A Fast Computational Method for Minimum Square Error Transform

Seongwhan Lee  Jin H. Kim  Frans C.A. Groen

Department of Computer Science  Department of Computer Systems
Korea Advanced Institute of Science and Technology  University of Amsterdam
P.O.Box 150, Cheongryang  P.O. Box 41882, 1009 DB Amsterdam
Seoul 130-650, Korea  The Netherlands

ABSTRACT

In this paper, we describe a basic minimum square error transform for point pattern matching and propose a fast computational method for minimum square error transform. The computational analysis revealed that the proposed method is faster than that of Groen et al. [3].

INTRODUCTION

Image matching is important in many applications, including registration, navigation, change detection, and stereo-mapping. The brute-force approach to image matching involves correlation, performed either in the space or spatial frequency domain; in either case, the cost of matching grows with the image area [1]. A more economical approach is to extract a discrete set of feature points from the two images that are to be matched, and match the resulting two point patterns. The cost of this process grows with the number of feature points rather than with the image area. In recent years, many researchers [1,2,3,4,5] have directed considerable attention to the problems of point pattern matching. In this paper, we describe the basic minimum square error (MSE) transform for point pattern matching and propose a fast computational method for MSE transform of Ref. [3].

We can say that two point patterns are equivalent when there exists an appropriate functional relation between the two coordinate systems

\[ x = f(r, s) \]  
\[ y = g(r, s) \]

which is satisfied by all point pairs of the two point patterns. In practical situations, however, two point patterns are seldom equivalent in the strict sense stated above. Only an approximate relation between the two point patterns exists, and it can be determined by finding a transform which minimizes some error criterion. The approximate relations \( f \) and \( g \) between the two points when there are \( n \) points in each point pattern are determined in a minimum-squares sense; i.e., by minimizing the residual error

\[ E = \sum w_i [(x_i - f(r_i, s_i))^2 + (y_i - g(r_i, s_i))^2] \]  

where \( w_i \) is a coefficient giving the possibility to weight the error with the likelihood of the match \((w_i > 0, \sum w_i = 1)\). When equal weights are applied, \( w_i = 1/n \).

EXISTING MSE TRANSFORM

Groen et al. [3] restricted \( f \) and \( g \) to be transforms which are defined by

\[ f(r_i, s_i) = s \cos \theta (r_i - r_0) + s \sin \theta (s_i - s_0) \]  
\[ g(r_i, s_i) = -s \sin \theta (r_i - r_0) + s \cos \theta (s_i - s_0) \]

where \( r_0, s_0 \) is the translation, \( \theta \) is the rotation, and \( s \) is the scale.

They defined:

\[ c_x = \sum w_i x_i \]  
\[ c_y = \sum w_i y_i \]  
\[ c_r = \sum w_i r_i \]  
\[ c_s = \sum w_i s_i \]
By setting the partial derivatives of E to zero, they obtained:

$$\frac{\partial E}{\partial \theta_0} = 0 :$$
$$r_0 = c_x - c_s / s \cos \theta + c_y / s \sin \theta$$

$$\frac{\partial E}{\partial s_0} = 0 :$$
$$s_0 = c_x - c_s / s \cos \theta + c_y / s \sin \theta$$

$$\frac{\partial E}{\partial \theta} = 0 :$$
$$\sin \theta (c_{ys} + c_{xr} - s_0 c_y - r_0 c_x) = \cos \theta (c_{xs} - c_{yr} - s_0 c_x + r_0 c_y)$$

From these equations the transform parameters can be calculated:

rotation:
$$\tan \theta = \frac{(c_{xs} - c_{yr} - c_x c_s + c_y c_r)}{(c_{xr} + c_{ys} - c_{sr} - c_x c_y + c_y c_x)}$$

scale:
$$s = \frac{(c_{xs} + c_{ys} - c_x c_s + c_y c_r) \cos \theta + (c_{xs} - c_y c_s + c_x c_r) \sin \theta}{(c_{sr} + c_{ys} - c_x c_y - c_y c_x)}$$

translation:
$$r_0 = c_x - c_s / s \cos \theta + c_y / s \sin \theta$$
$$s_0 = c_x - c_s / s \cos \theta + c_y / s \sin \theta$$

The transform error E equals:
$$E = c_{xs} c_{ys} c_x c_y c_s c_y s^2 (c_{sr} c_{ys} - c_x c_y c_r c_s)$$

**FAST COMPUTATIONAL METHOD FOR MSE TRANSFORM**

If we define $a_0$, $a_1$, $b_0$, and $b_1$ appropriately, Eq. (3a) and Eq. (3b) can be rewritten as:

$$f(r, s) = a_0 + a_1 r - b_1 s$$
$$g(r, s) = b_0 + b_1 r + a_1 s$$

By substituting $f$ and $g$ defined in Eq. (3a) and Eq. (3b) into Eq. (2), we get the residual error:

$$E = \sum w_i [x_i - (a_0 + a_1 r_i - b_1 s_i)]^2 + [y_i - (b_0 + b_1 r_i + a_1 s_i)]^2$$

In order to minimize $E$, we find the partial derivatives of $E$ with respect to $a_k$ and $b_k$, and set them equal to zero,

$$\frac{\partial E}{\partial a_k} = 0$$
$$\frac{\partial E}{\partial b_k} = 0$$

for $k = 0, 1$. Thus from Eq. (5a) we obtain:

$$\Sigma [x_i - (a_0 + a_1 r_i - b_1 s_i)] = 0$$
$$\Sigma [r_i [x_i - (a_0 + a_1 r_i)] + s_i [y_i - (b_0 + b_1 r_i)]] = 0$$

Similarly, we get from Eq. (5b):

$$\Sigma [y_i - (b_0 + b_1 r_i + a_1 s_i)] = 0$$
$$\Sigma [s_i [x_i - (a_0 + a_1 r_i)] + r_i [y_i - (b_0 + b_1 r_i)]] = 0$$

Then Eqs. (6a), (6b), (6c) and (6d) can be put into the following classical linear system of equations:

$$\begin{bmatrix}
    a_0 \\
    b_0 \\
    a_1 \\
    b_1
\end{bmatrix} =
\begin{bmatrix}
    \sum_i x_i^2 & \sum_i x_i y_i \\
    \sum_i x_i y_i & \sum_i y_i^2
\end{bmatrix}^{-1}
\begin{bmatrix}
    \sum_i x_i y_i \\
    \sum_i y_i^2
\end{bmatrix}$$

or, more compactly:

$$A = M^{-1}X$$

where $A$, $M$, and $X$ are defined appropriately.
The transform error $E$ equals:

$$E = \Sigma x_i^2 + \Sigma y_i^2 - A'X$$  \hspace{1cm} (9)$$

Parameters for translation $(t_0, s_0)$, rotation $(\theta)$, and scale $(s)$ can be determined as follows:

$$\theta = \tan^{-1}(-b_1, a_1)$$

$$s = \frac{a_1}{\cos \theta}$$

$$t_0 = \frac{(b_0 \sin \theta - a_0 \cos \theta)}{s}$$

$$s_0 = \frac{-(a_0 \sin \theta - b_0 \cos \theta)}{s}$$  \hspace{1cm} (10a, 10b, 10c, 10d)$$

Note that this MSE transform is much more efficient than the MSE transform proposed by Groen et al. [3]. The matrix $M$ in this transform depends only on the points extracted from the model point pattern. Therefore, $M^{-1}$ needs to be evaluated only once for each model point pattern. (A similar type of saving in computation time can be made in the case where a single observed point pattern is compared to many model point patterns.) The computational analysis of this fact will be given in the next section.

**COMPUTATIONAL ANALYSIS AND CONCLUSION**

Consider the MSE transform between $n$ pairs of corresponding points. Then, the MSE transform of Groen et al. [3] needs:

- $C_{x1}, \ldots, C_{x8}$: $8n$ multiplications and $12n$ additions
- $\theta$: $1$ multiplication, $6$ additions and $1 \tan^{-1}$
- $s$: $9$ multiplications, $10$ additions and $2 \sin / \cos$
- $E$: $1$ multiplication, $7$ additions

Therefore, in total, $8n+11$ multiplications, $12n+23$ additions and $3$ trigonometric functions are needed for $n$ pairs of corresponding points.

Our MSE transform needs:

- $X$: $4n$ multiplications and $6n$ additions
- $M^{-1}X$: $16$ multiplications and $12$ additions
- $\Sigma x_i^2 + \Sigma y_i^2$: $2n$ multiplications and $2n+1$ additions
- $AX$: $16$ multiplications and $12$ additions

Therefore, in total, $6n+32$ multiplications, $8n+25$ additions and no trigonometric functions are needed for $n$ pairs of corresponding points.

The speedup factor of our transform to that of Groen et al. [3] is as follows:

speedup factor for additions = $(12n +23) / (8n+25)$
speedup factor for multiplications = $(8n+11) / (6n+32)$

Currently, the proposed method is being used successfully in other research projects [6, 7]. The computational analysis revealed that the proposed method is faster than that of Groen et al. [3].

**REFERENCES**


