A Heat Release Model of Turbulent Premixed Flame Response to Acoustic Perturbations

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Abstract

The unsteady heat release characteristics play a significant role in combustion instabilities often observed in low emissions gas turbine combustors. Such combustion instabilities are often caused by coupling mechanisms between unsteady heat release rates and acoustic perturbations. A generalized model of the flame response to acoustic perturbations, including mixture ratio perturbations, is analytically formulated to extend the prior models by considering a distributed heat release along the mean flame front and using the flame’s kinematic model that incorporates the turbulent flame development. A specific example for the flame transfer function due to flame area fluctuation showed that, when a developing flame speed is used, the transfer function magnitude decreases faster at low Strouhal number and the transfer function phase increases more rapidly, as compared to the results with a constant flame speed. The results from both types of flame speed observe a well known \( n \tau \) model.

Nomenclature

- \( A_f \): flame area
- \( F \): flame transfer function
- \( \Delta h_g \): heat of reaction per unit mass of mixture
- \( l_t \): turbulent length scale
- \( Q \): heat release rate
- \( r \): radial coordinate
- \( r_b, r_t \): radius of the flame base and tip, respectively
- \( S_r \): Strouhal number, defined in Eq. (3)
- \( S_r \): turbulent flame speed
- \( t \): time
- \( T \): turbulent time \( (= l_t / u' \) )
- \( u, v \): axial and radial flow velocity
- \( x \): axial coordinate
- \( <> \): ensemble average
- \( \xi \): axial flame position
- \( \omega \): angular frequency
- \( \phi \): equivalence ratio
- \( \bar{\cdot} \): time average
- \( \gamma \): turbulent random fluctuation
- \( \gamma' \): acoustic oscillation

1. Introduction

Combustion instabilities often cause significant noise
and detrimental structural damages in low emissions, lean premixed gas turbine combustors. These instabilities are in many cases driven by a self-exciting feedback loop between heat release and acoustic perturbations in the combustion chamber, though the general mechanism includes vorticity and entropy modes in addition to aforementioned acoustic mode. Some of the mechanisms associated with instability mechanisms are flame area fluctuations driven by acoustic velocity perturbations [2, 3, 4] and coherent vortical structures convected downstream[5, 6], and reactive mixture composition perturbations [7, 8, 9, 10, 11].

A number of works related to heat release modeling has also been reported as some part of a larger model of the dynamics of unstable combustors to make an enhanced prediction of combustion instabilities. Models incorporating mixture ratio perturbations were developed by some researchers, among whom are Hubbard and Dowling [12], Dowling and Hubbard [13], and Cho and Lieuwen[11], which are based on laminar flame regimes. You et al.[14] incorporated turbulent flame characteristics into a heat release model using a triple decomposition technique and generalized the formulations of heat release response to acoustic perturbations, but the mathematical formulation of their model did not account for a distributed heat release due to a turbulent flame speed that develops along the flame front. Lipatnikov and Sathiah [15] implemented in their model a typical turbulent flame which is developing with the distance from the flame base. Their model is, however, restricted to the heat release response to flame area modulation due to acoustic velocity perturbations, not considering mixture ratio perturbations.

The objective of this study is to provide a generalized formulation of the turbulent flame response to all known flow perturbations with a developing turbulent flame speed being incorporated.

2. Analysis

2.1 Model for flame displacement perturbation

The analysis that follows is an extension of the heat release model based on laminar flame regime reported by Cho and Lieuwen[11] to the model including turbulent flame development [15] by employing a triple decomposition technique following You et al.[14]. The flame geometry in an axisymmetric combustor is shown in Fig. 1. The evolution of the turbulent flame surface is described by G-equation, assuming that chemical reactions are confined in a thin flamelet that separates unburned gases from burned gases. Using a triple decomposition technique that expresses flow variables as the sum of time-averaged, a periodic (coherent), and a random (turbulent) quantities, mean and periodic oscillation of an ensemble-averaged flame front can be derived from G-equation by retaining only linearized terms and neglecting the curvature effect, as demonstrated in detail by You et al. [14].

\[
\bar{\eta} - \nu \frac{d\bar{\xi}}{dr} \bar{\xi} - \frac{S_i}{\bar{r}} \left[ 1 + \left( \frac{d\bar{\xi}}{dr} \right)^2 \right]^{1/2} = 0 \tag{1}
\]

\[
\frac{\partial \xi^*}{\partial t} + \left[ \bar{\nu} + \frac{S_i}{\bar{r}} \right] \frac{\partial \xi^*}{\partial r} \left[ 1 + \left( \frac{d\xi^*}{dr} \right)^2 \right]^{1/2} \frac{\partial \xi^*}{\partial r} = \nu^* - \nu \frac{d\xi^*}{dr} S_i^* \left[ 1 + \left( \frac{d\xi^*}{dr} \right)^2 \right]^{1/2} \tag{2}
\]

Note that the \( S_i^* \) term in the right hand side of Eq. (2) is newly added to the expression of You et al. This additional term results from a slight modification of the periodic oscillation term of the flame front propagation, \( (S_i[\vec{V}G])^p \), to the form of \( S_i(-\vec{n} \cdot \nabla G^p) + S_i^*(-\vec{n} \cdot \nabla \vec{G}) \), the first term of which is related to the periodic oscillation of the flame front, as appears in the expression by You et al., and the second term is newly modeled to account for the periodic oscillation of a turbulent flame speed. This additional term is justified by two reasonings: one is based on the physical observation that the periodic oscillation of the flame front, \( \xi^* \), is determined by the competition between the periodic oscillation of flow velocities, \( \nu^* \) and \( \nu \), and that of a turbulent flame speed, \( S_i^* \). Thus flame kinematics of an oscillating flame front in Eq. (2) may be modeled better by including a turbulent flame speed oscillation in it. The other is based on the fact that the laminar counterpart of Eq. (2) can be obtained by replacing the turbulent flame speed, \( S_i \), with the laminar speed, \( S_i^* \), which is identical to the expression of Cho and Lieuwen[11], Eq. (3) therein). The solution of Eq. (2) is given in the following form with the assumption of harmonic oscillations for the fluctuating terms, \( x^*(r, t) = \text{Re}[x(r) \exp(-i\omega t)] \) with \( x = \{\xi, u, v, S_i\} \).

\[
\xi^*(r) = \int \frac{\beta(\eta) e^{i\beta(\eta)} \left( u^*_i(\eta) - S_i^*(\eta) \right)}{\xi} d\eta \tag{3}
\]

where
The mean flame shape is obtained by solving Eq. (1) for $\xi$ with no slip condition at the flame base, $\xi = 0$, to yield

$$\xi(r) = \frac{r - \rho \sqrt{\Sigma_r}(\rho^2 + \tilde{\Sigma_r}^2)^{1/2}}{\tilde{\Sigma}_r - \rho^2} \, dr$$

The positive or negative sign in front of $\tilde{\Sigma}_r$ represents a diverging or converging flame, respectively. A turbulent flame speed $\tilde{\Sigma}_r$ is modeled following Lipatnikov and Sathiah [15] (Eq. (3) in their publication).

$$\tilde{\Sigma}_r(r(t)) = \tilde{\Sigma}_r(r) = \tilde{\Sigma}_0 \left[ 1 + \frac{T}{\tau} \left( \exp \left( -\frac{t}{\tau} \right) - 1 \right) \right]^{1/2}$$

(5)

where $\tilde{\Sigma}_0$ is a fully developed turbulent flame speed. The time dependence of $\tilde{\Sigma}_r$ in Eq. (5) can be converted into a spatial (radial) dependence, which is required to conduct the integration over radial distance in Eq. (4), using the geometrical relation as depicted in Fig. 1, i.e., $dl = u_r dt$, $u_r = \tilde{u} \cos \theta$, $dl = dr / \sin \theta$, and $\sin \theta = \tilde{\Sigma}_r / \tilde{\Sigma}$, leading to the form

$$\frac{dr}{dt} = h(r, t) \cdot h(r, t) = \tilde{\Sigma}_r(r(t)) \left[ 1 - \left( \tilde{\Sigma}_r(r(t)) / \bar{\Sigma}(r) \right) \right]^{1/2}$$

(6)

where $r$ is a radius of flame front, $r_i$. The above expression is a general formulation as compared to the reduced expression, $dr_i / dt = \tilde{\Sigma}_r(t)$, used by Lipatnikov and Sathiah [15], in the case of $\tilde{\Sigma}_r << \bar{\Sigma}$. Solving the above equation yields the relation between $r_i$ and $t$. If the mean axial flow velocity $\bar{\Sigma}$ is uniform with a negligible mean radial velocity ($\bar{u} = 0$), then the mean shape of a diverging flame with a developing turbulent flame speed from Eq. (5) can be analytically obtained in an explicit form using Eqs. (4) and (6).

$$\tilde{\Sigma}(r(t)) = \tilde{\Sigma}(r) \left[ 1 - \left( \bar{\Sigma} / \tilde{\Sigma} \right)^2 \right]^{1/2} \, (dr / dt) \, dt$$

(7)

$$= \left[ 1 - \left( \tilde{\Sigma}_0 / \bar{\Sigma} \right)^2 \right] \bar{\Sigma} + \left( \tilde{\Sigma}_0 / \bar{\Sigma} \right)^2 \left[ \gamma + \Gamma(0, t / T) + \ln(t / T) \right]$$

where $\gamma = 0.577216$ is the Euler’s constant and the incomplete gamma function is defined as

$$\Gamma(0, x) = \int_0^x e^{-x} \, dx$$

### 2.2 Flame transfer function calculation

The total heat release response of the flame to flow disturbances can be examined by considering its global heat release rate $Q$ as follows.

$$Q(t) = \int_{\tilde{\Sigma}_0} \rho \tilde{\Sigma}_r \Delta h_s \, dA$$

(8)

This overall heat release rate accounts for distributed quantities in density, flame speed, heat of reaction along a flame surface. Note that this integral form of a distributed heat release over the flame surface makes the present analysis different from a lump type model used by You et al. [14] Taking the ensemble average of the above heat release rate eliminates random fluctuation due to turbulence, following You et al. [14].

$$\langle Q(t) \rangle = \int_{\tilde{\Sigma}_0} \rho \langle \Delta h_s \rangle \langle S_r \rangle \, dA$$

(9)

where only linear terms are retained and the definition of turbulent flame speed replaces $\langle S_r \rangle$. Subtracting the time average of Eq. (9) from Eq. (9) and neglecting nonlinear acoustic terms gives the following expression for the total heat release response to flow fluctuations,
normalized by the time-mean heat release rate, 
\[ \bar{Q} = \int_{a_{r=T}} \bar{\rho} \Delta h_{\text{a}} \bar{S}_{r} dA_{r}, \]
as follows.

\[
\frac{\partial^2 \bar{Q}}{\partial t^2} = \int_{a_{r>T}} \left( \rho^s \Delta h_{\text{a}} \bar{S}_{r} + \bar{\rho} (\Delta h_{\text{a}})^s \right) dA_{r} + \int_{a_{r=\partial T}} \bar{\rho} \Delta h_{\text{a}} \bar{S}_{r} dA_{r} \]
\[
= \int \left( \rho^s \Delta h_{\text{a}} \bar{S}_{r} + \bar{\rho} (\Delta h_{\text{a}})^s \right) dA_{r}
\]
which consists of the separate contributions of density, heat of reaction, flame speed, and flame area fluctuations. These fluctuations are associated with fluctuations in pressure, entropy, velocity, and equivalence ratio: The density fluctuation is determined by pressure and entropy fluctuations, 
\[ \rho^s / \bar{\rho} = \rho^s / (\bar{\rho}) - s^s / \bar{S}_{r}, \]
as utilized by You et al. [14]. The fluctuations in heat of reaction and flame speed are expressed in terms of equivalence ratio fluctuation, i.e.,
\[ \Delta (\Delta h_{\text{a}})^s = \frac{d((\Delta h_{\text{a}})^s)}{d\phi} \phi^s, \quad S_{r}^s = \frac{d(S_{r})}{d\phi} \phi^s. \]

Flame surface area is expressed in terms of a flame position, i.e., the mean flame surface area is given by 
\[ dA_{r} = 2\pi \beta(r) dr \]
and the fluctuating surface area is given by 
\[ dA_{r} = \frac{2\pi r \frac{\partial \xi^s}{\partial r}}{\beta(r)} \frac{d\xi^s}{dr} dr. \]
Then Eq. (10) is rearranged to the form

\[
\frac{\partial^2 \bar{Q}}{\partial t^2} = \int_{a_{r>T}} \left( \rho^s \Delta h_{\text{a}} \bar{S}_{r} + \bar{\rho} (\Delta h_{\text{a}})^s \right) dA_{r} + \int_{a_{r=\partial T}} \bar{\rho} \Delta h_{\text{a}} \bar{S}_{r} dA_{r}
\]
\[
+ \int \left( \rho^s \Delta h_{\text{a}} \bar{S}_{r} + \bar{\rho} (\Delta h_{\text{a}})^s \right) dA_{r}
\]

where the gradient of a fluctuating flame front was replaced by the fluctuations in velocities and flame position using Eq. (2), i.e.,
\[ \frac{\partial \xi^s}{\partial r} = \left( u^s_r - S_{r}^s \right) \beta(r) + i\omega \xi^a. \]

The flame front fluctuation \( \xi^a \) is further expressed by the fluctuations in velocity and equivalent ratio as shown in Eq. (3). Direct comparison of Eq. (11) with the expression from Cho and Lieuwen [11](Eq. (14) therein), which is the laminar flame counterpart of Eq. (11), implies that they are identical except that the laminar flame speed is replaced with turbulent one and the density fluctuation term is added to the present analysis. (Note that the two expressions utilize a slight different definition of \( \beta \).)

3. Results and discussion

The mean flame shape calculated from Eq. (7) are depicted in Fig. 2. The values of the turbulent length and velocity scales are: \( l_t = 0.2 h, \; u' = 0.05 \bar{u} \). The flame shape for \( S_{r} = 0.09 \) is more elongated than that for \( S_{r} = 0.3 \) because of a smaller value of \( \bar{S}_{r} = 0.09 \). The solid lines are calculated using Eq. (6) while the circles are approximated by \( dr / dt = \bar{S}_{r} \) for \( \bar{S}_{r} \ll \bar{u} \). They are in a good agreement though the tail part of the flame for \( S_{r} = 0.3 \) shows a slight deviation. Fig. 3 plots the flame front fluctuations with 90° phase difference each.

![Fig. 2 Mean flame shapes with a developing turbulent flame speed. Solid lines : calculated using Eq. (6); Circles : approximated by \( dr / dt = \bar{S}_{r} \).](image)

![Fig. 3 Mean and fluctuating flame shapes for \( S_{r} = 0.3 \).](image)

Though the generalized heat release modeling was formulated in the previous section, the results in this study focus on the heat release response to the flame area fluctuation only and the equivalence ratio fluctuation is neglected. The effect of density fluctuation is also known
to be negligible as long as the square of the mean flow Mach number, \( M^2 \), is small relative to unity. Under these assumptions, the flame transfer function due to flame area perturbation is then obtained by using the flame front fluctuation in Eq. (3) and dividing the resultant expression by a normalized axial velocity perturbation to give

\[
F = \frac{Q'/\bar{Q}}{u'/\bar{u}} \left[ \frac{r \rho \Delta h_s}{f(r)\beta(r)} \frac{d\xi}{dr} \right] 1 + iSr \left[ \frac{\int \frac{\bar{S}_r \bar{a}}{r - r_a} e^{i\phi_T(r,\eta)} d\eta}{\int \frac{\bar{S}_r}{\bar{a}} \beta(r) dr} - 1 \right]
\]

where it is assumed that the axial velocity perturbation \( u' \) is uniform with a negligible radial velocity perturbation. Note in the above equation that the flame transfer function depends mainly on the Strouhal number and the mean flame shape. In fact, if a turbulent flame speed \( \bar{S}_r \) and other mean quantities are assumed to be uniform throughout the combustor with no mean radial flow velocity, then the flame transfer function \( F \) is simplified to yield the following form with \( r_a = a \) and \( r_i = b \).

\[
F = \frac{2}{1 + a/b} \left[ \frac{1 - a/b}{\bar{S}_r^2} \left( e^{i\phi} - 1 \right) \frac{1}{iSr} \left( e^{i\phi} - a/b \right) \right]
\]

which is identical to the formulation by Schuller et al. [4] for a V-flame transfer function with uniform velocity perturbation (Eq. 30 therein). The transfer function in an integral form in Eq. (12) was evaluated by conducting numerical integrations using a trapezoidal rule. This numerical result obtained from a constant flame speed, \( \bar{S}_r \), with uniform mean quantities was compared with the above analytical solution with \( b = 2a \), as shown in Fig. 4 where the two results are in a good agreement. Fig. 5 shows the effect of the turbulent flame development upon the flame transfer function. For low Strouhal numbers (St < 5) a developing flame speed, plotted as solid line, causes transfer function magnitude to decrease faster than a fully developed constant flame speed, plotted as dotted line. This basically arises from the fact that the Strouhal number for a developing flame speed is smaller than that for a fully developed constant flame speed at a given frequency because \( |f|_m \) appearing at the denominator of the Strouhal number takes on a higher value for a developing flame speed. With further increasing Strouhal number, the transfer function magnitudes for the two types of flame speed cross each other repeatedly, as shown in Fig. 5(a). The reduced oscillation observed for a developing flame speed is due to the smoothing effects of integration. The phase of the transfer function for a developing flame speed increases more rapidly than that for a constant flame speed, as shown in Fig. 5(b). The phase of both cases increases linearly, in an averaged sense, with Strouhal number, which implies that their heat release characteristics can be described by a well known \( n-\tau \) model, \( Q'(t) = nu'(t - \tau) \), where \( n \) is an amplification factor and \( \tau \) is the time delay between the velocity disturbance and the corresponding heat release fluctuation.

4. Conclusions
A generalized model of the turbulent flame response
to acoustic perturbations, including mixture ratio perturbations, is analytically formulated by accounting for a distributed heat release along the mean flame front with a kinematic model that incorporates the turbulent premixed flame development. A specific example for the flame transfer function due to flame area fluctuation showed that the transfer function magnitude with a developing flame speed decreases faster at low Strouhal number, as compared to the case with a constant flame speed, and oscillates with smaller oscillation amplitudes. The transfer function phase for a developing flame speed increases more rapidly than that for a constant flame speed. Both cases observe a well known $n$-$\tau$ model. The future study will demonstrate detailed examples of the overall perturbation effect including the mixture ratio upon the flame transfer function.

Fig. 5. The effect of the turbulent flame development upon the flame transfer function. Solid line: a developing flame speed from Eq. (5); dashed line: a constant flame speed $\bar{u}^* = 0.1\bar{u}$, $\bar{s}_0 = 0.09\bar{u}$

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References