I. INTRODUCTION

Information processing with quantum systems enables what seems to be impossible with its classical counterpart [1–5].

Quantum key distribution (QKD) [6–9] is one of the most interesting and important quantum information processing protocols. QKD seems to have become the first practical quantum information processor [10]. Although security of the Bennett-Brassard 1984 (BB84) QKD [6] had been widely conjectured based on the no-cloning theorem [11,12], it is quite recently that its unconditional security was shown [13–15]. In particular, Shor and Preskill [15] showed the security of BB84 scheme by elegantly using the connections among several basic ideas in quantum information processing protocols, i.e., quantum error correcting codes (QECCs) [16,17] and entanglement purification [18].

A. Long code problem and concatenated BB84 protocol

The Shor-Preskill proof affords security for a certain type of BB84 protocol [15] where we perform error correction and privacy amplification by a classical code associated with Calderbank-Shor-Steane (CSS) quantum code [16,17,19]. For a chosen code and a given error rate of the communication channels, Alice (the sender) and Bob (the receiver) can calculate Eve’s (eavesdropper’s) information on the final key. What Alice and Bob would want is to make Eve’s information on the whole final key less than what they regard as negligible, for example, 1 bit. This can be done by choosing an appropriate code for the given error rate.

However, the Shor-Preskill proof does not place any bounds on the amount of effort Alice and Bob must put in decoding the classical error-correcting code associated with the CSS code [19]. In particular, when the code length is large, the decoding is not so simple [15]. One might say that we need not worry about the problem so much because we can do it by a code with a moderate length. However, we can easily see that it is risky. Let us consider a case where the number of bits is \( n \) and the error rate is \( r(0 \leq r \leq 1) \). Then the probability distribution of the error rate will be peaked at \( r \) with standard deviation \( \sigma = \sqrt{r(1-r)/n} \). When the number of bits \( n \) is one thousand and the error rate \( r \) is 10\%, the standard deviation is about 0.949\%. This means that the probability that the real error rate will be more than 12.4\% \((= r + 2.57\sigma)\) is 1\%. This implies that if they had assumed that a channel with 10\% error rate is below a threshold, say 12.4\%, for the one thousand bits, then the probability that Alice and Bob will be cheated is 1\%. Note that this is too high a risk for a cryptographic task where we must achieve an exponentially small probability to be cheated. Thus, they have to permit a large room, say 20\(\sigma\), between the error rates of the channel and the threshold. Then the threshold must be larger than 29.0\% in this case. However, no code is found to work yet if it exceeds 26.4\% [20]. Therefore, the number of bits \( n \) must be large, say 10\(^6\), in order to obtain the exponential security. However, as noted previously, decoding such a long code might be a difficult task [15].

On the other hand, concatenation is a very useful and interesting idea in both classical and quantum error corrections [21]. In this method, already-encoded bits are used as unit bits to encode new bits. The iterative processes can be repeated as we like. The more they perform the concatenation processes, the shorter the encoded bits become exponentially. Since the concatenated codes are not included in the original codes, it is nontrivial to adopt the concatenated codes for certain tasks. For example, it is by using concatenated coding that the fault tolerant quantum computation becomes possible [22].

In this paper, we show that the long code problem can be avoided by the idea of using concatenated BB84 protocol. First, Alice and Bob generate raw keys. Next, using a classical code associated with CSS code they perform error correction and privacy amplification on the raw keys, as prescribed in Ref. [15]. As a result they get the first key. The first key is not so much secure (possibly due to the long code...
problem). They perform the error correction and privacy amplification again in the same way on the first key and then they obtain the second key. They repeat this process until the estimated leaked information on the final key is negligible.

In Sec. II, first we give the concatenated modified Lo-Chau protocol. This protocol reduces to the concatenated CSS code protocol that reduces to the concatenated BB84 protocol. In Sec. III, we discuss the security of the concatenated schemes and how the concatenated protocol solves the long code problem. In Sec. IV, we give discussion and conclusion.

B. Notation

In this paper, we use mostly the notations used in Refs. [15,23].

The canonical basis of a qubit consists of $|0\rangle$ and $|1\rangle$. We define another basis as follows. $|\bar{0}\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|\bar{1}\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$. $H$ is the Hadamard transform. This transformation interchanges the bases $|0\rangle$, $|1\rangle$ and $|\bar{0}\rangle$, $|\bar{1}\rangle$. $I$ is the identity operator and $\sigma_x$, $\sigma_y$, $\sigma_z$ are the Pauli operators. $\sigma_{a(i)}$ denotes the Pauli operator $\sigma_a$ acting on the $i$th qubit, where $a = 0, x, y, z$. For a binary vector $s$, we let $\sigma_a^{[s]} = \sigma_{a(1)}^{s_1} \sigma_{a(2)}^{s_2} \cdots \sigma_{a(n)}^{s_n}$, where $s_i$ is the $i$th bit of $s$ and $\sigma_0^a = I$, $\sigma_1^a = \sigma_a$.

The Bell basis are the four maximally entangled states, $|\Psi^+\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$ and $|\Phi^\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$.

Let us consider two classical binary codes $C_1$ and $C_2$ such that $\{0\} \subset C_2 \subset C_1 \subset F_2^n$. If $C_1$ is the binary vector space of $n$ bits. A set of basis for the CSS code can be obtained from vectors $v \in C_1$ as follows: $v \rightarrow (1/|C_2|^{1/2}) \sum_{w \in C_2} |v+w\rangle$. Note that $v_1$ and $v_2$ give the same vector if $v_1-v_2 \in C_2$. $H_1$ is the parity check matrix for the code $C_1$ and $H_2$ is that for $C_2^\perp$, the dual of $C_2$. $Q_{2^n, n}$ is a class of QECCs. For $v \in C_1$, the corresponding code word is $v \rightarrow (1/|C_2|^{1/2}) \sum_{w \in C_2} (-1)^{s \cdot w} |x + v + w\rangle$.

II. CONCATENATED BB84 PROTOCOL

First, we give the concatenated modified Lo-Chau protocol. This scheme reduces to the concatenated CSS codes protocol that reduces to the concatenated BB84 protocol.

We consider a doubly concatenated scheme of the $[[n_1, k_1, d_1]]$ and $[[n_2, k_2, d_2]]$ CSS codes. It is clear that it can be generalized to multiple concatenation in the same way.

A. Protocol A: Concatenated modified Lo-Chau protocol

1. The first stage

(1) Alice creates $2n_1n_2$ Einstein-Podolsky-Rosen (EPR) pairs in the state $|\Phi^+\rangle^\otimes(2n_1n_2)$.

(2) Alice selects a random $(2n_1n_2)$-bit string $b$. She performs a Hadamard operation on second half of each EPR pair for which the component of $b$ is 1.

(3) Alice sends the second half of each EPR pair to Bob.

(4) Bob receives the qubits and publically announces this fact.

(5) Alice announces the bit string $b$.

(6) Bob undoes the Hadamard operations in step (2).

(7) Alice selects randomly $n_1n_2$ of the $2n_1n_2$ EPR pairs to serve as check bits.

(8) Alice and Bob each measure their halves of the $n_1n_2$ check EPR pairs in the $\{0\}, \{1\}$ basis and share the results. If too many of these measurements disagree, they abort the scheme.

(9) Alice randomly chooses $n_1$ unencoded qubits. She repeats this until all code bits are used. As a result, she gets $n_2$ sets of code bits whose number of elements is all $n_1$. She announces her choices to Bob.

(10) For each set, they make the measurements of $\sigma_z^{[r]}$ for each row $r \in H_1$ and $\sigma_x^{[r]}$ for each row $r \in H_2$ of the $[[n_1, k_1, d_1]]$ CSS code. Alice and Bob share the results, compute the syndromes for bit and phase flips, and then transforms their state so as to obtain $k_1k_2$ once-encoded high-fidelity EPR pairs.

In the next stage, Alice and Bob do essentially the same operations on the $k_1k_2$ once-encoded EPR pairs, as they have done on unencoded EPR pairs in the previous stage.

2. The second stage

(1) Alice randomly chooses $n_2$ once-encoded qubits. She repeats this until all code bits are used. As a result, there are $k_1$ sets whose number of elements is $n_2$. She announces her choice to Bob.

(2) For each set, Alice and Bob make the measurements of $\sigma_z^{[r]}$ for each row $r \in H_1$ and $\sigma_x^{[r]}$ for each row $r \in H_2$ of the $[[n_2, k_2, d_2]]$ CSS code. They share the results, compute the syndromes for bit and phase flips, and then transforms their state so as to obtain $k_1k_2$ doubly encoded EPR pairs.

(3) Alice and Bob measure the EPR pairs in the doubly encoded $\{0\}, \{1\}$ basis to obtain $k_1k_2$-bit final key.

Let us now consider reduction of the protocol. The same arguments used in Ref. [15] apply here. The difference is that they measure the syndrome twice at both levels of concatenation, that is, unencoded qubits and once-encoded qubits.

B. Protocol B: Concatenated CSS code protocol

1. The first stage

(1) Alice creates $n_1n_2$ random check bits and a random $(2n_1n_2)$-bit string $b$.

(2) Alice chooses $n_2$ $n_1$-bit string $x_i$ and $z_i$ at random ($i = 1, 2, \ldots, n_2$).

(3) Alice prepares $k_1n_2$ once-encoded bits in a state $|00 \ldots 0\rangle$ using $[[n_1, k_1, d_1]]$ CSS code $Q_{x_i, z_i}$.

2. The second stage

(1) Alice creates a random $k_1k_2$-bit key string $k'$.

(2) Alice chooses $k_1$ $n_2$-bit string $x'_i$ and $z'_i$ at random ($i = 1, 2, \ldots, k_1$).

(3) Alice encodes her key $|k'\rangle$ using the once-encoded qubits prepared in step (3) of the first stage with the
$[[n_2,k_2,d_2]]$ CSS code $Q_{x_j',z_j'}$.

(4) Alice randomly chooses $n_1n_2$ positions out of $2n_1n_2$ and puts the unencoded check bits in these positions. She randomly permutes the qubits of the doubly encoded code bits of the previous step. She puts them in the remaining positions.

(5) Alice performs the Hadamard operation on each qubit for which the component of $b$ is 1.

(6) Alice sends the resulting state to Bob. Bob acknowledges the receipt of the qubits.

(7) Alice announces the string $b$, the positions and values of unencoded check bits, the random permutation, and each $x_i$, $z_i$ and $x_j'$, $z_j'$.

(8) Bob undoes the Hadamard operation and random permutation.

(9) Bob measures unencoded check bits in the $\{|0\rangle,|1\rangle\}$ basis and announces the results to Alice. If too many of these measurements disagree, they abort the scheme.

(10) Bob measures the qubits in the doubly encoded $\{|0\rangle,|1\rangle\}$ basis to obtain $k_1k_2$-bit final key with near-perfect security.

Let us consider reduction of the concatenated CSS code protocol to the concatenated BB84 protocol. We can apply the same arguments used in Ref. [15]. The difference is that reduction processes involved in Eq. (4) in Ref. [15] are performed twice at both levels of concatenation.

C. Protocol C: Concatenated BB84 protocol

1. The first stage

(1) Alice creates $4n_1n_2(1+\delta)$ random bits.

(2) Alice chooses a random $4n_1n_2(1+\delta)$-bit string $b$. For each bit, she creates a state in the $\{|0\rangle,|1\rangle\}$ and $\{|0\rangle, |\bar{1}\rangle\}$ basis, if the corresponding components of the bit $b$ are 0 and 1, respectively.

(3) Alice sends the resulting qubits to Bob.

(4) Bob receives the $4n_1n_2(1+\delta)$ qubits, measuring each in the $\{|0\rangle,|1\rangle\}$ and $\{|0\rangle, |\bar{1}\rangle\}$ basis at random.

(5) Alice announces $b$.

(6) Bob discards any results where he measured a different basis than Alice prepared. With high probability, there are at least $2n_1n_2$ bits left. Alice decides randomly on a set of $2n_1n_2$ bits to use for the protocol and chooses at random $n_1n_2$ of these to be check bits.

(7) Alice and Bob announce the values of their check bits. If very few of these values agree, they abort the protocol.

(8) Alice announces $u+v$, where $u$ is a string consisting of randomly chosen code bits, and $v$ is a random code word in $C_1$ of the $[[n_1,k_1,d_1]]$ code. Alice announces the $n_1$ positions of the randomly chosen code bits.

(9) Repeat the previous step $n_2$ times, until all code bits are consumed.

(10) Bob subtracts each $u+v$ from each of his code bits, $u+v$, and corrects the result, $u+\epsilon$, to a code word in $C_1$ of the $[[n_1,k_1,d_1]]$ code.

(11) Alice and Bob use the coset of each $u+C_2$ as the key. In this way, they obtain $k_1n_2$-bit string.

2. The second stage

(1) Alice announces $u+v$, where $v$ is a string consisting of randomly chosen code bits, and $u$ is a random code word in $C_2$ of the $[[n_2,k_2,d_2]]$ code. Alice announces the $n_2$ positions of the randomly chosen code bits.

(2) Repeat the previous step at $k_1$ times, until all code bits are consumed.

(3) Bob subtracts each $u+v$ from his each code qubits, $v+\epsilon$, and corrects the result, $u+\epsilon$, to a code word in $C_1$ of the $[[n_2,k_2,d_2]]$ code.

(4) Alice and Bob use the coset of each $u+C_2$ as the key. In this way, they obtain the $k_1k_2$-bit key.

III. SECURITY OF THE PROTOCOL

What we want to show is the security of protocol C, the concatenated BB84 protocol. However, it is sufficient for us to show the security of protocol A, since protocol A reduces to protocol C. Accordingly, arguments in the following are for protocol A that accompanies entanglement purification.

The security of modified Lo-Chau protocol is based on the idea of random sampling [15,24]: Based on the measured error rate in the check bits, Alice and Bob estimate actual error rate in the code bits. Since the check bits are randomly chosen, they can do it with a high reliability. Thus, they can correct errors with a high reliability. (It is not misleading to use notations of QECCs in the discussion of entanglement purification because they are equivalent here.)

It is not difficult to intuitively understand how protocol A is secure. When the estimated actual error rate in the code bits is not low enough, they cannot obtain EPR pairs with a fidelity that is satisfactorily high. However, in most cases the fidelity has become higher. By iterating the entanglement purification, they can make the fidelity become higher and higher. Protocol A is secure when EPR pairs have high enough fidelity, since the almost perfect fidelity implies the almost perfect security [25,15]. This is indeed the original idea of quantum privacy amplification [26].

Let us give a more detailed description. The probability distribution of the number of actual errors forms an approximate Gaussian distribution whose deviation depends on the length of the code. The longer the code is, the smaller the deviation. A rigorous relevant equation regarding random sampling for this estimation is given in Ref. [24]. Let us consider the example considered in Introduction. The error rate $r$ is 10%. The number of code-bits $n$ is one thousand and the standard deviation is about 0.949%. This means that the probability that the real error rate will be less than $12.4\%\ (r=2.57\sigma)$ is 99%. Therefore, if the threshold of the error correcting code is greater than 12.4%, the error rate of purified EPR pairs will be less than 1%, which is much smaller than the original error rate, 10%. By repeating this operation, the fidelity can be made arbitrarily small. Let us...
estimate how error rate $r$ decreases with iteration of purification. For a given length of code $n$, the deviation $\sigma$ is approximately proportional to $\sqrt{r}$. (See Introduction.) Let us denote $i$th error rate and deviation by $r_i$ and $\sigma_i$, respectively, where $i$ is positive integer. Then we can get a relation $r_{i+1} \sim \exp(-T^2/\sigma_i^2) = \exp(-T^2/r_i)$ when $r \sim 0$. Here constant $T$ is the threshold of the code. We can see that $i$th error rate $r_i$ superexponentially decreases with respect to $i$. Fidelity is proportional to $1 - r$ and thus almost perfect fidelity can be obtained.

In protocol A, Alice randomly chooses sets of qubits to be used for syndrome measurement. This corresponds to random choice of code bits to be used for error correction and privacy amplification [in step (8) of the first stage and step (1) of the second stage] in Protocol C. It should be noted that the randomness in the choice is essential in the proposed protocol: the error rate is what is averaged for all $n_1n_2$ check bits. Unless the choice for the code bits is random, Eve can successfully cheat by correlating positions of the errors in the code bits.

IV. CONCLUSION

In a certain class of BB84 protocol with proven security [15], long classical error correcting codes are desired in order to obtain sufficient security. However, it is not easy to decode long codes. If they use intermediate length codes, they cannot obtain enough security. We have shown that this long code problem can be resolved by concatenation of the BB84 protocol. We have shown the security of the concatenated BB84 protocol: the concatenated BB84 protocol can be derived from the concatenated modified Lo-Chau protocol. We have described how the latter protocol is secure.

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