A Pseudo Metric on Fuzzy Sets Based on the Satisfaction Function with Viewpoint

Young-il Kim1, Kwang H. Lee1
Division of Computer Science, Dept. of EECS, KAIST
373-1 Kusong, Yusong, Taejon 305-701, South Korea

Abstract

Deciding a proper metric for crisp values are well established and the results are clear, but the operation on fuzzy values are inherently ambiguous because the fuzzy values include uncertainty. This paper deals with this distance metric on a set of crisp/fuzzy values with different viewpoints.

A bounded pseudo-metric function (distance function) for a set of crisp/fuzzy values is proposed. The distance function is based on the satisfaction function on continuous domain with a viewpoint. The viewpoint was adopted to incorporate user's preferences on the domain of discourse. The proposed distance function on a set of fuzzy values becomes a bounded metric function when the set of fuzzy values only contains crisp values. Then the role of viewpoints on the distance function is discussed.

Finally, the relation between the proposed distance function and defuzzification is shown.

Keywords : fuzzy distance, pseudo-metric, defuzzification
Technical Area : T5 Mathematics

1 Introduction

Fuzzy theory has been applied to many areas which need to manage uncertain and vague data. Such areas range over approximate reasoning, control, optimization, decision making, and so on [2] [3] [17] [16]. Deciding a proper metric for crisp values are well established and the results are clear, but the operation on fuzzy values are inherently ambiguous because the fuzzy values include uncertainty.

There are two approaches to deal with this ambiguity. The first one is to define a fuzzy valued metric on a set of fuzzy values[15]. However, this fuzzy valued metric has problems in its interpretation and applications. The second approach is to define a pseudo-metric on a set of fuzzy values [9] [8] [10] [11] [12] [13]. In this paper, the second approach is adopted.

In the literature, some researchers used the distance measure as a metric function, and some used it as a difference measure [15] [9] [8] [10] [11] [12] [13]. A distance measure between two fuzzy sets is a measure that describes the difference between these fuzzy sets. A lot of researchers have used it without any definitive axioms before [13]. But deciding how far two fuzzy values are apart from each other is different from measuring the difference between two fuzzy sets. Metric function on a set is a distance function which satisfies the triangular inequalities.

There are relatively fewer works found in the literature that deal with distance as a metric on a set of fuzzy values [9] [14] [11]. They have used the same term distance as a metric on a set.

In this paper, a bounded pseudo-metric function on a set of crisp/fuzzy values on which a viewpoint is given, is proposed. The distance function is based on the satisfaction function on continuous domain with a viewpoint [1] [2] [3]. Since the distance function is based on the satisfaction function with a viewpoint, it can incorporate user's preferences on the domain of discourse.

The role of viewpoints on the distance function is to manipulate the scaling factors on specific ranges. By changing the shape of the viewpoint on a specific range, user can scale up or down the discriminating factor on the range.

This paper is organized as follows. In section 2, the previous works on the satisfaction function and the notion of viewpoints are briefly reviewed. In section 3, a bounded pseudo-metric function on a set of fuzzy values with a viewpoint is proposed. In section 4, a relation between the proposed distance function and defuzzification is investigated. In the last section, the paper is summarized and concluded.

2 Background and Related Works

2.1 Satisfaction function with a viewpoint

For crisp numbers, the arithmetic comparison is straightforward, but in the case of fuzzy sets, the comparison result is not always clear. There are many comparison measures of fuzzy sets generating a value in [0, 1], 0 as false and 1 as true, as a comparison result[1] [2] [4] [5] [6] [7] [8].

The satisfaction function (SP) is a measure which estimates the satisfaction degree of arithmetic comparison relations between two normalized fuzzy sets [1]. The basic idea of the SP starts from the geometrical interpretation of the comparison between two fuzzy numbers. To get a satisfaction degree, two given fuzzy values are sampled over a discrete grid on 2 dimensions and the sampled fuzzy values on each grid are compared to aggregate the comparison result giving the satisfaction degree. Figure 1 shows the geometrical interpretation of the comparison between two fuzzy values.

The SP is extended on a continuous domain by Lee[2] [3] by sending γ to zero, i.e., \( \lim_{\gamma \to 0} S_\gamma \).

Definition 2.1 A fuzzy value \( A \) is a fuzzy set which satisfies the following conditions:
1. The membership function \( \mu_A \) is piecewise continuous.
2. \( \int_{-\infty}^{\infty} \mu_A(x)dx \) exists.
3. \( \int_{-\infty}^{\infty} \mu_A(x)dx \neq 0 \)
Definition 2.2 (SFC) The SFC(Satisfaction Function on a Continuous domain) for two fuzzy values A and B, S(ARB) is defined as follows:

\[ S(ARB) = \lim_{\Delta \to 0} S_{\Delta}(ARB) \]

where \( \Delta \in \{<, >, =\} \) and \( S_{\Delta}(ARB) \) is the satisfaction function defined by Lee[1].

From the definition 2.2, \( S(ARB) \) can be written as follows [3]:

\[
S(A < B) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_A(x) \mu_B(y) \, dx \, dy \\
S(A > B) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_A(x) \mu_B(y) \, dy \, dx \\
S(A = B) = 0
\]

where \( \mu \) is a T-norm operator.

\( S \) has several properties. For any two fuzzy values A and B, and c, d \( \in \mathbb{R} \), and \( \Delta \in \{<, >, =\} \). For more detailed proof of the following properties, refer to [2] [3].

1. \( S(A < B) + S(A > B) = 1 \).
2. \( 0 \leq S(ARB) \leq 1 \).
3. \( S(A < B) = S(B > A) \).
4. The comparison between a crisp number and a fuzzy value is given as follows:

\[
S(A > c) = S(c < A) = \int_{-\infty}^{c} \mu_A(x) \, dx \\
S(c < d) = \begin{cases} 
1 & \text{if } c < d \\
0.5 & \text{if } c = d \\
0 & \text{otherwise}
\end{cases}
\]

Lee[2] proposed a ranking method for fuzzy numbers using the SFC with a notion of a viewpoint. In his method, he compared a given set of fuzzy numbers with a reference fuzzy set, called a viewpoint, and used the evaluation result of the arithmetic comparison as a ranking index. By changing the shape of a viewpoint, user can apply his/her preferences on the domain of discourse in the ranking procedure [2].

Definition 2.3 (Viewpoint) A viewpoint, \( V \), for a set of fuzzy values, \( F \), is a fuzzy value which satisfies the following conditions:

\[
\mu_V(x) = \begin{cases} 
fv(x) & \text{if } V_{\min} \leq x \leq V_{\max} \\
0 & \text{otherwise}
\end{cases}
\]

Support(\( V \)) = \( [V_{\min}, V_{\max}] \)

Support(\( F \)) = \( \cup_{\alpha \in F} \text{Support}(\alpha) \)

\( V_{\min} \leq \inf \{ \alpha \in \text{Support}(\alpha) \} \)

\( V_{\max} \geq \sup \{ \alpha \in \text{Support}(\alpha) \} \)

\( 0 < fv(x) \leq 1 \) for \( x \in \text{Support}(V) \)

So a viewpoint is a fuzzy value which covers the set of fuzzy values to be ranked. Then each fuzzy value in \( F \) is compared to the viewpoint giving the evaluation result as follows.

Definition 2.4 The evaluation of A in a viewpoint V, ev(A), in an ascending order, is defined as:

\[
ev(A) = S(A < V)
\]

and if the ranking is in a descending order,

\[
ev(A) = S(A > V)
\]

2.2 Metric functions

As preliminaries, a few definitions for metric functions are given in this subsection.

Definition 2.5 (Bounded pseudo-metric function)[10] A bounded pseudo-metric function on a domain U is a function such that

1. \( d : U \times U \to [0, 1] \)
2. \( d(u, u) = 0 \)
3. \( d(u, v) = d(v, u) \)
4. \( d(u, v) \leq d(u, w) + d(w, v), \forall w \in U \)

With another condition, a pseudo-metric function becomes a metric function.

Definition 2.6 (Bounded metric function) A bounded metric function on a domain U is a function such that

1. \( D \) is a bounded pseudo-metric function on U.
2. \( D(u, v) > 0 \) if and only if \( u \neq v \)

For some of the well known pseudo-metric functions, refer to [14].

3 A bounded pseudo-distance metric based on the satisfaction function with a viewpoint

A pseudo-metric function based on the satisfaction function with a viewpoint(SFV)[2] is proposed in this section. It becomes a bounded metric for a set of crisp numbers.

Remark For the rest of this proposal, multiplication will be used for the T-norm operator used in definition 2.2.
3.1 A bounded pseudo-distance metric on a set of fuzzy values

Definition 3.1 Bounded pseudo-metric on a set of fuzzy values

\[ \text{Dist}_V(A, B) = |S(A > V) - S(B > V)| \]

where \( F \) is a set of fuzzy values, \( A, B \in F \) and \( V \) is a viewpoint on \( F \).

Proposition 3.1 \( \text{Dist}_V(A, B) \) is a bounded pseudo-metric function on a domain \( F_V \) where \( A, B \in F \) and \( V \) is a viewpoint on \( F \).

Proof

1. \( 0 \leq \text{Dist}_V(A, B) \leq 1 \) by definition.
2. \( \text{Dist}_V(A, A) = 0 \) by definition.
3. \( \text{Dist}_V(A, B) = \text{Dist}_V(B, A) \) by definition.

4.

\[
\begin{align*}
\text{Dist}_V(A, C) + \text{Dist}_V(C, B) & = |S(A > V) - S(C > V)| + |S(C > V) - S(B > V)| \\
& \geq |S(A > V) - S(C > V)| + |S(C > V) - S(B > V)| \\
& = |S(A > V) - S(B > V)| \\
& = \text{Dist}_V(A, B).
\end{align*}
\]

Proposition 3.2 If \( C \) is a set of crisp numbers and \( V \) is a viewpoint on \( C \), \( \text{Dist}_V(a, b) \) is a bounded metric on \( C_V \) where \( a, b \in C \).

Proof

Evidently, \( \text{Dist}_V \) is a pseudo-metric.

From the equation 4,

\[
S(a > V) = \int_{-\infty}^{+\infty} \mu_V(x) \, dx
\]

Because \( \forall x \in \text{Support}(V), \mu_V(x) > 0 \) , \( f(x) \) is strictly increasing in \( \text{Support}(V) \). It means \( f : \mathbb{R} \rightarrow \mathbb{R} \) is 1-to-1. Thus, \( \text{Dist}_V(a, b) = 0 \) means \( a = b \).

Thus, \( \text{Dist}_V \) is a bounded metric function on \( C_V \).

3.2 Numeric Examples and Relation to Defuzzification

In order to show how the proposed method works and how a viewpoint affects the distance metric, an examples is presented. Then a relation of the proposed method to defuzzification is described in this subsection.

Figure 2: The role of viewpoint, \( F_1, F_2, F_3, \) and \( F_4 \) are evenly spaced and of the same shape

<table>
<thead>
<tr>
<th>( \text{Dist}_V(A, B) )</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Dist}_V(F_1, F_2) )</td>
<td>1/3</td>
</tr>
<tr>
<td>( \text{Dist}_V(F_1, F_3) )</td>
<td>2/3</td>
</tr>
<tr>
<td>( \text{Dist}_V(F_1, F_4) )</td>
<td>2/3</td>
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<tr>
<td>( \text{Dist}_V(F_2, F_3) )</td>
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<td>( \text{Dist}_V(F_2, F_4) )</td>
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</tr>
<tr>
<td>( \text{Dist}_V(F_3, F_4) )</td>
<td>1/3</td>
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</tbody>
</table>

Table 1: \( \text{Dist}_V \) among \( F_1, F_2, F_3, \) and \( F_4 \)

3.2.1 Numeric Examples

Figure 2 shows four fuzzy values, \( F_1, F_2, F_3, \) and \( F_4 \). They are evenly spaced. The membership function of them are shown in the below:

\[
\begin{align*}
\mu_{F_1}(x) &= \begin{cases} 
-2\frac{3}{2} - x + 1 & \text{if } 1 \leq x \leq 2, \\
0 & \text{otherwise},
\end{cases} \\
\mu_{F_2}(x) &= \begin{cases} 
-2\frac{3}{2} - (x - 2) + 1 & \text{if } 3 \leq x \leq 4, \\
0 & \text{otherwise},
\end{cases} \\
\mu_{F_3}(x) &= \begin{cases} 
-2\frac{3}{2} - (x - 4) + 1 & \text{if } 5 \leq x \leq 6, \\
0 & \text{otherwise},
\end{cases} \\
\mu_{F_4}(x) &= \begin{cases} 
-2\frac{3}{2} - (x - 6) + 1 & \text{if } 7 \leq x \leq 8, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

The viewpoint \( V \) is given as the following:

\[
\mu_V(x) = \begin{cases} 
\frac{1}{3} & \text{if } 0 \leq x < \frac{3}{2}, \\
1 & \text{if } \frac{3}{2} \leq x \leq 3, \\
0 & \text{otherwise}.
\end{cases}
\]

The distances between each pair of them are shown in the table 1. If \( V \) is a uniform viewpoint, then the distance is all the same for all pairs, but because the \( V \) gives doubled discrimination on the right half side of the domain, the distance becomes twice bigger for \( F_3 \) and \( F_4 \).

3.2.2 Defuzzification with a viewpoint

Defuzzification is a process of converting a fuzzy set to a crisp number [16]. In fuzzy logic control, defuzzification is necessary because the final control variable is mostly in crisp numbers.

It would be a reasonable property for a pseudo-metric on a set of fuzzy sets that the pseudo-metric gives a metric for crisp subset of the set, i.e., we can always determine a proper distance when there is no uncertainty involved.
Figure 3: Dist\(_v\)(A, c) = 0

For any fuzzy set A in F\(_V\), there exists a unique crisp number, c, which has the same distance from the origin point as A, i.e., Dist\(_v\)(A, c) = 0.

In the figure 3 (a), a uniform viewpoint is used and let \(\mu(x) = k\) where \(x \in \text{Support}(V) = [a, b]\).

From the definition of Dist\(_v\)(A, c) = 0,

\[
\begin{align*}
S(c > V) &= S(A > V) \\
\int_{-\infty}^{\infty} \mu(x)dx &= \int_{-\infty}^{\infty} \mu(x)dx \\
\frac{c-a}{b-a} &= 1 \\
S(A > V) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x)\mu(y)dxdy \\
&= \frac{c-a}{b-a} \int_{-\infty}^{\infty} \mu(y)dy
\end{align*}
\]

The proposed method can be used for areas which require a distance function between fuzzy values. A candidate application is clustering fuzzy values. Further research is required for the use of the proposed method in the clustering of fuzzy values.

References


