left( \tilde{u}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{v}_n^c \right) / \sin^2 \theta \]

\[ + \tilde{w}_{n,\phi} \tilde{w}_n^c \cot^2 \theta + \left( \tilde{v}_{n,\phi} \tilde{w}_n^c + \tilde{v}_{n,\phi} \tilde{w}_n^c \right) / \sin \theta \]

\[ - \left( \tilde{v}_{n,\phi} \tilde{w}_n^c + \tilde{w}_{n,\phi} \tilde{w}_n^c \right) \cot \theta \]

where comma indicates the partial differentiation; \( k \) and \( \mu \) are bulk modulus and shear modulus. Note that \( \lambda^c_{1} (-\delta^c_1 + 1) \) is the complementary eigenvalue and \( \tilde{w}_n^c (\theta, \phi; \delta^c_1) \), \( \tilde{v}_n^c (\theta, \phi; \delta^c_1) \) and \( \tilde{v}_n^c (\theta, \phi; \delta^c_1) \) its corresponding eigenfunctions in \( r, \theta \) and \( \phi \) coordinates, respectively. When the complementary eigenvalues satisfy the relation of \( \delta^c_1 = -3 - \delta_1 = -3 - \delta_n \) (or \( \lambda^c_{1} = -1 - \lambda_1 = -1 - \lambda_n \)), the two terms in Eq. (16) are independent of the radial coordinate \( r \) for real eigenvalues, but the second term, involving the power \( \delta_n + \delta^c_1 + 3 = 2 \text{Im}[\delta_n] = -2 \text{Im}[\delta^c_1] \), appears to be \( r \)-dependent when \( \delta_n \) and \( \delta^c_1 \) are complex. However, the argument of path or surface independence for \( M^{(A,B)} \) shows that this term should disappear.

References


