Ecology-inspired Evolutionary Algorithm using Feasibility-based Grouping for Constrained Optimization

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Abstract: When evolutionary algorithms are used for solving numerical constrained optimization problems, how to deal with the relationship between feasible and infeasible individuals can directly influence the final results. This paper proposes a novel ecology-inspired EA to balance the relationship between feasible and infeasible individuals. According to the feasibility of the individuals, the population is divided into two groups, feasible group and infeasible group. The evaluation and ranking of these two groups are performed separately. The number of parents from feasible group has a sigmoid relation with that of feasible individuals, and then rank the infeasible individuals with feasible individuals together [1], [2], [3]. Some other methods [4] directly assume that feasible individuals are always fitter than infeasible ones. However, this kind of view ignores one important thing that evolutionary algorithm is a probabilistic and recurrent method. It is possible that some of the infeasible individuals carry more useful information than feasible individuals during evolution process. Moreover, quite often the system can reach the optimal point more easily if it is possible to “cross” an infeasible region (especially in non-convex feasible search space).

There have been some work on adjusting the relationship between feasible and infeasible individuals. In [5], [6], feasible and infeasible individuals were evaluated with different criteria. Other method like GENOCOP III [7] repaired the infeasible individuals for evaluation. However, for each particular problem, a specific repair strategy needs to be designed.

In this paper, based on the idea of fully utilizing the useful information of infeasible individuals, a novel evolutionary algorithm using feasibility-based grouping (EA_FG) is proposed for constrained optimization problems. In each generation, according to the feasibility of the individuals, the whole population is divided into two groups: feasible group and infeasible group. Evaluation and ranking of these two groups are performed in parallel and separately. The best individuals from feasible and infeasible groups are selected together as parents. The number of feasible parents could be thought to have better fitness values than infeasible individuals (here “better” means to have more chance to survive and reproduce). Since all constrains of feasible individuals have already been satisfied, the only aim left is to find \( \vec{x} \) minimize \( f(\vec{x}) \). Most of the existing evolutionary algorithms for constrained optimization problems follow such an idea and, more or less, underrated the importance of the infeasible individuals. For example, nearly all of the penalty methods add some “penalties” to the fitness functions of the infeasible individuals, and then rank the infeasible individuals with feasible individuals together [1], [2], [3]. Some other methods [4] directly assume that feasible individuals are always fitter than infeasible ones.

When evolutionary algorithms are adopted to solve constrained optimization problems, intuitively, feasible individuals could be thought to have better fitness values than infeasible individuals (here “better” means to have more chance to survive and reproduce). Since all constraints of feasible individuals have already been satisfied, the only aim left is to find \( \vec{x} \) minimize \( f(\vec{x}) \). Most of the existing evolutionary algorithms for constrained optimization problems follow such an idea and, more or less, underrated the importance of the infeasible individuals. For example, nearly all of the penalty methods add some “penalties” to the fitness functions of the infeasible individuals, and then rank the infeasible individuals with feasible individuals together [1], [2], [3]. Some other methods [4] directly assume that feasible individuals are always fitter than infeasible ones.

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1 Introduction

The general constrained optimization problem \((P)\) is to find \( \vec{x} \) so as to

\[
\min_{\vec{x}} f(\vec{x}), \quad \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n
\]

(1)

where \( \vec{x} \in F \subseteq S \). The objective function \( f \) is defined on the search space \( S \subseteq \mathbb{R}^n \) and the set \( F \subseteq S \) defines the feasible region. Usually, the search space \( S \) is defined as an \( n \)-dimensional rectangle in \( \mathbb{R}^n \) (domains of variables defined by their lower and upper bounds):

\[
l(j) \leq x_j \leq u(j), \quad j = 1, \ldots, n
\]

(2)

where the feasible region \( F \subseteq S \) is defined by a set of \( m \) additional constraints (\( m \geq 0 \)):

\[
g_k(\vec{x}) \leq 0, \quad k = 1, \ldots, l,
\]

(3)

\[
h_k(\vec{x}) = 0, \quad k = l + 1, \ldots, m.
\]

(4)

Any point \( \vec{x} \in F \) is called a feasible solution, otherwise, \( \vec{x} \) is an infeasible solution.

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Any existing evolutionary algorithms for constrained optimization problems, which evaluate and rank feasible and infeasible individuals together, can be incorporated into EA_FG to improve the performance. In this paper, a dynamic penalty method is incorporated into EA_FG to test the effectiveness of EA_FG. The initial study on the EA_FG can be found in [8], [9], [10].

This paper is organized as follows. Section II presents a detailed description of the overall structure of EA_FG. Also, an ecology-inspired parent selection mechanism is discussed. In Section III, experimental results on 13 benchmark problems are presented and compared with a dynamic penalty method. Finally, Section IV concludes with some remarks.

2 EA_FG for Constrained Optimization

2.1 Overall Structure of EA_FG

Fig. 1 shows the flowchart of EA_FG. In the beginning of every generation, the whole population is divided into two groups: feasible group and infeasible group according to the feasibility of every individual. Then, the evaluation and ranking of these two groups are performed in parallel and separately. In order to share information of these two groups, the best feasible and infeasible individuals are selected as parent population. Parent population reproduces and generates offspring population. The offspring population is to be divided into feasible and infeasible groups again. The iteration keeps on working until the termination condition is satisfied.

When EA_FG is used to solve constrained optimization problems, two aspects should be noted. The first one is how to perform evaluation and ranking of infeasible groups. The second one is how to select parents from feasible and infeasible groups, in another words, how to decide the number of feasible parents and the number of infeasible parents. For the first aspect, any existing evaluation and ranking method for the whole population, can be adopted to evaluate and rank the infeasible group. In this paper, the dynamic penalty method [1] will be tried. For the second aspect, a novel method inspired by the natural population growth in a confined space will be used.

2.2 Ecological-inspired Parent Selection

In nature, the growth of a simple population in a confined space, where resources are not unlimited, is simply described by a graph that always looks sigmoid (Fig. 2(a)) [11]. In the early stage, resources are abundant, the death rate is minimal and the reproduction can take place as fast as possible. The population increases geometrically until an upper limit is approached. This upper limit, or saturation value is a constant for a particular set of conditions in a par-
paticular habitat and is called the carrying capacity \((k)\). The population growth rate declines to zero as the population becomes more crowded and the population size stabilizes at the maximum that the environment can support, reaching an equilibrium population density. Fig. 2(b) shows one observed example: paramecium population growth.

Inspired by this phenomenon, a novel parent selection strategy is generated for EA\(_{FG}\). The number of feasible parents which are selected from the feasible group and the number of feasible individuals will follow a sigmoid relation as follows:

\[
\text{i) } \text{numFeaPar} = \text{ceil}(\text{sig}(\text{numFeaInd})),
\]

\[
\text{if } \text{numFeaPar} > \text{numPar}, \text{numFeaPar} = \text{numPar}. \tag{5}
\]

\[
\text{ii) } \text{numInfeaPar} = \text{numPar} - \text{numFeaPar}. \tag{6}
\]

where \(\text{numFeaPar}\) represents ‘number of feasible parents,’ \(\text{numFeaInd}\) ‘number of feasible individuals,’ \(\text{numPar}\) ‘number of parents,’ \(\text{numInfeaPar}\) ‘number of infeasible parents,’ \(\text{sig}(x)\) denotes a sigmoid-type equation, \(\text{ceil}(x)\) rounds the elements of \(x\) to the nearest integers towards infinity. Note that (6) restricts \(\text{numFeaPar}\) not to exceed the predefined number of parents (\(\text{numPar}\)). Once numbers of feasible and infeasible parents are decided, the corresponding numbers of best individuals are selected from the feasible and infeasible groups, respectively, according to their ranking.

To explain this strategy more clearly, assume that (30, 200)-ES is used for computation. Fig. 3 shows an example of the sigmoid relation between the number of feasible parents and the number of feasible individuals. (5) can be calculated by the following equation:

\[
\text{numFeaPar} = \text{ceil}\left(\frac{30}{1 + 30 \times \exp(-0.05 \times \text{numFeaInd})}\right),
\]

where 30 is the limit of \(\text{numFeaPar}\), and -0.05 is set to make the curve similar to a natural population growth curve like Fig. 2(b). If \(\text{numFeaInd} = 20\), (8) gives \(\text{numFeaPar} = 3\). And by (7), \(\text{numInfeaPar} = 30 - 3 = 27\). Therefore, the best 3 individuals are selected from the feasible group and best 27 individuals from the infeasible group, thus to form 30 parents for reproduction. Also, if \(\text{numFeaInd} = 180\), (8) gives \(\text{numFeaPar} = 30\). And by (7), \(\text{numInfeaPar} = 30 - 30 = 0\). In this case, all 30 parents are from the feasible group.

3 Experimental Studies

Since EA\(_{FG}\) is an open structure algorithm, any evolutionary algorithm for numerical constrained optimization problems can be employed for the infeasible group. In this section, a dynamic penalty method was adopted to rank and evaluate the infeasible group of EA\(_{FG}\), which would be called ‘EA\(_{FG}\_D\)’ in the later part of this paper. EA\(_{FG}\_D\) was tested and the results were compared with those of the dynamic penalty method in [12] on 13 benchmark functions. The details of these functions are listed in Appendix. Problems G2, G3, G8 and G12 are maximization problems. They were transformed into minimization problems using \(-f(\vec{x})\). Problems G3, G5, G11 and G13 include one or several equality constraints. All of these equality constraints were converted into inequality constraints, \(|h(\vec{x})| - \delta \leq 0\), using the degree of violation \(\delta = 0.0001\).
A \((\mu, \lambda)\)-ES was employed for recombination and mutation. For impartial comparison, all parameters of the ES used here were the same as those of [12]. For each of the benchmark problems, 30 independent runs were performed using \((30, 200)\)-ES. The termination condition was set to be 1,750 generations. The initial population of \(x\) was generated according to a uniform \(n\)-dimensional probability distribution over the search space \(S\). The ecology-inspired parent selection in Fig. 3 was adopted.

Table 1 and Table 2 show the simulation results with a dynamic penalty method and EA\(_{FG,D}\), respectively. The median number of generations for finding the best solution in each run is indicated by \(g_m\) in the tables. The tables also show the known ‘optimal’ solution for each problem and statistics for the 30 independent runs. Best, median, worst and mean values of 30 runs were used as performance criteria. 13 problems totally have \(13 \times 4 = 52\) criteria. Among the 52 criteria,

- EA\(_{FG,D}\) performed better on 27 criteria,
- EA\(_{FG,D}\) performed worse on 6 criteria, and
- EA\(_{FG,D}\) performed the same on 19 criteria.

For problems G1, G8 and G11, both algorithms performed well and found the optimal solutions for all 30 runs. For problem G10, both algorithms failed to find the optimal solution. For problem G2, EA\(_{FG,D}\) provided ‘similar’ results to the dynamic penalty method. It performed better on \(\text{worst}\) and \(\text{mean}\), but worse on \(\text{best}\) and \(\text{median}\) than the dynamic penalty method. For the rest of problems, EA\(_{FG,D}\) outperformed the dynamic penalty method except problem G13. For problem G3, \(\text{best}\) of the dynamic penalty method was \(-0.583\), while \(\text{best}\) of EA\(_{FG,D}\) could reach the optimal value \(-1.000\). For problems G4, G6 and G9, EA\(_{FG,D}\) performed significantly better in terms of all four criteria. It should be noted that for problem G5, EA\(_{FG,D}\) could find a feasible solution 3 times out of 30 runs, while the dynamic penalty method failed to find the solution.

4 Conclusion

This paper proposed a new constraint handling technique: ecology-inspired evolutionary algorithm using feasibility-based grouping. This method divides the population into two groups, feasible group and infeasible group according to the feasibility of the individuals. The evaluation and ranking of these two groups are performed in parallel and separately. The number of feasible parents has a sigmoid-type relation with that of feasible individuals which is inspired by the ecological population growth in a confined space in nature. In addition, a dynamic penalty method was modified and included into EA\(_{FG,D}\) to evaluate and rank the infeasible group. EA\(_{FG,D}\) was tested on a set of 13 benchmark problems. Experimental results showed EA\(_{FG,D}\) could improve the performance of the dynamic penalty method on most problems.

5 Acknowledgments

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Bibliography


Table 2: Experimental Results of EA_FG_D; The Subscript 4 in the Function Name G5(3) Indicates 3 Feasible Solutions Found Among 30 tests; “−” Means no Feasible Solutions Were Found.

<table>
<thead>
<tr>
<th>Function</th>
<th>optimal</th>
<th>best</th>
<th>median</th>
<th>worst</th>
<th>mean</th>
<th>st. dev.</th>
<th>$g_m$</th>
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</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>−15.000</td>
<td>−15.000</td>
<td>−15.000</td>
<td>−15.000</td>
<td>−15.000</td>
<td>6.3E−06</td>
<td>676</td>
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<tr>
<td>$G_2$</td>
<td>−0.803619</td>
<td>−0.803567</td>
<td>−0.785643</td>
<td>−0.767080</td>
<td>−0.787585</td>
<td>1.1E−02</td>
<td>1110</td>
</tr>
<tr>
<td>$G_3$</td>
<td>−1.000</td>
<td>−1.000</td>
<td>−0.088</td>
<td>−0.001</td>
<td>−0.121</td>
<td>1.8E−01</td>
<td>1743</td>
</tr>
<tr>
<td>$G_4$</td>
<td>−30665.539</td>
<td>−30665.457</td>
<td>−30561.593</td>
<td>−30185.263</td>
<td>−30540.911</td>
<td>1.2E+02</td>
<td>545</td>
</tr>
<tr>
<td>$G_{5(3)}$</td>
<td>5126.498</td>
<td>5208.553</td>
<td>5377.300</td>
<td>5462.911</td>
<td>5349.588</td>
<td>1.1E+02</td>
<td>1267</td>
</tr>
<tr>
<td>$G_6$</td>
<td>−6961.814</td>
<td>−6961.062</td>
<td>−6651.928</td>
<td>−6179.194</td>
<td>−6652.052</td>
<td>2.2E+02</td>
<td>13</td>
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<tr>
<td>$G_8$</td>
<td>−0.095825</td>
<td>−0.095825</td>
<td>−0.095825</td>
<td>−0.095825</td>
<td>−0.095825</td>
<td>0.0E+00</td>
<td>369</td>
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<tr>
<td>$G_9$</td>
<td>680.630</td>
<td>680.630</td>
<td>680.637</td>
<td>680.677</td>
<td>680.643</td>
<td>1.3E−02</td>
<td>1632</td>
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<tr>
<td>$G_{10}$</td>
<td>7049.331</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$G_{11}$</td>
<td>0.750</td>
<td>0.750</td>
<td>0.750</td>
<td>0.750</td>
<td>0.750</td>
<td>2.2E−07</td>
<td>610</td>
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<tr>
<td>$G_{12}$</td>
<td>−1.000000</td>
<td>−1.000000</td>
<td>−0.999997</td>
<td>−0.9999905</td>
<td>−0.999968</td>
<td>2.3E−05</td>
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<tr>
<td>$G_{13}$</td>
<td>0.053950</td>
<td>0.742678</td>
<td>0.997744</td>
<td>0.998500</td>
<td>0.984344</td>
<td>4.7E−02</td>
<td>1749</td>
</tr>
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</table>


A Test Function Suite

Problem $G_1$: Minimize

$$f(\vec{x}) = 5 \sum_{j=1}^{4} x_j - 5 \sum_{j=1}^{4} x_j^2 - \sum_{j=5}^{13} x_j,$$

subject to

$$g_1(\vec{x}) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0,$$

$$g_2(\vec{x}) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0,$$

$$g_3(\vec{x}) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0,$$

$$g_4(\vec{x}) = -8x_1 + x_{10} \leq 0,$$

$$g_5(\vec{x}) = -8x_2 + x_{11} \leq 0,$$

$$g_6(\vec{x}) = -8x_3 + x_{12} \leq 0,$$

$$g_7(\vec{x}) = -2x_4 - x_5 + x_{10} \leq 0,$$

$$g_8(\vec{x}) = -2x_5 - x_7 + x_{11} \leq 0,$$

$$g_9(\vec{x}) = -2x_8 - x_9 + x_{12} \leq 0,$$

and bounds

$$0 \leq x_j \leq 1 \ (j = 1, \ldots, 9),$$

$$0 \leq x_j \leq 100 \ (j = 10, 11, 12),$$

$$0 \leq x_{13} \leq 1.$$

The global minimum is at $\vec{x}^* = (1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 1)$, and $f(\vec{x}^*) = -15$.

Problem $G_2$: Maximize

$$f(\vec{x}) = \left| \sum_{j=1}^{n} \cos x_j - 2 \prod_{j=1}^{n} \cos^2 x_j \right|,$$

$$\sqrt{\sum_{j=1}^{n} x_j^2}.$$
subject to
\[ g_1(\vec{x}) = 0.75 - \sum_{j=1}^{n} x_j \leq 0, \]
\[ g_2(\vec{x}) = \sum_{j=1}^{n} x_j - 7.5n \leq 0, \]
and bounds
\[ 0 \leq x_j \leq 10 \quad (j = 1, \ldots, n), \]
where \( n = 20 \). The global maximum is unknown; the known solution is \( f(\vec{x}^*) = 0.803619 \).

Problem G3: Maximize
\[ f(\vec{x}) = (\sqrt{n})^n \sum_{j=1}^{n} x_j, \]
subject to
\[ h_1(\vec{x}) = \sum_{j=1}^{n} x_j^2 - 1 = 0, \]
and bounds
\[ 0 \leq x_j \leq 1 \quad (j = 1, \ldots, n), \]
where \( n = 10 \). The global minimum is at \( x_j^* = 1/\sqrt{n} \quad (j = 1, \ldots, n) \), and \( f(\vec{x}) = 1 \).

Problem G4: Minimize
\[ f(\vec{x}) = 5.3578547x_2^3 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141, \]
subject to
\[ g_1(\vec{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \leq 0, \]
\[ g_2(\vec{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \leq 0, \]
\[ g_3(\vec{x}) = 80.51249 + 0.0071317x_3x_5 + 0.0029955x_1x_2 + 0.0021813x_2^3 - 110 \leq 0, \]
\[ g_4(\vec{x}) = -80.51249 - 0.0071317x_3x_5 - 0.0029955x_1x_2 - 0.0021813x_2^3 + 90 \leq 0, \]
\[ g_5(\vec{x}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \leq 0, \]
\[ g_6(\vec{x}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \leq 0, \]
and bounds
\[ 78 \leq x_1 \leq 102, 33 \leq x_2 \leq 45, 27 \leq x_j \leq 45, \quad (j = 3, 4, 5). \]
The optimal solution is at \( \vec{x}^* = (78, 33, 29.995256025682, 45, 36.775812905788) \), and \( f(\vec{x}^*) = -30665.539 \).

Problem G5: Minimize
\[ f(\vec{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_3^2, \]
subject to
\[ g_1(\vec{x}) = -x_4 + x_3 - 0.55 \leq 0, \]
\[ g_2(\vec{x}) = -x_3 + x_4 - 0.55 \leq 0, \]
\[ h_1(\vec{x}) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x_4 - 0.25) + 894.8 - x_1 = 0, \]
\[ h_2(\vec{x}) = 1000\sin(x_3 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 894.8 - x_2 = 0, \]
\[ h_3(\vec{x}) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 = 0, \]
and bounds
\[ 0 \leq x_1 \leq 1200, 0 \leq x_2 \leq 1200, \]
\[ -0.55 \leq x_3 \leq 0.55, -0.55 \leq x_4 \leq 0.55. \]
The best known solution is at \( \vec{x}^* = (679.9453, 1026.067, 0.1188764, -0.3962336) \), and \( f(\vec{x}^*) = 5126.4981 \).

Problem G6: Minimize
\[ f(\vec{x}) = (x_1 - 10)^3 + (x_2 - 20)^3, \]
subject to
\[ g_1(\vec{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0, \]
\[ g_2(\vec{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0, \]
and bounds
\[ 13 \leq x_1 \leq 100, 0 \leq x_2 \leq 100. \]
The known global solution is \( \vec{x}^* = (14.095, 0.84296) \), and \( f(\vec{x}^*) = -6961.81388 \).

Problem G7: Minimize
\[ f(\vec{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45, \]
subject to
\[ g_1(\vec{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 0, \]
\[ g_2(\vec{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0, \]
\[ g_3(\vec{x}) = -8x_1 + 2x_2 + 5x_3 - 2x_{10} - 12 \leq 0, \]
\[ g_4(\vec{x}) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0, \]
\[ g_5(\vec{x}) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \leq 0, \]
\[ g_6(\vec{x}) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_4 - 6x_6 \leq 0, \]
\[ g_7(\vec{x}) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_3^2 - x_7 - 30 \leq 0, \]
\[ g_8(\vec{x}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \leq 0, \]
and bounds
\[ -10 \leq x_j \leq 10 \quad (j = 1, \ldots, 10). \]
The optimal solution is at \( \vec{x}^* = (2.171996, 2.363683, 8.773926, 5.095894, 0.9906548, 1.430574, 1.321644, 9.828726, 8.280092, 8.375927) \), and \( f(\vec{x}^*) = 24.3062901 \).

Problem G8: Maximize
\[ f(\vec{x}) = \frac{\sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^3(x_1 + x_2)}, \]
subject to
\[ g_1(\vec{x}) = x_1^2 - x_2 + 1 \leq 0, \]
\[ g_2(\vec{x}) = 1 - x_1 + (x_2 - 4)^2 \leq 0, \]
and bounds
\[ -10 \leq x_1 \leq 10, -10 \leq x_2 \leq 10. \]
The optimal solution is at \( \vec{x}^* = (1.2279713, 4.2453733) \),
and \( f(\vec{x}^*) = 0.095825. \)

Problem G9 : Minimize
\[ f(\vec{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 \]
\[ + 10x_5^2 + 7x_6^2 + x_7^4 - 4x_7x_8 - 10x_9 - 8x_7, \]
subject to
\[ g_1(\vec{x}) = -127 + 2x_1^2 + 3x_2^3 + x_3 + 4x_4^2 + 5x_5 \leq 0, \]
\[ g_2(\vec{x}) = -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0, \]
\[ g_3(\vec{x}) = -196 + 23x_1 + x_2^2 + 6x_3^2 - 8x_7 \leq 0, \]
\[ g_4(\vec{x}) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3 + 5x_6 - 11x_7 \leq 0, \]
and bounds
\[ -10 \leq x_j \leq 10 (j = 1, \ldots, 7). \]
The known global solution is at \( \vec{x}^* = (2.330499, 1.951372, \)
\[ -0.4775414, 4.365726, -0.6244870, 1.038131, 1.594227), \]
and \( f(\vec{x}^*) = 680.6300573. \)

Problem G10 : Minimize
\[ f(\vec{x}) = x_1 + x_2 + x_3, \]
subject to
\[ g_1(\vec{x}) = -1 + 0.0025(x_4 + x_6) \leq 0, \]
\[ g_2(\vec{x}) = -1 + 0.0025(x_5 + x_7 - x_4) \leq 0, \]
\[ g_3(\vec{x}) = -1 + 0.01(x_8 - x_5) \leq 0, \]
\[ g_4(\vec{x}) = -x_1x_6 + 833.332522x_4 + 100x_1 - 8333.33333 \leq 0, \]
\[ g_5(\vec{x}) = -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \leq 0, \]
\[ g_6(\vec{x}) = -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \leq 0, \]
and bounds
\[ 100 \leq x_1 \leq 10000, 1000 \leq x_j \leq 10000 (j = 2, 3), \]
\[ 10 \leq x_j \leq 1000 (j = 4, \ldots, 8). \]
The optimal solution is at \( \vec{x}^* = (579.3167, 1359.943, \)
\[ 5110.071, 182.0174, 295.5985, 217.9799, 286.4162, 395.5979), \]
and \( f(\vec{x}^*) = 7049.3307. \)

Problem G11 : Minimize
\[ f(\vec{x}) = x_1^2 + (x_2 - 1)^2, \]
subject to
\[ h_1(\vec{x}) = x_2 - x_1^2 = 0, \]
and bounds
\[ -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1. \]