Robust Time-Optimal Control for Flexible Structures by Augmented Dynamics

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Abstract — Robust optimal control problem for flexible structures using on-off type discrete actuators has been a subject of intensive research. Optimization by switching time parameterization can be used to minimize the maneuver time subject to equality boundary constraints. Sensitivity of the switching times with respect to modal parameters could be a critical factor degrading the performance of the time optimal solutions. A new approach for the robustness enhancement of switching times with respect to modal parameter uncertainty is introduced. The so-called augmented dynamic model is added to account for the modal uncertainty. The proposed approach turns out to be similar to some previous methods, but provides some new results.

INTRODUCTION

The time optimal control for the attitude maneuver of flexible space structures can be achieved by control input with a maximum level. Some input shaping methods were proposed to reduce residual vibration due to model uncertainty. In particular, for a rest-to-rest maneuver, anti-symmetric control torque profile with symmetric switching times lead to vibration control as well as attitude changes. It is well known that for a linear undamped mechanical system with N modes, there are at most 2N − 1 switching instants for the control of every mode. Various approaches have emerged to find out the switching instants in such a way that the maneuver time is minimized while the induced vibration is effectively suppressed. One of the most popular methods to find the switching times is parameter optimization technique. The switching times are parameterized as unknowns, and then they are computed by an optimization algorithm with equality constraints. The cost function to be optimized is the final maneuver time, while the equality constraints should be satisfied simultaneously. The equality constraints represent conditions for the final attitude by rigid motion and zero residual vibration. In some cases, other forms of cost function such as combined time-fuel optimal control may be employed.

The time optimal control solution as derived by parameter optimization may be subject to robustness with respect to the modal characteristics of the system. Robust approaches have been investigated to overcome the issue of sensitivity of the control command with respect to the modal characteristics in some previous studies. It turns out that adding more switching times than the exact optimal solution can lead to responses being robust about model uncertainties. The number of equality constraints increases due to the extra switching times. Enhanced robustness is achievable by this approach.

In this study, we use the so-called augmented dynamics method to design robust switching profiles for time optimal control solution. In order to overcome the sensitivity problem, a virtual dynamic model with a modal frequency perturbed from the actual value is augmented. The virtual dynamic model does not really exist, but it is primarily used to provide robustness in modal characteristics. The augmented dynamic model also results in increased switching times and corresponding robustness. The new approach is essentially similar to the previous ones where the sensitivity of the modal responses with respect to the natural frequencies is set to zero. In the augmented dynamic model approach, the modal parameter is selected by designer's choice. Hence, the new method leaves some room for shaping responses while still maintaining enhanced robustness by virtue of appropriate selection of the modal parameters. The modal frequencies for the augmented system can be tailored to produce a desired performance. It is also verified in this study that the previous approach of using sensitivity constraint equation could be considered as a special case of the proposed idea in this study.

TIME OPTIMAL CONTROL

First, a linearized second order mechanical system with a rigid-body mode and flexible vibrational modes are described as

\[ M\ddot{q} + Kq = Fu \]  (1)
where \( q = [q_1, q_2, \cdots, q_N]^T \) is a generalized coordinate vector, \( M, K \) are mass and stiffness matrices, and \( F, u \) are input influence and control input, respectively. For simplicity, no damping term is introduced in the governing equations of motion. The control input is assumed to be a single input subject to the following magnitude limitation.

\[
|u(t)| < N \tag{2}
\]

The modal coordinates can be used to replace the governing equation in the modal coordinates(7) form as

\[
\eta_1 + \omega_n^2 \eta_1 = \phi_1 u
\]

\[
\vdots
\]

\[
\eta_N + \omega_n^2 \eta_N = \phi_N u
\]

where \( \eta_i \) usually corresponds to the rigid body mode with \( \omega_n \) equal to zero. For a time-optimal attitude maneuver, in particular, rest-to-rest maneuver for a flexible model with \( N \) modes, the boundary conditions are prescribed as

\[
\eta_i(0) = \dot{\eta}_i(0) = 0, \quad \eta_i(T) = \dot{\eta}_i(T) = 0 \quad \text{at} \quad t = 0
\]

\[
\eta_i(T) = \eta_i, \quad \dot{\eta}_i(T) = 0, \quad \eta_i(T) = \dot{\eta}_i(T) = 0 \quad \text{at} \quad t = T
\]

where \( i = 1, 2, \cdots, N \), \( T \) is the final maneuver time and \( \eta_f \) represents the final target displacement of the rigid body mode.

The performance index for the time-optimal control is defined as

\[
J = \int dt
\]

It is well known that for a flexible structural system with \( N \) modes of motion, the time-optimal rest-to-rest maneuver strategy should be a bang-bang input. The actuator input is saturated at the maximum level with multiple switching times. According to the previous studies, there should be at most \( 2N - 1 \) switching times distributed symmetrically about the half maneuver time. The control input can be represented in terms of step functions such as

\[
\eta_i(t) = \frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2N} (T - t_j)^2 M_j
\]

\[
\eta_i(t) = -\frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2N} M_j \cos \omega_i (t - t_j), i = 2, \cdots, N
\]

where \( t \geq T \). The time-optimal control switching times can be formulated into a parameter optimization problem. The cost function in Eq. (5), the total maneuver time, is minimized subject to the equality constraints. The equality constraints consist of rigid-body mode displacement and zero-residual oscillation for the flexible modes such as

\[
\eta_f - \frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2N} (T - t_j)^2 M_j = 0
\]

\[
-\frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2N} M_j \cos \omega_i (t - t_j) = 0, i = 2, \cdots, N
\]

The second equation of Eq. (9) can be rewritten, by the anti-symmetric input profile, as

\[
\sum_{j=0}^{2N} M_j \cos \omega_i (t - t_j) = 0, i = 2, \cdots, N
\]

A parameter optimization algorithm can be applied to solve the optimization problem for the optimal switching times \( t_i, i = 1, 2, \cdots, 2N - 1 \). There are S/W tools available for the nonlinear parameter optimization problem with equality constraints. The MATLAB Optimization Toolbox could be a candidate for moderate sizes of problems.

**ROBUST TIME-OPTIMAL CONTROL**

It is well known that the time-optimal control solution is dependent upon the modal parameters of the system. This is quite obvious from the equality constraints; a nonlinear function of modal parameters. The switching times are sensitive to the modal parameters as it is shown in the optimization formulation. Thus some robust measure should be taken for parameter insensitive control input design. A promising approach introduced in the previous studies is to establish extra constraints for the purpose of enhancing robustness. For instance, the sensitivity of the modal coordinate responses to the natural frequencies is set to zero.

\[
\frac{\partial \eta_i}{\partial \omega_i} = \frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2N} \cos \omega_i (t_j - t_i) M_j \sin (t_j - t_i)
\]

\[
= 0
\]

The above condition can be rewritten as

\[
\sum_{j=0}^{2N} (t_j - t_i) M_j \sin \omega_i (t_j - t_i) = 0 \quad \text{for} \quad i = 2, \cdots, N
\]

Similarly, high order robustness constraints such as \( \frac{\partial^n \eta_i}{\partial \omega_i^n} = 0, m = 2, 3, \cdots \) can be imposed. This approach is discussed in detail in Ref. 8.
Due to the robustness constraint, the number of constraint equations increases. Hence, the original constraint by the modal coordinate responses are combined with the robustness constraints for complete solution. In order to satisfy the extra constraints, the number of parameters, i.e., switching times should increase in proportion. In Refs. 8 and 9, the number of switching parameters was increased to satisfy additional constraints established by the robustness constraints. If there exist \( r \) constraints for robustness, then the switching times should increase by \( 2r \). Hence, the control input in Eq. (6) with additional constraints needs to modified as
\[
\begin{align*}
\sum_{j=0}^{2(N+r)} u(t) &= \sum_{j=0}^{N} M_j U_j (t - t_j) \\
&= \sum_{j=0}^{N+r} M_j \cos \bar{\alpha}_j (t - t_j) \\
&= \sum_{j=0}^{N+r} M_j \cos \bar{\alpha}_j (t - t_j) + \sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_j (t - t_j)
\end{align*}
\]

The augmented modal coordinate should also satisfy the terminal boundary conditions such as \( \bar{\eta}_k (t) = 0, t \geq T \).

In other words,
\[
\sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_k (t_j - t_k) = 0
\]

The new constraint equation is added to the original constraint equation (Eq. (10)) covering the augmented dynamic mode also. In order to help us to understand the physical meaning of the new constraint by the virtual dynamic mode, let us consider
\[
\bar{\eta}_i (t) = \phi \eta
\]

For instance, consider a \( k \)th flexible mode dynamic described as
\[
\begin{align*}
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_1 u \\
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_2 u
\end{align*}
\]

where \( \bar{\eta}_k \) represents the augmented dynamic mode and \( \bar{\omega}_k \) is the natural frequency perturbed from the original \( \omega_k \). Hence, the total dynamic system model including the virtual dynamic mode is written as
\[
\begin{align*}
\bar{\eta}_i + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_1 u \\
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_2 u \\
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_3 u \\
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_4 u \\
\bar{\eta}_k + \bar{\omega}_k^2 \bar{\eta}_k &= \phi_5 u
\end{align*}
\]

The number of modal coordinates increases due to the augmented mode. It is assumed that \( \bar{\omega}_k = \omega_k \pm \delta \omega \), where the perturbation parameter \( \delta \omega \) needs to be selected by a designer. Now, the control input profile should be modified to accommodate the augmented dynamic mode. Thus the number of switching parameters naturally increases by two for each virtual mode. The state of the virtual dynamics introduce additional constraint given by
\[
\bar{\eta}_k (t) = -\frac{\partial}{\partial t} \int_{-\infty}^{t} \sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_j (t - t_j)
\]

From Eq. (18) and the terminal constraint on \( \eta_k \), one can easily show that Eq. (19) can be simplified into
\[
\sum_{j=1}^{2(N+r)} M_j (t_j - t_k) \sin \omega_k (t_j - t_k) = 0
\]

The new constraint equation essentially is equivalent to the zero derivative condition in Eq. (12). Hence, the augmented dynamic approach may be regarded as generalization for the constraint on the sensitivity given by Eq. (12). Further expansion of Eq. (20) for higher-order terms in \( \delta \omega \) yields
\[
\begin{align*}
\sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_j (t_j - t_k) &= \sum_{j=0}^{N+r} M_j \cos \omega_k (t_j - t_k) + \sum_{j=0}^{2(N+r)} M_j \bar{\omega}_k \cos \omega_k (t_j - t_k) \\
\sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_j (t_j - t_k) &= \sum_{j=0}^{N+r} M_j \cos \omega_k (t_j - t_k) + \sum_{j=0}^{2(N+r)} M_j \bar{\omega}_k \cos \omega_k (t_j - t_k)
\end{align*}
\]

The number of switching parameters was increased to satisfy additional constraints established by the robustness constraints. If there exist \( r \) constraints for robustness, then the switching times should increase by \( 2r \). Hence, the control input in Eq. (6) with additional constraints needs to modified as
\[
\begin{align*}
\sum_{j=0}^{2(N+r)} u(t) &= \sum_{j=0}^{N} M_j U_j (t - t_j) \\
&= \sum_{j=0}^{N+r} M_j \cos \bar{\alpha}_j (t - t_j) \\
&= \sum_{j=0}^{N+r} M_j \cos \bar{\alpha}_j (t - t_j) + \sum_{j=0}^{2(N+r)} M_j \cos \bar{\alpha}_j (t - t_j)
\end{align*}
\]
augmented modes. Let us introduce the following notations for the uncertain model parameter.

\[ c^i_j = \omega_k(t_j - t_h) \]  

It can be rewritten in a matrix form as

\[
\begin{bmatrix}
\omega_k \\
\omega_k
\end{bmatrix}
\begin{bmatrix}
t_j - t_h \\
t_j - t_h
\end{bmatrix} = \begin{bmatrix}
c^i_j \\
c^i_j
\end{bmatrix}
\]

One can see that the above form is equivalent to a least-square problem. The switching times are sought in a way that the parameter perturbation in the augmented mode is computed in the least-square sense.

The proposed analysis could be used to claim the robustness of the sufficient condition. The modal coordinate equation in Eq. (3) can be cast into a state space form.

\[ \dot{x} = Ax + Bu \]

where the state vector \( x \) consists of modal coordinates, as \( x = [q_1, q_2, \ldots, q_N] \) and the system matrices are given by

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
-\omega_1 & -\omega_2 & \cdots & -\omega_N
\end{bmatrix},

B = \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_N
\end{bmatrix}
\]

For the given cost function in Eq. (5) and the state equation in Eq. (23), the Hamiltonian of the system can be expressed as

\[ H = 1 + \frac{1}{2} \dot{\lambda}^T(Ax + Bu) \]

where \( \lambda \) represents a costate vector with components given as \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_N] \). The costate vector satisfies the following differential equation

\[ \dot{\lambda} = -A^T \lambda \]

One can also see that the Hamiltonian is not an explicit function of time, hence \( H(t) = 0 \) should hold over the optimal trajectory. The Pontryagin’s maximum principle with the Hamiltonian produces the following control law

\[ u(t) = -N \text{sgn}[B^T \lambda(t)] \]

where \( \text{sgn}(x) \) function represents a signum function with \( \text{sgn}(x) = 1 \) if \( x > 0 \) and \( \text{sgn}(x) = -1 \) if \( x < 0 \). Each component of the costate vector satisfies the following differential equations

\[ \dot{\lambda}_i = 0, \quad \dot{\lambda}_i = -\lambda_i, \quad \dot{\lambda}_i = \lambda_i, \quad \dot{\lambda}_i = -\omega_i^2 \lambda_i \]

for \( i = 2, 3, \ldots, N \). It was shown that the costate vector at the half-maneuver time is prescribed as

\[ \lambda(t_h) = [\lambda_1(t_h), 0, \lambda_2(t_h), 0, \ldots, \lambda_N(t_h), 0]^T \]  

The solution to the differential equations for the costate, therefore, becomes

\[ \lambda_i(t) = \lambda_i(t_h), \quad \lambda_i(t) = -(t - t_h) \lambda_i(t_h), \quad \lambda_i(t) = \lambda_i(t_h) \cos \omega_i(t - t_h), \quad \lambda_i(t) = -\lambda_i(t_h) \sin \omega_i(t - t_h) \]

for \( i = 2, 3, \ldots, N \). From the costate vector solution, the switching function of the system is defined from Eq. (30) as follows

\[ \xi(t) = \lambda^T(t)B \]

By using the costate equation solution, the switching function can be written as

\[ \xi(t) = \phi_1 \lambda_1(t_h)(t - t_h) + \sum_{i=2}^{N} \phi_i \lambda_i(t_h) \sin \omega_i(t - t_h) \]

For sufficient condition of optimality, the switching function should be equal to zero for \( t = t_1, t_2, \ldots, t_N \).

Now it is assumed that the virtual dynamic model is added for the \( k \)th modal coordinate. An modal coordinate \( (e_k) \) is added and corresponding change is made in the state and costate equations. Thus, the new switching function including the \( k \)th virtual dynamic mode is given by

\[ \xi(t) = \phi_1 \lambda_1(t_h)(t - t_h) + \sum_{i=2}^{N} \phi_i \lambda_i(t_h) \sin \omega_i(t - t_h) \]

\[ + \phi_k \lambda_k(t_h) \sin \omega_k(t - t_h) \]

where \( \lambda_k(t_h) = \omega_k \pm \delta \omega \) represents the perturbed natural frequency of the \( k \)th flexible mode and \( \lambda_k(t_h) \) is a costate.

The new switching function should also satisfy

\[ \xi(t) = 0 \quad \text{for} \quad t = t_{k+1}, t_{k+2}, \ldots, t_{k+N} \]

together with

\[ 1 + \phi_k \lambda_k(t_h)(t - t_h) + \sum_{i=2}^{N} \phi_i \lambda_i(t_h) \sin \omega_i(t - t_h) = 0 \]

Eqs. (36) and (37) can be used to compute \( \lambda_1(t_k), \lambda_2(t_k), \ldots, \lambda_N(t_k) \) and \( \lambda_k(t_k) \). The sufficient condition for the optimality thus can be proved by examining the new shaping function. The augmented dynamic method thus leads to the sufficient condition of optimality. It should be noted that in the previous
approaches where the sufficient condition was not dealt with. This is because extra switching times were just introduced to satisfy necessary boundary and robust constraints. Hence, the feedback control law becomes
\[ u(t) = -N \text{sgn}[\hat{B}^T \dot{x}(t)] \] (39)
where \( \hat{B} \) is modified from the original \( B \) to accommodate the new model (\( \bar{\eta}, \)).

SIMULATION RESULTS
Sample simulation results are conducted for a two-mass model connected by a spring as Fig. 1.

![Two-mass spring model](image)

Fig. 1 A two-mass spring model

This model has been used in Ref. 8, and the same model is adopted in this study for benchmark test. The governing equations of motion described by displacement of each mass is
\[ m_i \ddot{x}_i + k(x_i - x_j) = u \]
\[ m_j \ddot{x}_j - k(x_i - x_j) = 0 \] (35)
The mass and stiffness properties are given as \( m_1 = m_2 = 1 \text{kg}, \) and \( k = 1 \text{N/m}, \) respectively. The rest-to-rest maneuver is specified by the following boundary conditions
\[ x_i(0) = x_i(T) = 0, \quad x_j(0) = x_j(T) = 1 \]
The modal coordinate equations are written as
\[ \ddot{\bar{\eta}} + \omega^2 \bar{\eta} = \Phi u \] (36)
where it can be easily computed as \( \omega_1 = 0, \)
\[ \omega_2 = \sqrt{2k/m} = \sqrt{2} \text{ rad/sec} \] There are two modes of motion including the rigid-body motion. The number of switching times for the normal minimum-time rest-to-rest maneuver is therefore equal to three as illustrated in Fig. 2.

![Control input profile for the example model](image)

The symmetric property for the switching times can be used to reduce the parameter to two. The equality constraints by the boundary conditions for the modal coordinates are prescribed by
\[ 2 + 2t_1^2 + t_2^2 - 4t_1t_2 = 0 \] (37)
\[ 1 - 2\cos \omega_2 (t_2 - t_1) + \cos \omega_2 t_2 = 0 \] (38)
The time optimal control solution for the total maneuver time with \( 2 \tau \) can be solved in conjunction with the inequality constraints by using MATLAB optimization Toolbox. For the given parameters, it can be shown that the solution is, exactly same as Ref. 8.

Simulation results for the displacement \( x_2 \) of the second mass by the control input with optimized switching times are presented in Fig. 3. Different values of the material parameter \( k \) are tested to examine the robustness of the system response. As it can be shown, the system response is dependent upon \( k \) to a significant extent. In fact, Fig. 3 looks almost identical to that of Ref. 8 for benchmarking purpose.

![Simulation results with optimized switching times](image)

As we can see, the response satisfies exactly the time-optimal rest-to-rest maneuver conditions with a perfect model parameter information. With model parameter change, residual oscillations are induced. With significant model error, large amount of residual error energy results. In fact, this result is identical to that in Ref. 8.

Now we test the robust control approach by assuming uncertainty in the natural frequency \( \bar{\omega}_2 \). The virtual model approach is taken again with the augmented modal parameter \( \tilde{\omega}_2 \) introduced. The number of switching times increased by two, so it is equal to five as Fig. 4.
Again, symmetric property for the switching times reduces the number of switching parameters to three. Constraint equation including the virtual dynamic mode

\[
2 + 2t_2^2 - 2t_1^2 - t_2^2 - 4t_3 = 0
\]

\[
\cos \omega t_3 - 2 \cos \omega (t_1 - t_2) + 2 \cos \omega_1 (t_1 - t_2) - 1 = 0
\]

The MATLAB Optimization Toolbox is used to solve the optimization problem. Several values of the model parameter \( k \) are also used for simulation. The natural frequency of the augmented dynamic is set to \( \tilde{\omega}_2 = \sqrt{1.8} \) rad/sec to check the performance of the proposed method. The switching times obtained are used to plot the responses of modal coordinates. Simulation results are presented in Fig. 5.

As it can be shown in Fig. 5, the system response looks more robust than Fig. 3 with the help of the augmented dynamic mode. The perturbation of the modal parameter for the augmented dynamic mode is made in the direction consistent with change in \( k \). Next, we select the modal parameter perturbation in the direction opposite to the change in \( k \). Hence, it is selected as \( \tilde{\omega}_2 = \sqrt{2.2} \) rad/sec. Switching times are obtained again by optimization, and simulation using the optimized switching times are displayed in Fig. 6.

Fig. 6 Simulation results by using robust switching times \( \tilde{\omega}_2 = \sqrt{2.2} \) rad/sec

The residual oscillations in Fig. 6 is slightly greater than those of Fig. 5. This is because the modal parameter perturbation is assumed in the wrong direction. Thus, it may be important to decide the modal parameter perturbation consistent with actual parameter. However, still noticeable improvement in the transient response in Fig. 6 over Fig. 3 is observed. Thus modal parameter perturbation method shows inherent robustness.

Next, we assume that the modal parameter perturbation is exactly matched with the worst case uncertain model parameter \( k = 0.7 \). Thus, the perturbed natural frequency of the second mode is selected as \( \tilde{\omega}_2 = \sqrt{2k / m} = \sqrt{1.4} \) rad/sec. Simulation result based upon the optimization result by taking the new \( \tilde{\omega}_2 \) is presented in Fig. 7.

Fig. 7 Simulation results by the perturbed modal parameter exactly matching the worst case model uncertainty

The transient response in Fig. 7 is quite small, and it is because of the modal perturbation parameter \( \tilde{\omega}_2 \) exactly matching the worst case model parameter change.
In the simulation results, robustness of the response with respect to the model parameter change has improved significantly by augmented dynamic. Hence, a proper choice for the perturbed modal parameter (\( \tilde{\omega}_j \)) can lead to satisfactory performance in the presence of significant model error. Theoretically, if one choose \( \tilde{\omega}_j \) in such a way as to exactly match the parameter \( k \), then the best performance is assured. It should be noted only the first order condition is considered subject to a small perturbation \( \Delta \omega \). For a large value in \( \Delta \omega \), high order constraints in Eq. (23) should be imposed.

**CONCLUDING REMARKS**

Augmenting a virtual dynamic mode can increase robustness arising from uncertain model parameter. It turns out that the new method leads to results similar to the previous approaches with constraint equations for output sensitivity with respect to model parameter. In the new method, one can choose the perturbed model parameter like a design parameter. The augmented dynamic approach results in robust output response over a moderate range of the design parameter. More design capacity is provided, therefore, by the proposed approach for robust performance of flexible spacecraft maneuver and vibration control.

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