Karhunen–Loéve expansion of Burgers’ model of turbulence

D. H. Chambers, R. J. Adrian, P. Moin, D. S. Stewart, and H. J. Sung
Department of Theoretical and Applied Mechanics, University of Illinois, Urbana, Illinois 61801

(Received 8 December 1987; accepted 16 May 1988)

Characteristics of the Karhunen–Loéve expansion of a strongly inhomogeneous random process possessing small viscous length scales and a large outer scale have been investigated in relation to the application of the expansion to turbulent flow fields. Monte Carlo simulations of a randomly forced Burgers’ equation with zero velocity boundary conditions generate the random process numerically and the Karhunen–Loéve (KL) eigenfunctions and the eigenvalue spectra are computed for different Reynolds numbers. The eigenfunctions possess thin viscous boundary layers at the walls and are independent of Reynolds number in the core, where the random process is quasihomogeneous. The eigenfunctions and eigenvalues of the outer, large scale motions obey a principle of Reynolds number similarity. Eigenvalue spectra contain much of the energy in the first few modes, but they are as broad as ordinary trigonometric power spectra. The rate at which the expansion converges to within 90% of the total energy decreases with increasing Reynolds numbers and the expansion of the mean plus the fluctuation converges more rapidly than the expansion of the fluctuation alone.

I. INTRODUCTION

A random process on a spatial domain has a unique expansion in terms of orthogonal functions with random coefficients, called the Karhunen–Loéve (KL) expansion. The KL expansion (or orthogonal decomposition) minimizes the mean-square error of any partial sum of the expansion, and hence the expansion is the most rapidly converging orthogonal expansion in the mean-square sense for a given random process. A requirement of this representation is the need to measure or calculate a sufficiently resolved correlation tensor for the process.

In 1967, Lumley suggested that turbulent flows could be represented by the KL expansion in concert with the shot-noise decomposition. In this scheme the basic functions of the KL expansion ("characteristic eddies") were interpreted to correspond to well-defined (coherent) structures. Applications of the expansion to evaluation of structure have been reported by Payne and Lumley, Bakewell and Lumley, Glauser, Leib, and George, Long and Arndt, and Herzog. Sampling error has limited experimental studies to the calculation of the first few terms of the expansion. Application of the expansion to the results of a numerical simulation of channel flow was reported by Moin in 1984. The basis functions were computed as functions of the coordinate normal to the channel wall for the domain over full channel width and domains spatially limited to the wall. More recently, Aubry used the KL expansion in a Galerkin solution scheme for the dynamical modeling of turbulence close to a wall.

The application of the Karhunen–Loéve expansion to three-dimensional turbulent flows has been hindered by the size of the computational problem and the difficulty of obtaining accurate correlation functions for the full domain of \( u(x) \). Experimental measurements have been limited to the first few eigenfunctions by the capabilities of the experimental apparatus. Calculations of the KL eigenfunctions from numerical simulations are limited by the large size of the array, which represents the correlation and the statistical stability of the correlation estimates. The application of the representation for the whole flow in three spatial dimensions requires large amounts of computer memory and long averaging times. The issues of convergence, scaling properties, and Reynolds number dependence of the expansion for the whole velocity field have yet to be addressed in a complete way.

In this paper we discuss the properties of the KL expansion for a standard turbulence model with some features of real turbulent flows, and we carry out a parametric study. Our model is one dimensional and is based on the forced Burgers’ equation. While it is significantly simpler than real, three-dimensional turbulence, the model possesses several dynamical characteristics that make it suitable for investigating the properties of the KL expansion applied to turbulent flows. Specifically, the statistics of the model solutions are strongly inhomogeneous at the boundaries and the inhomogeneity occurs in thin viscous wall layers, much like three-dimensional wall turbulence. There is also a wide range of scales of motion, ranging from large inviscid motions to thin internal viscous layers. These features allow us to test the ability of the KL expansion to represent functions that have both slow and rapid spatial variations in a relatively small number of expansion terms. The relatively well established structure of the Burgers’ equation also permits us to relate properties of the KL expansion to the underlying physics of the flow.

If the Karhunen–Loéve expansion is to make a major contribution to the theory and computation of turbulence, it must significantly reduce the total number of terms needed for a satisfactory representation. Investigations of various turbulent processes have shown that the first several terms of
the KL expansion often contain a large fraction of the signal energy as defined by the mean square. However, the manner in which the rate of convergence in the mean-square norm depends upon dynamical parameters such as the Reynolds number has not been studied systematically, nor has the number of terms needed to represent derivatives been determined. We shall evaluate the rate of convergence of the expansion and determine how it depends on the Reynolds number. All expansions are for the full interval of the solution. We determine the structure of the eigenfunctions and evaluate their Reynolds number dependence. We show that the eigenfunctions possess wall boundary layers attached to outer structures that (at high Reynolds numbers) are Reynolds number independent. We also show that the spectrum of eigenvalues is broad at large Reynolds number, requiring many terms to represent higher-order derivatives of the function.

II. BURGERS’ EQUATION MODEL

Burgers’ equation for the velocity \( u \) in the x direction is given by

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},
\]

(1)

and represents a balance of time dependence, nonlinear convection, and diffusion. The parameter \( \nu \) represents the viscosity. This equation has been studied extensively both theoretically and numerically.

Solutions of Burgers’ equation at high Reynolds numbers exhibit a characteristic N-wave structure consisting of a positive increasing ramp and a negative increasing ramp separated by a thin viscous “shock layer.” Both wall layers and internal traveling shocks can develop depending on the specific details of the initial and boundary conditions and forcing. The thicknesses of the shock layers and the wall layers are typically of the order of an inverse Reynolds number defined by the amplitude of a characteristic average velocity and average length scale identified from the initial conditions, boundary conditions, or forcing. Each shock moves at a velocity determined by the arithmetic mean of the values of \( u \) on either side of the shock. Large amplitude shocks travel faster than shocks with smaller amplitudes and grow by overtaking and assimilating smaller shocks. Shocks encountering a boundary at which the solution is held fixed are trapped. In this way wall layers develop which are sustained by a steady influx of new shocks produced in the interior.

The shocks and wall layers are regions in which viscosity becomes important. Outside these regions, the solution is essentially inviscid, in analogy with real turbulent flows. In general, the larger scales of the solutions of Burgers’ equation are described by the inviscid equation (\( \nu = 0 \)), while the small scales are influenced by the viscous term. The formation of shocks results in a net flow of the spectral energy from low to high wavenumbers: a one-dimensional energy cascade. The N waves and the wall layers are the organized structures that dominate the dynamics of solutions to Burgers’ equation.

In this study, Burgers’ equation is solved on a finite interval of length \( l_0 \) with boundary conditions of zero at each end. As an initial value problem, Burgers’ equation for these conditions would eventually decay to zero from any bounded initial condition. In order to obtain stationary random solutions, a random forcing term is added to the equation. The resulting equation is similar to that studied by Jeng and Hosokawa and Yamamoto. The forcing function is a white noise random process in \( x \) with zero mean and is constructed as a Fourier series in \( x \) with stationary, random, time-varying coefficients. These coefficients are mutually uncorrelated white noise processes in time that have been low-pass filtered to obtain a nonzero integral time scale. The mean-square value of the dimensional forcing, \( \sigma^2 \), defines a velocity scale \( u_0 = \sqrt{\sigma^2} \), where \( l_0 \) is the length of the computational domain. Burgers’ equation in nondimensional form using \( u_0 \) and \( l_0 \) as the scales can be written as

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{Re} \frac{\partial^2 u}{\partial x^2} + \chi(x,t),
\]

(2)

where \( \chi \) is the dimensionless random forcing function and \( Re \) is the Reynolds number \( u_0 l_0 / \nu \). The construction of \( \chi(x,t) \) is designed to create a relatively featureless process (maximum entropy) in order to minimize the creation of structures in \( u(x,t) \) through the action of the forcing function.

Equation (2) is solved with \( \langle \chi^2 \rangle = 1 \). The boundary and initial conditions are \( u(0,t) = u(1,t) = 0 \) and \( u(x,0) = 0 \). The numerical method uses second-order spatial differencing and a third-order predictor-corrector scheme for time stepping. The spatial grid consisted of 555 points with spacing varying from \( \frac{1}{30} \) in the interior of the interval to \( \frac{1}{30} \) near \( x = 0 \) and \( x = 1 \). Details regarding the construction of the forcing and the numerical scheme can be found in Ref. 15. The computations were performed on a Cray XMP-48 at the National Center for Supercomputing Applications at the University of Illinois.

Ensembles of 40,960 samples were computed for \( Re = 413.5, 1654, \) and 6616. Averages computed over an ensemble are equivalent to time averaging using 3200 integral times of data. Samples of the solution \( u(x,t) \) at an instant in time are shown in Fig. 1. Wall layers and internal shocks with thicknesses on the order of \( 1/Re \) can be seen for Reynolds numbers 1654 and 6616. The smallest Reynolds number of 413.5 is too low for sharp shocks to appear, but distinct wall layers are observed. The wall layers are antisymmetric, negative for \( x = 0 \), and positive for \( x = 1 \). This is expected since only shocks with negative mean values and negative velocities will encounter the boundary at \( x = 0 \). Shocks with positive velocities move toward the wall at \( x = 1 \). This antisymmetry is seen for all Reynolds numbers and is reflected in the mean and skewness shown in Figs. 2 and 3.

The statistical moments in Figs. 2 and 3 possess distinct symmetries around the midpoint of the interval. The odd moments are antisymmetric while the even moments are symmetric. This reflects the property that Burgers’ equation is invariant under the transformation \( x \rightarrow -x \), \( u \rightarrow -u \).
FIG. 1. A sample of the solution to Burgers' equation $u(x,t)$, where $x$ is measured from the midpoint of the interval. This invariance holds for Burgers' equation with no forcing and with symmetric boundary and initial conditions. The statistical properties of the forcing used here are also invariant under this transformation. Small deviations from this symmetry are attributed to sampling error, which, despite the large ensemble of samples, is noticeable because the fluctuations around the average were large.

The root mean square of $u(x,t)$ is constant over the interior region between the two wall layers (Fig. 2). The rms of the fluctuations around the mean is constant over a smaller interior region. The flatness (Fig. 3) is nearly constant in the interior region and, while the skewness necessarily changes sign at the midpoint, its magnitude is relatively small throughout the interior region. (These curves bear some qualitative resemblance to the skewness and flatness of the velocity fluctuations normal to the wall in the direct numerical simulation of channel flow by Kim, Moin, and Moser.) Thus the statistical structure of the interior region of the Burgers' equation solution is quasihomogeneous. The effects of the shocks on the statistical moments in the interior are distributed uniformly over the whole interior region. This quasihomogeneity makes the rms of $u(x,t)$ a meaningful characteristic velocity scale for the whole solution and motivates the use of the Reynolds number based on this scale. The wall layers are the primary regions of significant statistical spatial inhomogeneity.

The statistical homogeneity of the interior region is especially evident in the plots of the correlation and covariance for $Re = 1654$ [Figs. 4(a)–4(d)]. The correlation $R(x,x') = \langle u(x,t)u(x') \rangle$ shows a sharp crest running along the line $x = x'$. On either side of this crest, the correlation decreases slowly toward the boundaries, then drops rapidly in the region close to the wall where the structure is determined by the dynamics of the viscous wall layer. The symmetry $R(x,x') = R(x',x)$ follows from the definition of the correlation function. The symmetry around the line $x' = 1 - x$ is a consequence of the symmetry of Burgers' equation around the midpoint of the interval. The nearly constant value of $R(x,x')$ along the lines of constant $x - x'$ in the interior, indicates an approach to statistical homogeneity of the second-order moments in this region. The covariance $C(x,x') = R(x,x') - \langle u(x) \rangle \langle u(x') \rangle$ [Fig. 4(c)] shows similar behavior with a smoothly varying interior region and narrow regions along the edges corresponding to the viscous wall layers. In the interior region, equal covari-

FIG. 2. Mean rms of $u(x,t)$, and rms of fluctuation of $u(x,t)$ for $Re = 413.5, 1654$, and 6616.

FIG. 3. Skewness and flatness of $u(x,t)$ for $Re = 413.5, 1654$, and 6616.
ance contours [Fig. 4(d)] are nearly straight $45^\circ$ lines, and the covariance is closely approximated by a function of $x-x'$ consistent with statistical homogeneity. Since the mean of the fluctuation is a constant (zero) the fluctuation is wide-sense stationary (i.e., homogeneous) in $x$. Strict sense stationarity is precluded by the variation of the third-order moments. The regions near the corners $x=0, x'=1$ and $x=1, x'=0$ are slightly negative, primarily a consequence of the mean.

Figures 5(a) and 5(b) are contour plots of the correlations for $Re = 6616$ and 413.5. Their form is basically the same as for $Re = 1654$, the main difference being the width of the wall regions. The thickness of the wall region scales with the inverse Reynolds number as suggested by the behavior of the rms in Fig. 2. The similarity of the correlation function for each Reynolds number shows that the correlation is only weakly dependent on $Re$ in the interior. The scaling of the solution with the interval length indicates that the length of the interval is an appropriate outer length scale. The large scale behavior of the randomly forced Burgers' equation with this scaling is nearly independent of Reynolds number. The main Reynolds number effects are seen in the small scale viscous wall layers.

Table I lists dimensionless values of the Taylor microscale $\lambda_{\tau}$, integral length scale $l_i$, root-mean-square velocity $\sigma_v$, and associated Reynolds number evaluated in the quasi-homogeneous region $0.25 < x < 0.75$. The Taylor microscale is of the order of the viscous shock layer thickness.

III. KARHUNEN–LOÉVE EXPANSION

The Karhunen–Loéve expansion (proper orthogonal decomposition) is a generalized Fourier expansion of a random field using a set of orthogonal basis functions chosen to minimize mean-square error. The expansion can be derived by formally writing the random process $u(x)$ on the interval $I$ as a sum of orthonormal functions $\phi_n$ with random Fourier coefficients $a_n$, where
to arbitrary variations of the integrable basis functions, a Fredholm integral equation for the basis functions is obtained:

$$\int_I R(x,x') \varphi_n(x') dx' = \lambda_n \varphi_n(x),$$

(6)

where the kernel $R(x,x') = \langle u(x)u(x') \rangle$ is the correlation function of $u(x)$, and $\varphi_n$ and $\lambda_n$ are eigenfunctions and eigenvalues. Fredholm theory for non-negative, symmetric kernels guarantees that the eigenvalues are non-negative, countable, and bounded.\(^{18}\) In addition, there are only a finite number of linearly independent eigenfunctions associated with a single eigenvalue. Mercer’s theorem states that the kernel can be expanded as a single sum of eigenfunctions,

$$R(x,x') = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \varphi_n(x').$$

(7)

From this result, it can be shown that the random coefficients are statistically orthogonal random variables with mean-square values equal to the eigenvalues $\lambda_n$,

$$\langle a_m a_n \rangle = \lambda_m \delta_{mn},$$

(8)

where $\delta_{mn}$ is the Kronecker delta.

The properties which have generated the most interest in this expansion as a representation of turbulent flows, are the statistical orthogonality of the modes [Eq. (8)] and the speed of convergence. The procedure of minimizing the mean-square error $e_N$ identifies the set of orthogonal functions, which converge faster than any other generalized Fourier expansion. This means that for any finite number of terms, the Karhunen–Loève expansion will have smaller mean-square error than any other expansion using the same number of terms.

An important limit of the KL expansion occurs when $u(x)$ is stationary. The domain of integration in (6) for this case is $I = (-\infty, \infty)$ and the correlation $R(x,x')$ is a function only of the difference $x' - x$. The solutions for the eigenfunctions are sinusoids and the KL expansion reduces to the familiar Fourier–Stieltjes representation of a stationary random process.

An alternative to representing $u(x)$ with a KL expansion is to expand only the fluctuating part of $u$, $u'(x) = u(x) - \langle u(x) \rangle$. The development of this expansion is identical to the previous expansion and each shares the same mathematical properties:

$$u'(x) = \sum_{n=1}^{\infty} b_n \Psi(x),$$

(9)
\[ \int_{I_0} C(x,x') \Psi_n(x') dx' = \eta_n \Psi_n(x), \]  
(10)

\[ C(x,x') = \sum_{n=1}^{\infty} \eta_n \Psi_n(x) \Psi_n(x'), \]  
(11)

\[ \langle b_n b_m \rangle = \eta_m \delta_{mn}. \]  
(12)

The kernel of the integral equation in this case is the covariance 
\[ C(x,x') = \{u(x) - \langle u(x) \rangle \} \{u(x') - \langle u(x') \rangle \}. \]

The eigenfunctions of the covariance are not equal to the eigenfunctions of the correlation, except in the event that the mean \( \langle u(x) \rangle \) itself an eigenfunction of the correlation.

The KL eigenfunctions and eigenvalues have been obtained numerically from the correlation and covariance of the solutions to the Burgers’ equations for \( I = (0,1) \) and for each Reynolds number. The integral equations were solved by reducing them to matrix eigenvalue problems using the trapezoidal rule and symmetrized using the procedure found in Ref. 7. The resulting eigenvalues \( \lambda_n \) and \( \eta_n \) are shown in Fig. 6 for each Reynolds number, ordered according to the sequence of their corresponding eigenfunctions. Sequence is a generalized wavenumber (frequency) equal to the number of times an eigenfunction changes sign on the interval \( I \). The concept of sequence allows the eigenvalue spectra to be interpreted like Fourier power spectra. The high sequence eigenvalues generally correspond with small scale structures and low sequence eigenvalues correspond with large scale structures.

The low sequence eigenvalues for both the covariance and the correlation exhibit striking independence of the Reynolds number (see Fig. 6). The nondimensionalization of length using the computational interval \( I_0 \) is equivalent to scaling with an outer length scale. Significant Reynolds number effects are seen in the high sequence eigenvalues (small scale structures). The overall effect is a broadening of the eigenvalue spectrum with increasing Reynolds number. The same low wavenumber behavior is found in the trigonometric power spectra of high Reynolds number, self-preserving turbulent flows.

The sequence ordered eigenvalue spectra found in Figs. 6(a) and 6(b) decrease monotonically, indicating dominance by the lowest-order modes. Interestingly, the first two modes of the correlation have almost identical energies, and the third mode is almost ten times less energetic, indicating the prominence of the lowest-order modes in the expansion. However, beyond the second mode, the rate of decrease is not precipitous and the eigenvalue spectra are relatively broad. It must be remembered that the importance of the higher-order modes is amplified in the expansions of the derivatives of \( u \), so that the existence of small eigenvalues does not necessarily imply that it is safe to neglect higher-order terms in the KL expansion.

The eigenvalue spectra are compared with a trigonometric power spectra computed for the quasihomogeneous interior region between \( x = 0.25 \) and \( x = 0.75 \). The power spectra are computed for Reynolds numbers \( 1654 \) and \( 6616 \) along with the moments for Burgers’ equation using the same number of samples. The eigenvalue spectra are very similar in form to the power spectra for sequences \( s \) in the approximate band \( 7 \leq s \leq 40 \), differing only by a constant factor slightly greater than two. If \( u(x) \) were strictly homogeneous we would expect the difference to be exactly a factor of \( 2 \) since the interval from \( x = 0.25 \) to \( x = 0.75 \) would contain only one-half of the total energy. In this band the power spectra decay at a rate proportional to \( 1/s^2 \), analogous to the \( k^{-2} \) spectrum reported for numerical solutions of the unforced Burgers’ equation.11,13,14,19 In these solutions the \( k^{-2} \) behavior was associated with the random locations of the sharp viscous shock front. The similarities between the eigenvalue spectra and the power spectra suggests that the KL expansions of both \( u \) and \( u' \) are similar to trigonometric Fourier expansions. This might be expected since both \( u \) and \( u' \) are nearly statistically homogeneous in the interior region.

Significant differences between the correlation and the eigenvalue spectra occur only for low sequences \( (s < 10) \). The two types of spectra are essentially identical for sequences greater than \( 20 \). Since the difference between the correlation and the eigenvalue function depends only on the mean \( \langle u(x) \rangle \), it follows that the contribution of the mean to the correlation must be well represented by the lowest sequence modes. Thus the mean, including the thin wall layers with rapid spatial variation, is a large scale structure in terms of the KL expansion.

The speed of convergence of the KL expansion has been determined using two tests. These are derived from the expansions of the correlation and covariance in terms of their eigenfunctions, Eqs. (7) and (11). If \( x' \) is set equal to \( x \) and one considers a partial sum of \( N \) terms in Eq. (7), an approximate expansion for the mean square of \( u \) is obtained,

\[ \langle u^2(x) \rangle = \sum_{n=1}^{N} \lambda_n \varphi_n^2(x). \]  
(13)

Increasing the upper limit \( N \) of the sum gives better approximations to the mean square and the sum should equal the mean square when \( N \) approaches the maximum number of eigenvalues. This test will be applied to the eigenfunctions.

![FIG. 6. Eigenvalues of \( \langle u^2(x) \rangle \) the correlation \( R(x,x') \), (b) the covariance \( C(x,x') \) of \( u(x,t) \) for \( Re = 413.5, 1654, \) and \( 6616 \), and (c) the power spectrum of the interior region of \( u(x,t) \) for \( Re = 1654 \) and \( 6616 \).](image)
later. An expression for the integrated kinetic energy of $u$ is obtained by integrating (13) over the interval $I$ to obtain

$$E = \sum_{n=1}^{N} \lambda_n.$$

(14)

The normalization of the eigenfunctions results in a sum that depends only on the eigenvalues. This can be used to determine the number of modes in the KL expansion needed to represent a given fraction of the total integrated energy.

Figure 7 shows the convergence of (14) to the total energy as the number of modes in the sum is increased. The inset shows that the number of modes needed to represent at least 90% of the total energy increases with Reynolds number, from four for $Re = 413.5$ to nine for $Re = 6616$. It is clear that convergence is slower for larger Reynolds numbers. This trend is also found in Fourier representations, where it is well known that the bandwidth of the spectrum increases with increasing Reynolds number. We can nominally define the bandwidth of the KL spectrum as the number of terms to reach 90% of the total energy.

The convergence of the KL expansion for the fluctuation $u'(x)$ can be determined using the same tests. The partial sum approximation for the mean-square fluctuation is

$$\langle u'^2(x) \rangle = \sum_{n=1}^{N} \eta_n \Psi_n^2(x).$$

(15)
Integrating this sum over \( x \) gives the expansion for the total integrated fluctuation energy,

\[
E_{u'} = \sum_{n=1}^{N} \eta_n.
\]  

(16)

Figure 8 shows the convergence of the sequence of partial sums of (16) to the fluctuation energy, and the inset shows the number of terms needed to represent 90% of the energy as a function of Reynolds number. As before, the bandwidth increases with Reynolds number from five terms for \( \text{Re} = 413.5 \) to ten terms for \( \text{Re} = 6616 \). The convergence here is slightly slower than the previous case. In addition to the slower convergence, the mean must be included in any representation of \( u(x) \) using the expansion for \( u'(x) \). Thus the direct expansion of \( u(x) \) using the correlation eigenfunctions is more efficient than expansion of the fluctuations, with a smaller number of terms required for given accuracy.

The first five eigenfunctions of the correlation, Fig. 9(a), are characterized by approximately sinusoidal behavior in the interior region and viscous wall layer behavior at the boundaries. The wall layer regions are thin for the Reynolds numbers studied here and they provide the mechanism for accommodating the interior sinusoids to the wall boundary conditions. The sinusoidal behavior is a consequence of the quasihomogeneous structure of the correlation function in the interior. Eigenfunctions of odd order (even sequence) are symmetric about the midpoint of the interval, and those of even order (odd sequence) are antisymmetric. In terms of mode number the sequence is \( s = m - 1 \).

Each eigenfunction exhibits a boundary layer structure in which the sinusoidal behavior occurs in an outer region and the rapid variations occur in thin viscous wall layers. It is clear that matched asymptotic analysis of these structures would offer the possibility of new insights into the representations of turbulence close to a wall, but such work will not be undertaken here. The structure of the eigenfunctions of the covariance is very similar to that of the correlation, Fig. 9(b), except for \( m = 2 \).

The eigenfunctions of the correlation in the wall layer are shown in more detail in Fig. 10. In this region the eigenfunctions depend strongly upon the Reynolds number, but much of the dependence can be removed by rescaling the distance from the wall with the thickness of the viscous layer. The wall layer thickness is found to be proportional to \( \text{Re}^{-1} \).

The most remarkable aspect of the eigenfunctions in Fig. 9 is their invariance in the outer region with respect to changes in the Reynolds number. Outside the wall layers, the eigenfunctions are nearly identical for Reynolds numbers spanning a range of 16:1. This Reynolds number independence is also found to be a property of higher-order eigenfunctions, as shown in Fig. 11 for modes up to \( m = 18 \). (The small differences between the higher-order eigenfunctions, shown for different Reynolds numbers, are of the or-
Reynolds number similarity of turbulent quantities requires that the quantities be independent of the Reynolds number when scaled by proper scales of length and velocity. For the KL expansion, Reynolds number invariance of the eigenfunctions is paralleled by Reynolds number invariance of the lower-ordered eigenvalues when each quantity is scaled on the outer length scale. As noted earlier, the eigenvalues in Figs. 6(a) and 6(b) are nearly identical for the two largest Reynolds numbers up to $m = 10$ or more. The lowest Reynolds number flow produces similar eigenvalues over a much smaller range, up to perhaps $m = 5$. It represents a condition where viscous effects begin to manifest themselves in the outer flow. At the lowest Reynolds number, the wall layer thickness is approximately one-fourth of the integral scale, and the Taylor microscale is more than one-half of the integral scale. Under these conditions the viscous effects in the outer layer begin to damp the energies of the higher-order modes. The loss of similarity is attributed primarily to interior viscous shock layers and less to the thickening of the viscous wall layers. As Reynolds number increases, the range over which eigenvalue similarity occurs increases correspondingly.

The foregoing observations provide the first evidence of which we are aware, in support of a principle of large scale similarity for the KL eigenfunctions of a two-length scale process: In the physical region outside of the wall layers and in the range of sequences where the eigenfunction wavelength is large compared to the viscous length scale, both the structure of the eigenfunctions and the spectrum of the eigenvalues are independent of Reynolds number when scaled on outer variables.

This similarity principle is, of course, well established for the case of homogeneous turbulence wherein all eigenfunctions are sinusoids for any Reynolds number and power spectral similarity occurs for low wavenumbers when scaled on the outer variables. The import of the present result lies in the extension to structural invariance of the KL eigenfunctions. While the evidence presented here pertains strictly to Burgers' equation, we conjecture that the KL eigenfunctions of three-dimensional turbulence will possess similar properties.

The convergence of the series (13) to $u^2(x)$ is shown in Fig. 12 for all three Reynolds numbers. The convergence is slower for higher Reynolds numbers. The direct numerical result for $u^2(x)$ is the curve labeled $\omega$. Note that the convergence appears to be uniform. Uniform convergence means that the error of a truncated KL representation of $u(x)$ is approximately constant throughout the whole interval. No greater error is incurred in the viscous wall layers than in the homogeneous interior region. The same property is seen in the convergence of (15) to the mean-square fluctuation $\langle u^2(x) \rangle$ (Fig. 13). Uniform convergence of the Karhunen-Loève expansion for both wall layers and the interior region offers a significant advantage over other expansions such as Fourier representations.

It should be noted that the $N$-wave structures of the Burgers' model are not well represented by any single eigenfunction, indicating that the eigenmodes by themselves do not unambiguously define the coherent structures of a random field even if there is a high concentration of energy in one or two modes.

IV. CONCLUSION

In summary, Burgers' equation with random forcing produces solutions that have several properties common to real turbulent flows, including viscous wall layers, internal viscous layers, large scale structures ($N$ waves), and a spectral energy cascade. When the equation is scaled with the interval length the large scale structures in the solution are independent of Reynolds number. Reynolds number dependence occurs for the small scale structures, i.e., the shocks and viscous wall layers. The statistical structure of the solutions in the large interior region is approximately homogeneous. These properties clearly define a process with two greatly separated length scales and make the Burgers'
equation model a reasonable test of the Karhunen–Loéve expansion for representation of inhomogeneous turbulence with a large, inviscid outer length scale and a small viscous inner scale.

The spectrum of eigenvalues is broad and its bandwidth increases with increasing Reynolds number. It is independent of Reynolds number for low sequencies. This is analogous to low wavenumber spectral similarity for real turbulent flows since the sequency defined in our calculations is analogous to a dimensionless wavenumber scaled with the outer length scale, e.g., the interval length. We believe this similarity may also appear in KL expansions of three-dimensional turbulence. The spectrum for high sequencies depends on Re and closely resembles a Fourier power spectrum of the quasihomogeneous interior region. The number of KL modes containing 90% of the energy ranges from four to ten, increasing with Re. This number is slightly larger for the expansion of the fluctuation $u'(x)$. Since the expansion of the fluctuations would require the addition of the mean to represent the total velocity completely, it is more efficient to deal with the direct KL expansion of the total velocity than the expansion of the fluctuation. The width of the eigenvalue spectrum implies that many terms are needed to represent the derivatives of $u(x)$. Thus the convergence of the mean square is not a sufficient criterion to determine the number of terms needed for expansion of $u(x)$ if the expansion is to be used in a numerical scheme. It is sufficient if one is only interested in interpreting the energy containing structures of turbulence physically.

The eigenfunctions also exhibit viscous boundary layers at the end points of the interval where they go to zero. The eigenfunctions are approximately sinusoidal and independent of Re in the interior region. The sinusoidal behavior results from the near homogeneity of $u(x)$ in the interior as seen from the form of the correlation function. This observation, combined with the similarity of the low sequency eigenvalue spectrum, leads us to propose the existence of large scale similarity of the Karhunen–Loéve expansion in the outer region. We conjecture that this large scale Reynolds number similarity will also hold for KL expansions of three-dimensional turbulent flows.

**ACKNOWLEDGMENTS**

This research was supported by NASA Ames Research Center Joint Technical Interchange NCSC 2-131, and by the University of Illinois National Center for Supercomputing Applications (NCSA). The NCSA is supported by the National Science Foundation. HJS is pleased to acknowledge the support of the Korean Science and Engineering Foundation and the University of Illinois.

---

5. D. F. Long and R. E. A. Arndt, in Ref. 4, p. 4.27.