An analytical solution for transient temperature distribution in fillet arc welding including the effect of molten metal

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Abstract: This paper presents an analytical solution to predict the transient temperature distribution in fillet arc welding, including the effect of the molten metal generated from the electrode. The analytical solution is obtained by solving a transient three-dimensional heat conduction equation with convection boundary conditions on the surfaces of an infinite plate with finite thickness, and mapping an infinite plate on to the fillet weld geometry, including the molten metal with energy equation. The electric arc heat input on the fillet weld and on the infinite plate is assumed to have a travelling bivariate Gaussian distribution.

To check the validity of the solution, flux cored arc (FCA) welding experiments were performed under various welding conditions. The actual isotherms of the weldment cross-sections at various distances from the arc start point are compared with those of the simulation result. As the result shows a good accuracy, this analytical solution can be used to predict the transient temperature distribution in the fillet weld of finite thickness under a moving bivariate Gaussian distributed heat source.

The simplicity and short calculation time of the analytical solution provides the rationale for using the analytical solution to model the welding control systems or to obtain an optimization tool for welding process parameters.

Keywords: fillet weld, mapping, analytical solution, bivariate Gaussian distribution

NOTATION

<table>
<thead>
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<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>a</td>
<td>constant</td>
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<tr>
<td>A_n</td>
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<td>c</td>
<td>specific heat of weldment</td>
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<tr>
<td>C, C_1</td>
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<td>d</td>
<td>thickness of infinite plate</td>
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<td>effective convection coefficients</td>
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<td>q(t)</td>
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<td>q_i</td>
<td>intensity of point heat source</td>
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<td>q_o(t)</td>
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<td>U’, V’</td>
<td>coordinates of point heat source</td>
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<td>x, y, z</td>
<td>moving coordinates in fillet weld</td>
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<td>σ_u, σ_v</td>
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1 INTRODUCTION

Arc welding is a process of fusing and joining metals using the heat generated by the electric current between the electrode and the weldment. Even though this process has been widely applied in most industries, especially in shipbuilding, severe heat, light emission and fumes make many people think it an unpleasant job.

Automation of the welding process is therefore strongly required. For automation the welding process needs to be well understood but, as the welding process causes very complex physical and metallurgical changes in the work-piece, it is very difficult to establish an exact mathematical model for it. The material non-linearities, such as the temperature-dependent characteristics of the physical parameters of the weldment, make the problem more intangible.

Although numerical analysis is a powerful tool for understanding the welding process, it requires a very large calculation time when it is used as a model for the control system of the welding process. In this regard, an analytical model has many advantages for optimization and control of the welding process because of its flexibility and simplicity.

Rosenthal (1) was the first to bring the problem down to a manageable size by assuming temperature-invariant characteristics of the physical parameters. He derived the temperature distribution equations by utilizing the heat conduction equation for the quasi-stationary state. Although his mathematical formulations qualitatively revealed the effects of the welding parameters on the weld pool shape, the solution incorporates rather large errors in temperature distribution because of many assumptions. Since then, to reduce the errors, a number of modifications have appeared with more realistic assumptions, such as distributed heat source (2), change of phase (3) and circulation in the weld pool (4).

However, all these are valid only in the quasi-stationary state; the static models are unsuitable to use in optimization and control of the welding process. Eager and Tsai (5) derived a transient model of the temperature field in a semi-infinite thickness plate subjected to a travelling Gaussian distributed heat source. Although this model illustrates the transient behaviour of the temperature distribution, it produces errors in predicting the depth of penetration, especially in full penetration and in thin plate, due to the assumption of the semi-infinite thickness. Boo and Cho (6) derived an analytical model of the arc welding process which describes the three-dimensional temperature field more accurately in a finite thickness plate subjected to a Gaussian distributed travelling heat source.

However, all the works described in the above have been developed for a bead on plate welding. In industries the bead on plate is scarcely used but, instead, fillet welding is widely used in most fabrications of welded structures. For example, more than 80 per cent of welding joints are fabricated by a fillet weld in the shipbuilding industry (7). Therefore, it would be very useful to derive an analytical solution that could predict the transient temperature distribution in fillet arc welding.

As the geometrical shape and arc distribution of the fillet weld are very complex compared with those of a bead on plate, it is very difficult to derive temperature distribution in the fillet weld, analytically, from the governing equation and boundary conditions of a fillet weld. So far, efforts to derive an analytical solution of the temperature distribution in a fillet weld have not been actively made.

Recently, Jeong and Cho (8) developed a new method to derive the analytical solution of a fillet weld and to solve the technical difficulties described above. In the paper the analytical solution is obtained through three steps. Firstly, the arc distribution in the fillet weld is assumed to have a bivariate Gaussian distribution. Secondly, an analytical solution is derived which predicts the transient temperature distribution in the infinite plate of finite thickness subjected to a bivariate Gaussian distributed heat source which is moving along the welding line with convection boundary conditions on the surfaces of the infinite plate. Finally, the mapping function is derived to map the infinite plate of finite thickness on to the fillet weld shape. In mapping, the energy equation is considered to satisfy the condition that the heat flow through any mapping region of the transformed coordinates must be the same as that of the non-transformed coordinates. Therefore, using the derived equations, such as the analytical solution for the temperature distribution in an infinite plate, mapping function and energy equation, the transient temperature distribution of the fillet weld subjected to a bivariate Gaussian distributed moving heat source is obtained.

However, the results of calculations made using this analytical solution incorporate rather large errors, especially as the calculated penetration in the throat is much deeper than that of the actual welding. This is due to the weld bead which is not taken into consideration in the model of the fillet weld, as the shape of the fillet weld is assumed in this paper to be a right-angled bend. Another cause of error is the latent heat of molten metal during cooling and discontinuity in the weldment which are not taken into consideration. Therefore, in this paper the fillet weld is assumed to be a right-angled bend including the shape of the molten metal. The latent heat of molten metal is also taken into consideration to improve the accuracy of the analytical solution.

To verify the validity of the solution, a series of fillet welding experiments was performed under various welding conditions. The isotherms in the cross-section of the weldment were compared with those of the calculated results. This comparison showed that the analytical solution provided better accuracy than that of previous research (8) and also showed a good tendency to predict the size of the molten pool and heat-affected zone (HAZ) regardless of full or partial penetration in the fillet weld.

These results indicate that the proposed analytical solution could be applied to estimate the transient temperature changes, which often become an important task in automatic welding.
2 TRANSIENT TEMPERATURE DISTRIBUTION IN THE FILLET WELD

2.1 Overview

The coordinate system for the weldment and electrode in a fillet weld is defined as shown in Fig. 1. \( X, Y, Z \) denote the coordinates fixed at the origin, 0, a point on the surface of the weldment, while \( x, y, z \) denote the moving coordinates with the origin fixed at the centre of the arc \( (X_a, Y_a, 0) \) moving along the welding line.

During welding, the heat at the surfaces of the weldment is dissipated into the atmosphere due to the forced convection by the flow of shielding gas and natural convection. Therefore, the convection boundary conditions are adopted on the surfaces of the weldment.

As the geometrical shape and arc distribution of the fillet weld are very complex to use when solving the temperature distribution analytically, an alternative method is taken in this study. The temperature distribution is first solved in the infinite plate, to avoid mathematical complexity, and then transformed using mapping in the infinite plate into the fillet weld shape including molten metal.

The transformed coordinate system, the infinite plate, is as shown in Fig. 2. On the other hand, to achieve not only geometrical mapping but also physical mapping, the energy equation is considered to satisfy the physical condition that the heat flow through any mapping region of the transformed coordinates must be the same as that of the non-transformed coordinates.

2.2 The temperature distribution in transformed coordinates

The transformed coordinate system, the infinite plate, is as shown in Fig. 2. In the figure, \( U, V, W \) are the coordinates fixed on the surface of plate and \( u, v, w \) are the moving coordinates with the origin at the centre of the arc \( (U_a, V_a, 0) \). During welding, the heat conduction equation of solid steel must be satisfied when the heat is applied to the weldment (9):

\[
\rho c \frac{\partial T_u}{\partial t} = \nabla (k \nabla T_u)
\]

where \( T_u \) is the temperature in the transformed coordinates of the weldment, \( \rho, c \) and \( k \) are the density, the specific heat and the thermal conductivity at temperature \( T \), respectively, and \( t \) is time. In previous researches (2–6) the material properties are assumed to be constant. Therefore, equation (1) can be rewritten as

\[
\frac{\partial T_u}{\partial t} = \alpha \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2} \right) T_u
\]

where \( \alpha = k/\rho c \) is the thermal diffusivity. Heat at the top surface of the plate is dissipated into the atmosphere due to the forced convection by the flow of shielding gas and
natural convection at the bottom surface during welding. Then, the boundary condition on the top surface is given by (9)

\[-k \frac{\partial T_u(U, V, 0)}{\partial W} + h_1[T_u(U, V, 0) - T_0] = 0\]  

(3)

and

\[k \frac{\partial T_u(U, V, d)}{\partial W} + h_2[T_u(U, V, d) - T_0] = 0\]  

(4)

on the bottom surface of the plate. In the above, \(h_1\) and \(h_2\) are the effective convection coefficients, \(T_0\) is the room temperature and \(d\) is the thickness of the plate. The boundary condition at an infinite distance from the heat source is

\[\lim_{r \to \infty} T_u(U, V, W, t) = T_0\]  

(5)

where \(r = \sqrt{[(U - U')^2 + (V - V')^2 + W^2]}\) is the distance from the point heat source. The point heat source is located at \((U', V', 0)\). The initial condition is given by

\[T_u(U, V, W, 0) = T_0\]  

(6)

The distribution of the arc is assumed to be a bivariate Gaussian distribution, as shown in Fig. 3 (10). Putting \(\sigma^2 = \sigma_u \sigma_v\), the distribution is expressed as

\[Q(u, v, t) = \frac{q(t)}{2\pi\sigma^2} \exp \left( -\frac{\sigma_u^2 u^2 + \sigma_v^2 v^2}{2\sigma^2} \right)\]  

(7)

where \(\sigma_u\) and \(\sigma_v\) are the distribution parameters with the dimension of length, \(q(t)\) is the heat input at time \(t\), \(q(t) = \eta V \ell(t)\), \(\eta\) is the arc efficiency and \(V \ell(t)\) is the electrical power input.

By solving equations (2) to (7), the temperature in the infinite plate of finite thickness subjected to a moving bivariate Gaussian distributed heat source can be predicted.

It takes four steps to obtain the solution. For the first step, the temperature at a point, \(P(U, V, W)\), at time \(t\) in the infinite plate subjected to an instantaneous point heat source whose intensity is \(q_i\) at point \((U', V', 0)\) at time \(t_1\) is obtained from (6)

\[T_u(U, V, W, t) = T_0\]  

\[= \frac{q_i}{2\pi kd(t - t_1)} \exp \left\{ -\frac{(U - U')^2 + (V - V')^2}{4\alpha(t - t_1)} \right\} \]  

\[\times \sum_n A_n \exp \left\{ -\mu_n^2(t - t_1) \right\} \]  

\[\times \left[ \cos \left( \frac{\mu_n W}{\sqrt{\alpha}} \right) + \frac{\beta_1 \sqrt{\alpha}}{\mu_n} \sin \left( \frac{\mu_n W}{\sqrt{\alpha}} \right) \right]\]  

(8)

where the constants are given by

\[A_n = \frac{\mu_n^2}{\mu_n^2 + \alpha \beta_1^2 + 2\alpha \beta_1 d}\]  

\[\tan \left( \frac{\mu_n d}{\sqrt{\alpha}} \right) = \frac{\sqrt{\alpha} \mu_n (\beta_1 + \beta_2)}{\mu_n^2 - \alpha \beta_1 \beta_2}\]  

\[\beta_1 = \frac{h_1}{k}, \quad \beta_2 = \frac{h_2}{k}, \quad k = \rho c\alpha\]

For the second step, the temperature distribution due to the instantaneous bivariate Gaussian distributed heat source with its centre at the origin of fixed coordinates at time \(t_1\) is obtained by the superposition of a series of instantaneous point heat source solutions [equation (8)] over the region of the distributed heat source [equation (7)]. The solution is obtained from (8)

\[dT_u(t_1)\]  

\[= \frac{q(t_1)}{\pi \rho c d l_1} \frac{\sigma^2}{\sqrt{[\sigma^4 + 2\alpha(t - t_1)\sigma^2]} \sqrt{[\sigma^4 + 2\alpha(t - t_1)\sigma^2]} \]  

\[\times \exp \left\{ -\frac{[\sigma_u^2 + 2\alpha(t - t_1)]U^2 + [\sigma_v^2 + 2\alpha(t - t_1)]V^2}{2[\sigma_u^2 + 2\alpha(t - t_1)] [\sigma_v^2 + 2\alpha(t - t_1)]} \right\} \]  

\[\times \sum_n A_n \exp \left\{ -\mu_n^2(t - t_1) \right\} \]  

\[\times \left[ \cos \left( \frac{\mu_n W}{\sqrt{\alpha}} \right) + \frac{\beta_1 \sqrt{\alpha}}{\mu_n} \sin \left( \frac{\mu_n W}{\sqrt{\alpha}} \right) \right]\]  

(9)

For the third step, equation (9) is extended for the centre of the arc located at any point \((U_a, V_a, 0)\) at time \(t_1\). When considering the moving heat source, the total formation of the temperature distribution with respect to the distance from the centre of the arc at time \(t\) is obtained by summing the respective contributions of all the instantaneous bivariate Gaussian distributed heat sources for the time interval from \(t_1 = 0\) to \(t_1 = t\). Then, the temperature distribution
in the moving coordinate \((u, v, w, t)\) at time \(t\) due to the heat input is expressed as

\[
T_u(u, v, w, t) - T_0 = \int_0^{t_p} \pi pc d \left[ \frac{\sigma^2}{\sqrt{\sigma^4 + 2\alpha(t - t_1)\sigma_1^2}} \sqrt{\sigma^4 + 2\alpha(t - t_1)\sigma_2^2} \right] \times \exp \left( -\frac{1}{2} \frac{\sigma_1^2}{\sigma^2} (u + U_\alpha(t) - U_\alpha(t_1))^2 \right) \times \exp \left( -\frac{1}{2} \frac{\sigma_2^2}{\sigma^2} (v + V_\alpha(t) - V_\alpha(t_1))^2 \right) \times \exp \left( -\frac{1}{2} \frac{\sigma_3^2}{\sigma^2} (w + W_\alpha(t) - W_\alpha(t_1))^2 \right) \times \sum A_n \exp \left( -\mu_n^2 (t - t_1) \right) \times \left[ \cos \left( \frac{\mu_n}{\sqrt{\alpha}} \right) + \frac{\beta_1}{\sqrt{\alpha}} \sin \left( \frac{\mu_n}{\sqrt{\alpha}} \right) \right] \; dt_1 \tag{10}
\]

2.3 Mapping from the infinite plate on to the fillet weld

Equation (10) represents the temperature distribution in the infinite plate subjected to a bivariate Gaussian distributed moving heat source at time \(t\). To obtain the temperature in the fillet weld, the temperature, \(T_u(u, v, w, t)\), in the transformed coordinates has to be transformed to \(T(x, y, z, t)\) in the physical coordinates of the fillet weld by using mapping. Mapping in the direction of the welding line is \(x = u\); thus, the mapping from the \(vw\) plane on to the \(yz\) plane is required.

To derive the mapping, three complex planes are introduced as shown in Fig. 4. The complex variable \(W = v + wi\) represents the cross-section of the infinite plate, \(R = q + ri\) is a dummy for mapping and \(Z = y + zi\) is the cross-section of the fillet weld shape. As can be seen in the Appendix, the mapping from \(W\) to \(R\) is given by

\[
R = a e^{(x + iy)(2d_1)} + 1 \quad \frac{1}{e^{(x + iy)(2d_1)} - 1} \tag{11}
\]

Similarly, the mapping from \(R\) to \(Z\) is given by

\[
Z = K \int_0^{\pi/2} \frac{1}{2\pi} \frac{(\xi^2 - 1)^{1/4}}{\xi^2 - a^2} \, d\xi + \frac{L}{2} + \frac{L}{2} \tag{12}
\]

The constants \(a\) and \(K\) are obtained from the Appendix. Substitution of equations (11) and (12) will reject \(R\) and derive the mapping from the \(W\) to the \(Z\) planes. To show the validity of mapping, an illustration of the mapping from the \(vw\) plane on to the \(yz\) plane is introduced as shown in Fig. 5. This is the mapping from the infinite plate of thickness \(d\) on to the fillet weld of the same thickness and a leg length of \(\frac{3}{2}d\).

2.4 Distributed heat source in the transformed coordinate system

The bivariate Gaussian distributed heat source is defined with the total heat input \(q(t)\) and the distribution parameters \(\sigma_x\) and \(\sigma_y\). To obtain the temperature distribution in the fillet weld, the temperature distribution in the infinite plate has to be obtained in advance. Therefore, it is required to see how the heat source in the \(x, y, z\) coordinates is represented in the \(u, v, w\) coordinates.

Using the mapping function, equations (11) and (12), the distribution parameter in the transformed coordinates can be obtained. As the heat distribution obtained from the mapping function is very complex, it is also assumed that a bivariate Gaussian distribution can have another distribution parameter. Therefore, it is required to find the distribution parameters in the transformed coordinates (8).

As the mapping along the welding line is \(u = x\), the mapping of the distribution parameter is \(\sigma_u = \sigma_x\). The total heat inputs from the welding arc to the weldment in both the transformed and physical coordinate are the same, \(q_u(t) = q(t)\). The transformed value of \(\sigma_x\) in the physical
coordinates is adopted as \( j v \) in the transformed coordinates. By introducing the operator \( M \), the mapping from \( W \) to \( Z \) [equations (11) and (12)] is expressed in a simple form as
\[
Z = M(W).
\]
Then \( j v \) is given by
\[
 j v = M(j v).
\]

2.5 Temperature distribution in the physical coordinate system

The mapping obtained in the previous section is only for geometrical mapping, not for physical transformation. Therefore, for the last step in the process of obtaining the analytical solution in the fillet weld including molten metal, the energy equation is considered to satisfy the requirement that the heat flow through any mapping region of the transformed coordinates must be the same as that of the non-transformed coordinates. This means that the increase or decrease of the heat in any mapping regions of both the infinite plate and the fillet weld is to be the same. Therefore the following equation must be satisfied:
\[
T(x, y, z, t) = T(u, v, w, t) \left| \frac{dW}{dZ} \right|
\]

From the Appendix,
\[
\frac{dZ}{dR} = K \frac{(R^2 - 1)^{1/4}}{R^2 - a^2},
\]
\[
\frac{dW}{dR} = \frac{-2ad}{\pi(R - a)(R + a)}
\]
and
\[
R = a e^{(\pi d)(w - a)} + 1
\]
\[
R = a e^{(\pi d)(w - a)} - 1
\]
Finally, the temperature distribution in the fillet weld is given by
\[
T(x, y, z, t) = T(u, v, w, t) \left| \frac{-2ad}{\pi K(R^2 - 1)^{1/4}} \right| (14)
\]

3 COMPARISON BETWEEN THE ANALYTICAL SOLUTION AND THE EXPERIMENTAL RESULT

3.1 Simulation with the analytical solution

In this section, efforts are made to prove the validity of the analytical solution of the transient temperature distribution in the fillet weld, expressed by equations (10), (11), (12) and (14). To achieve this purpose the experimental results and the calculated results obtained by the analytical solution are compared with each other.

In practice, material properties of the weldment vary with temperature. In this study the properties at 973 K (700°C), the approximate mean of the melting point 1768 K (1495°C) and the room temperature were used for all calculations. The forced and natural convection coefficients, \( h_1 = 50 \text{ W/m}^2\text{K} \) and \( h_2 = 18 \text{ W/m}^2\text{K} \), in Table 1 are selected from the empirical relations in references (6, 8, 11, 12).

<table>
<thead>
<tr>
<th>Property</th>
<th>Notation (unit)</th>
<th>Weldment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity</td>
<td>( k ) (W/m K)</td>
<td>30.3</td>
</tr>
<tr>
<td>Specific heat</td>
<td>( c ) (J/kg K)</td>
<td>752</td>
</tr>
<tr>
<td>Density</td>
<td>( \rho ) (kg/m³)</td>
<td>7860</td>
</tr>
<tr>
<td>Forced convection heat coefficient</td>
<td>( h_1 ) (W/m² K)</td>
<td>50</td>
</tr>
<tr>
<td>Natural convection heat coefficient</td>
<td>( h_2 ) (W/m² K)</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 1 Physical properties of the weldment
The latent heat of molten metal is transferred to the weldment by conduction and dissipated into the atmosphere by convection during cooling. The latent heat of molten metal generated from the electrode per unit time is considered as an additional heat source. The distribution of the latent heat is the same as the shape of the bead, of which the leg length is $L$, as shown in Fig. 6. For the simplicity of calculation it is assumed to be a Gaussian distribution close to the shape of the bead. Its distribution parameter is $L/2\sqrt{3}$.

From reference (11) the distribution parameters $\sigma_x$ and $\sigma_y$ are selected as 3.25 mm. Keeping this distribution parameter, the value that best fits the shapes of the weld pool obtained from experiment and the analytical solution is chosen as the arc efficiency. These experiments were done with FCA welding with a selected arc efficiency of 78 per cent. The obtained arc efficiency is somewhat larger than the reported data, 66–71 per cent (11), because the heat input in the analytical solution contains the heat loss due to convection on the surfaces of weldment.

Simulation using the analytical solution takes three steps as follows. Firstly, the heat distributions $q_u$, $q_v$, and $q_u(t)$ in $u,v,w$ coordinates are obtained from given $\sigma_x$, $\sigma_y$, and $q(t)$ with equation (13). Secondly, the temperature distribution in the infinite plate of finite thickness is solved using equation (10) with the obtained heat distribution. Finally, the temperature distribution of the infinite plate is mapped on to the fillet weld using equations (11), (12) and (14).

### Table 2 Welding condition

<table>
<thead>
<tr>
<th>Condition</th>
<th>Unit</th>
<th>Value</th>
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<tbody>
<tr>
<td>Welding current</td>
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<td>240</td>
</tr>
<tr>
<td>Heat input</td>
<td>W</td>
<td>7200</td>
</tr>
<tr>
<td>Gun travel speed</td>
<td>mm/s</td>
<td>5</td>
</tr>
<tr>
<td>Contact tube to workpiece</td>
<td>mm</td>
<td>20</td>
</tr>
</tbody>
</table>

The welding conditions used in the simulation and experiment are shown in Table 2 and the chemical composition of the weldment is shown in Table 3.

### 3.2 Experiment

To prove the validity of the analytical solution and to obtain the arc efficiency, a series of FCA welds was carried out in the flat position using 100 per cent CO$_2$ gas and 1.2 mm electrode which is equivalent to AWS E71T-1. The weldment in this example was hot-rolled DNV class AH32 steel plate which is most widely used in shipbuilding. The dimensions of the specimen were 200 mm by 50 mm with a thickness of 9 mm.

A typical macrograph of the experimental results is shown in Fig. 9. After welding was completed the weldment was sectioned, polished and etched to reveal two distinct isothermal lines of solidus and a transformation temperature that represents the heat-affected zone (HAZ) of the weldment. Then the isothermal lines are compared with those of the calculated result, as shown in Fig. 10.

### 3.3 Results and discussions

Figure 7 shows the effect of the variations in distribution parameters (distance from arc start point = 35 mm, arc efficiency = 78 per cent, current = 240 A, voltage = 30 V, travel speed = 5 mm/s)
parameters on the temperature field of the weldment as a result of simulation. The solid lines and dots in the figure represent the weld pool and HAZ for three different distribution parameters. An increase of the distribution parameters increases the width and decreases the depth of penetration in the weld pool and HAZ. However, the figure shows that the variation of distribution parameters does not affect the temperature distribution very much. The result is somewhat different in actual welding. This is due to the fully formed bead shape in the analytical solution. In the analytical solution the electrical heat is applied on top of the fully formed bead and heat is conducted to the weldment through the bead. However, in reality the bead grows continuously with time due to the addition of molten metal.

Figure 8 shows that an increase in the arc efficiency increases the weld pool and HAZ. The figure shows that the arc efficiency is more dominant on the depth of penetration than on the width of the weld pool and HAZ.

All results discussed above are observed on the cross-section, which is at 35 mm from the commencement of welding for each 0.1 s time interval until the weld pool and HAZ are fully formed. Figure 9 shows the weld pool and HAZ at various distances from the welding start point in a weldment of 9 mm thickness with FCA welding. It shows that the depth and width of the weld pool and HAZ increase as the distance increases in the transient region.

Comparison between the calculated and experimental results is shown in Fig. 10. In general, the calculated and experimental results tend to be in good agreement. However, the calculated results have much smaller throat penetration than that of the experimental result. This is also due to the fully formed bead shape in the model of the analytical solution shown in Fig. 4. In the analytical solution the electrical heat is applied on top of the fully formed bead and heat is conducted to the weldment through the bead. In reality, however, heat is applied on the surface of the weldment directly at first and formation of the bead is continuous. For this reason less heat is transferred to the corner.

**Fig. 8** Temperature distribution for various arc efficiencies (distance from arc start point = 35 mm, distribution parameter = 3.25 mm, current = 240 A, voltage = 30 V, travel speed = 5 mm/s)

**Fig. 9** Cross-sectional views of weldment for various distances from the arc start point (current = 240 A, voltage = 30 V, travel speed = 5 mm/s)
of the fillet weld in the analytical solution than in actual welding. The existence of any discontinuity in the weldment during fit-up causes another error in the analytical solution. It makes both the shape of the weld pool and HAZ unsymmetrical. The shape of the weld pool and HAZ obtained using this analytical solution is symmetrical, as shown in Fig. 10. However, the experimental result shows discontinuity and a difference in the depth of penetration and HAZ as the existence of the gap in the welding line.

Even though the analytical solution shows some errors, it can present the relationship between the welding parameters and the size of the weld bead and penetration as a good indicator of welding quality. When considering the continuous generation of molten metal from the electrode, heat transfer through the molten metal and the existence of discontinuity in weldment, a more accurate result can be obtained using the analytical solution.

4 CONCLUSION

An analytical solution was derived to predict the transient temperature distribution in a fillet weld of constant section subjected to a bivariate Gaussian distributed heat source including the effect of molten metal. The analytical solution is obtained by mapping the temperature distribution in the infinite plate of finite thickness on to the fillet weld using the energy equation.

To check the validity of the analytical solution, FCA welding experiments were carried out and the result was compared with the calculated result. This comparison shows reasonable agreement with the shape of the weld pool and HAZ.

The latent heat of molten metal is assumed to be a Gaussian distributed additional heat source whose distribution is close to the shape of the bead. Since the physical mechanisms, such as continuous generation of the molten metal from the electrode and flow of molten metal in the weld pool, are not considered in this study some error is incurred in estimating the throat penetration. Another error in the HAZ shape is incurred due to the existence of discontinuity in the actual weldment.

Even though the proposed analytical solution has a simple form, it presents the relationship between the welding parameters and the size of the weld pool and penetration. Therefore the analytical solution is a convenient and reliable means of estimating process parameters and designing feedback control systems for fillet arc welding.

REFERENCES

Mapping of the infinite plate on to the fillet weld including molten metal

(a) **Mapping between the R and Z planes**

The Schwarz–Christoffel theorem (13) states that any polygon bounded by the straight lines in the $Z = y + zi$ plane can be transformed into the axis of $q$ in the $R = q + ri$ plane. Choose for the four points A, B, C and D the points $-1, 1, -a$ and $a$ on the axis of $q$, as shown in Fig. 4. Then

$$
\frac{dZ}{dR} = K \frac{(R^2 - 1)^{1/4}}{R^2 - a^2} \quad (a > 1)
$$

Therefore

$$
Z = K \int_{0}^{\xi} \frac{(\xi^2 - 1)^{1/4}}{\xi^2 - a^2} \, d\xi + C
$$

(15)

Putting $R = 0$, the value of $Z = L/2 + (L/2)i$ can be found for point E. So the constant $C = L/2 + (L/2)i$. For any point $R > a$ the real value of $Z$ is $-d$, and the constant $a$ is obtained from

$$
-d = \text{real} \left[ \int_{0}^{R} \frac{(\xi^2 - 1)^{1/4}}{\xi^2 - a^2} \, d\xi + C \right]
$$

(16)

For point A, $R = -1$ and $Z = L$, and the constant $K$ is given by

$$
K = \frac{L/2 - (L/2)i}{\int_{0}^{1} [(\xi^2 - 1)^{1/4}/(\xi^2 - a^2)] \, d\xi}
$$

(17)

Finally, the mapping is

$$
Z = K \int_{0}^{\xi} \frac{(\xi^2 - 1)^{1/4}}{\xi^2 - a^2} \, d\xi + \frac{L}{2} + \frac{L_1}{2}i
$$

(18)

(b) **Mapping between the W and R planes**

Use also the Schwarz–Christoffel theorem with points C, D and E:

$$
\frac{dW}{dR} = \frac{K_1}{(R - a)(R + a)}
$$

$$
W = \frac{K_1}{2a} \log \frac{R - a}{R + a} + C_1
$$

For point E, $R = 0$ and $W = 0$, and for point F, $R \to \infty$ and $W = di$. Therefore the constants $C_1 = di$ and $K_1 = -2ada$. Finally, the mapping is

$$
W = -\frac{d}{\pi} \log \frac{R - a}{R + a} + di
$$

(19)

$$
R = a \frac{e^{(\pi/d)(W-di)}}{e^{(\pi/d)(W-di)} - 1}
$$