Several Statistical Properties of MUSIC Null-Spectrum

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Abstract

A statistical performance analysis of the multiple signal classification (MUSIC) method is addressed in this paper. It is shown that the estimation error of the sample MUSIC null-spectrum can be decomposed into the biased and unbiased errors, and the statistical properties of the two estimation errors are obtained. We also obtain more exact expressions of the resolution threshold which is used to evaluate the resolution capability for two closely located signal sources.

1 Introduction

The performance study of the sample null-spectrum, e.g., the MUSIC sample null-spectrum, has been performed in [1] by simulation. In [2][3][4] also the statistical properties of the sample MUSIC null-spectrum have been analyzed. For example, the first and second order statistics of the sample MUSIC null-spectrum are obtained in [2], and a more exact expression of the second order statistic is obtained in [3][4].

In statistical performance analysis the resolution capability of the sample null-spectrum for the closely-located two signal sources is of importance, as shown in [2][3]. In this paper we mainly focus on the statistical properties and resolution capability of the sample MUSIC null-spectrum: we obtain a more exact expression for the resolution threshold (RT) without the assumption that two signal sources should be closely-located.

2 Signal Model and Assumptions

Let us consider an array of L sensors of unity gain (unweighted). The array output vector is denoted by $y(t) \in C^{L\times 1}$, where $C^{L\times 1}$ is the space of $L \times 1$ complex valued column vectors. For narrow-band sources, we assume the standard model of observation:

$$y(t) = Ax(t) + n(t), \quad t = 1, 2, \ldots, N, \quad (1)$$

where the column vector $x(t)$ is an $M \times 1$ zero mean complex random vector of source time series as observed at the array phase center, and $n(t)$ is the additive noise vector.

Assumptions on the signal source $x(t)$ and the noise $n(t)$ are as follows:

A1. The $x(t)$, $t = 1, 2, \ldots$, are independent zero mean circular normal random vectors with positive definite covariance matrix $E[x(t)x^H(t)] = R_x$.

A2. The $n(t)$, $t = 1, 2, \ldots$, are also independent circular normal random vectors with zero mean and covariance matrix σI.

A3. The two vectors $x(t)$ and $n(s)$ are statistically independent for any $t, s = 1, 2, \ldots$.

In (1) the matrix A is an $L \times M$ $(L > M)$ complex matrix having the particular structure $A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_M)]$, where θ_i is the DOA of the i-th signal. Here $a(\theta_i) \in C^{L\times 1}$

is called the steering or transfer vector.

The covariance matrix of $y(t)$ is

$$R_y = AR_xA^H + \sigma I,$$

where H denotes the Hermitian: that is, $A^H = (A^*)^T$, $*$ and T stand for the complex conjugate and transpose, respectively.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L$ denote the ordered eigenvalues of R_y. Let the normalized eigenvector corresponding to λ_i be denoted by e_i, $i = 1, 2, \ldots, L$, with which we define two matrices $S = [e_1, e_2, \ldots, e_M]$ and $G = [e_{M+1}, e_{M+2}, \ldots, e_L]$ of sizes $L \times M$ and $L \times (L - M)$, respectively. The ranges of the matrices S and G are called the signal and noise subspaces, respectively. The MUSIC null-spectrum $f(\theta)$ is then defined by

$$f(\theta) = a^H(\theta)GG^Ha(\theta). \qquad (2)$$

In practice, we can not obtain R_y from a finite observation of $y(t)$. Thus let us define $\hat{R}_y = \frac{1}{N}\sum_{t=1}^{N} y(t)y^H(t)$ to be the sample covariance matrix of $\{y(t), t = 1, 2, \ldots, N\}$, which is an estimate of R_y. As we have done in the eigendecomposition of R_y, let $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_L\}$ denote the normalized eigenvectors of $\hat{R}_y$ with the associated eigenvalues being arranged in the order of decreasing magnitude. Note that $\hat{e}_i$, $i = 1, 2, \ldots, L$, are random vectors. In addition let $\hat{S} = [\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_M]$ and $\hat{G} = [\hat{e}_{M+1}, \hat{e}_{M+2}, \ldots, \hat{e}_L]$ be the sample signal and noise subspaces, respectively. The MUSIC sample null-spectrum $\hat{f}(\theta)$ is then defined by

$$\hat{f}(\theta) = a^H(\theta)\hat{G}\hat{G}^Ha(\theta). \qquad (3)$$

3 Results

After a few steps we have the approximation

$$\hat{f}(\theta) \simeq f(\theta) + b(\theta) + v(\theta), \qquad (4)$$

where $b(\theta) = a^H(\theta)SS^H\hat{G}\hat{G}^HSS^Ha(\theta)$ and $v(\theta) = -a^H(\theta)[GG^H\hat{S}S^H + S\hat{S}^HGG^H]a(\theta)$. It is easy to see that for large N (or asymptotically) Equation (4) becomes an identity or

$$\hat{f}(\theta) = f(\theta) + b(\theta) + v(\theta). \qquad (5)$$

Lemma 1: *a)* The normalized error, $2b(\theta)/\sigma_r^2(\theta)$, is asymptotically χ^2 distributed with degree of freedom $2(L - M)$, where

$$\sigma_r^2(\theta) = \frac{1}{N}[\sum_{k=1}^{M} \frac{\lambda_k\sigma}{(\lambda_k - \sigma)^2}|a^H(\theta)e_k|^2]. \qquad (6)$$

b) The error $v(\theta)$ asymptotically has a normal distribution with zero mean and variance

$$\begin{aligned}\sigma_v^2(\theta) &= 2(\sum_{i=1}^{M}|z_i(\theta)|^2w^H(\theta)R_iw(\theta) \\ &+Re[\sum_{i=1}^{M}\sum_{k=1}^{M} z_i(\theta)z_k(\theta)w^H(\theta)V_{ik}w^*(\theta)]), \qquad (7)\end{aligned}$$

where $z_i(\theta) = e_i^Ha(\theta)$ and $w(\theta) = GG^Ha(\theta)$.

Theorem 1 : The asymptotic mean of the sample null-spectrum is given by

$$\begin{aligned}E[\hat{f}(\theta)] &= f(\theta) + \frac{(L-M)}{N}\cdot \\ &[\sum_{k=1}^{M} \frac{\lambda_k\sigma}{(\lambda_k - \sigma)^2}|a^H(\theta)e_k|^2]. \qquad (8)\end{aligned}$$

Theorem 2: When θ is between θ_1 and θ_2 and the difference between θ_1 and θ_2 is sufficiently small enough to $var[v(\theta)] \simeq 0$, we have

$$var[\hat{f}(\theta)] \simeq \frac{(L-M)}{N^2}[\sum_{k=1}^{M} \frac{\lambda_k\sigma}{(\lambda_k - \sigma)^2}|a^H(\theta)e_k|^2]^2. \qquad (9)$$

For $M = 2$ let us define $t_{ij} = a^H(\theta_i)e_j$, $i, j = 1, 2$, where $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$. From the orthogonal projection theorem [5], we have $t_{mj} = w_1t_{1j} + w_2t_{2j}$, where w_1 and w_2 are complex valued scalar quantities. In addition from the quantities t_{ij} we can also obtain the first and second order statistics of the MUSIC null-spectrum without eigendecomposition [6].

Denoting the RT [2][3] by $\bar{\Phi}$, we have $\min_i E[\hat{f}(\theta_m) - \hat{f}(\theta_i)] = 0$ for ASNR = $\bar{\Phi}$. If the ASNR is greater than $\bar{\Phi}$, we can discriminate two signal sources. Conversely, to discriminate two closely-located signal sources the ASNR should be greater than $\bar{\Phi}$. In [6] a more exact value of $\bar{\Phi}$ is obtained directly from the array geometry and parameters of the signal sources.

4 Simulation Result

Let us consider a uniform linear array of 10 sensors ($L = 10$) with sensor spacing half the wavelength. When the number of signal sources M is 2 with its DOA's being 15° and 20°, some illustrations of the sample null-spectra and mean null-spectrum are shown in Fig. 1. In Table 1 the RT's for various cases are shown, which are obtained based on our technique. Computer simulations show that the values of the RT is quite exact. In the table it is seen that the RT is increased when the correlation of the two signal sources is increased and/or the difference of the two DOA's becomes smaller. In addition when the ratio of the two signal source powers ν is smaller, the RT is increased.

5 Summary

Based on some approximations we can decompose the estimation error of the sample MUSIC null-spectrum into two errors, the biased and unbiased errors. We see that the biased and unbiased estimation errors have asymptotically χ^2 and normal distributions, respectively. This result implies that the sample MUSIC null-spectrum is a biased estimate of the MUSIC null-spectrum. The mean and variance of the sample MUSIC null-spectrum are obtained analytically, and confirmed by simulation results. It is seen that the asymptotic mean and variance for correlated signal sources are smaller than those for uncorrelated signal sources. We also obtain more exact expressions of the resolution threshold which is used to evaluate the resolution capability for two closely located signal sources.

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corr. coeff.	$\rho = 0 + i0$		$\rho = 0.5 + i0.5$	
$\nu = P_2/P_1$	$\nu = 1$	$\nu = 0.5$	$\nu = 1$	$\nu = 0.5$
RT (ASNR, dB)	38.18	42.11	43.25	46.42
$E[\hat{f}(\theta_1)]$	1.23e-04 (1.24e-04)	4.93e-05 (4.98e-05)	7.65e-05 (7.59e-05)	3.68e-05 (3.65e-05)
$E[\hat{f}(\theta_2)]$	1.23e-04 (1.25e-04)	9.89e-05 (1.01e-04)	7.65e-05 (7.81e-05)	7.36e-05 (7.53e-05)
$E[\hat{f}(\theta_m)]$	1.23e-04 (1.24e-04)	9.84e-05 (9.89e-05)	7.65e-05 (7.74e-05)	7.33e-05 (7.36e-05)
$E[\hat{f}(\theta_m) - \hat{f}(\theta_1)]$	2.47e-11 (4.92e-07)	4.92e-05 (4.91e-05)	9.27e-13 (1.53e-06)	3.65e-05 (3.71e-05)
$E[\hat{f}(\theta_m) - \hat{f}(\theta_2)]$	2.47e-11 (-1.09e-06)	-4.08e-07 (-2.34e-06)	9.27e-13 (-7.45e-07)	-3.51e-07 (-1.71e-06)

Table 1. The resolution threshold when two DOA's are 15° and 18° and the mean and variance of the sample null-spectrum at the resolution threshold (the values in parantheses are obtained by computer simulation with 100 trials and those without parantheses are theoretical values).*

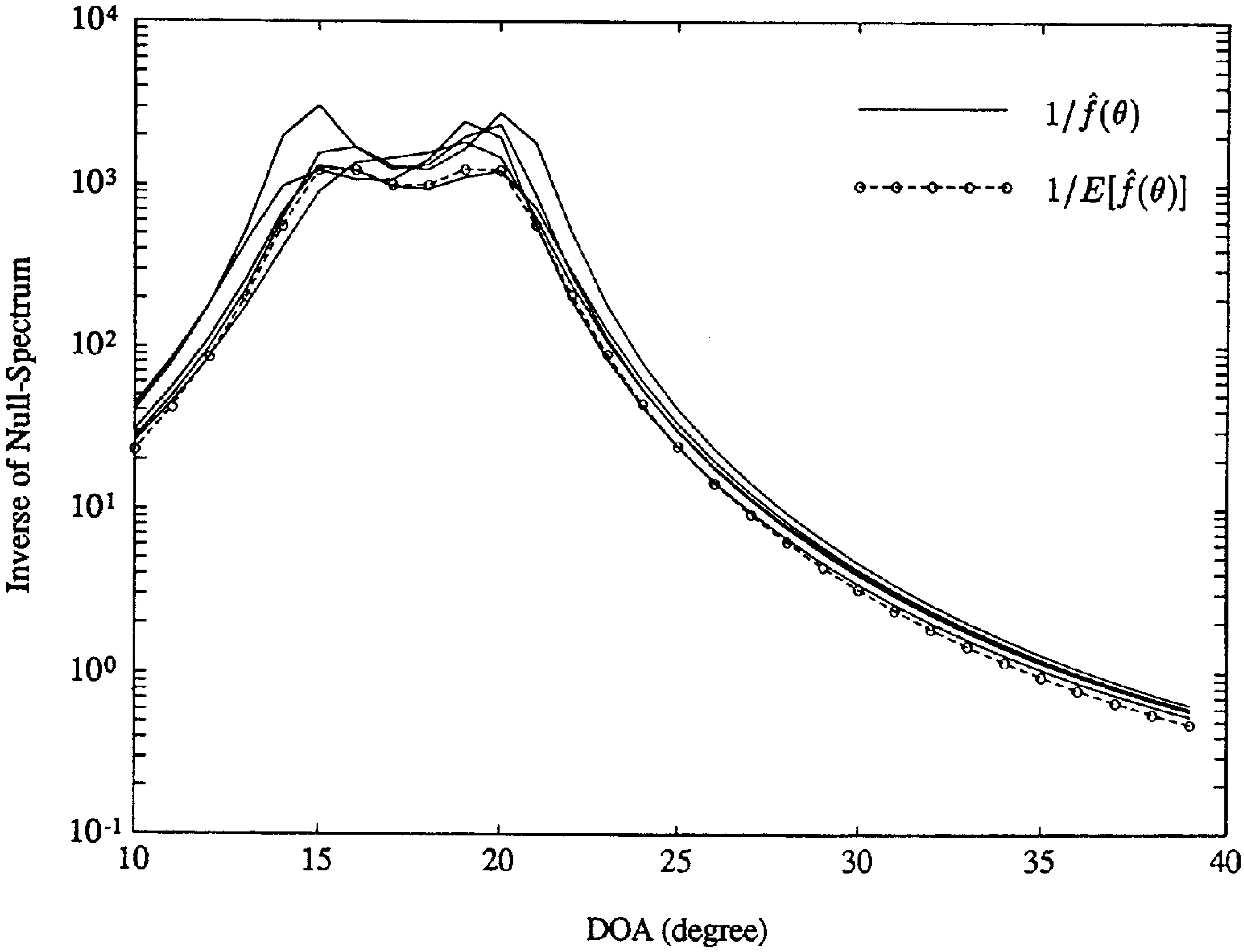

Figure 1. Inverse of the sample null-spectra and mean null-spectrum.