We investigate quantum teleportation through noisy quantum channels by solving analytically and numerically a master equation in the Lindblad form. We calculate the fidelity as a function of decoherence rates and angles of a state to be teleported. It is found that the average fidelity and the range of states to be accurately teleported depend on types of noises acting on quantum channels. If the quantum channels are subject to isotropic noise, the average fidelity decays to 1/2, which is smaller than the best possible value of 2/3 obtained only by the classical communication. On the other hand, if the noisy quantum channel is modeled by a single Lindblad operator, the average fidelity is always greater than 2/3.

Let us consider quantum teleportation through noisy channels as illustrated in Fig. 1. The top two qubits are taken by Alice and the bottom qubit is kept by Bob. Here, measurements are performed at the end of the circuit for computational convenience. Classical conditional operations can be replaced with corresponding quantum conditional operations [9]. Decoherence of an open quantum system is due to the interaction with its environment. Under the assumption of Markov and Born approximations and after tracing out the environmental degrees of freedom, the dynamics of an open quantum system is described by a master equation for the density operator of the quantum system alone, $\rho(t)$, in the Lindblad form [10,11]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_S,\rho] + \sum_{i,a} \left( L_{i,a} r L_{i,a}^\dagger \rho L_{i,a}^\dagger L_{i,a} - \frac{1}{2} L_{i,a}^\dagger L_{i,a} \rho \right),$$

where the Lindblad operator $L_{i,a} = \sqrt{\kappa_{i,a}(t)} \sigma_a(i)$ acts on the $i$th qubit and describes decoherence. Throughout this paper, $\sigma_a(i)$ denotes the Pauli spin matrix of the $i$th qubit with $\sigma = x,y,z$. The decoherence time is approximately given by $1/\kappa_{i,a}$. We could control noise by switching $\kappa_{i,a}(t)$ on and off. We take the Hamiltonian of a qubit system as an ideal model of a quantum computer, which is given by $[12]$

![FIG. 1. A circuit for quantum teleportation through noisy channels. The two top lines belong to Alice, while the bottom one belongs to Bob. $M$ represents measurement. The dotted boxes $A$, $B$, $C$, and $D$ denote noisy channels. Time advances from left to right. During the time interval corresponding to the width of the dotted box, the Lindblad operator is turned on.](image-url)
\[ H_S(t) = -\frac{1}{2} \sum_{i=1}^{N} \mathbf{B}^{(i)}(t) \cdot \mathbf{\sigma}^{(i)} - \sum_{i < j} J_{ij}(t) \mathbf{\sigma}_+^{(i)} \mathbf{\sigma}_-^{(j)}, \]  

(2)

where \( \mathbf{\sigma}^{(i)} = (\mathbf{\sigma}_x^{(i)}, \mathbf{\sigma}_y^{(i)}, \mathbf{\sigma}_z^{(i)}) \) and \( \mathbf{\sigma}_\pm^{(i)} = \frac{1}{2} (\mathbf{\sigma}^{(i)} \pm i \mathbf{\sigma}^{(i)}) \). In solid-state qubits, various types of couplings between qubits \( i \) and \( j \) are possible such as the X-Y coupling given above, the Heisenberg coupling, and the Ising coupling \( J_{ij}\mathbf{\sigma}_z^{(i)} \mathbf{\sigma}_z^{(j)} \) in nuclear magnetic resonance. The various quantum gates in Fig. 1 could be implemented by a sequence of pulses, i.e., by turning on and off \( \mathbf{B}^{(i)}(t) \) and \( J_{ij}(t) \). We develop the simulation code that solves Eq. (1), the set of differential equations for the density matrix \( \rho_{\text{out}}(t) \), based on the Runge-Kutta method \[\text{[13]}.\] Equation (1) shows \( \text{Tr}_p(t) = 1 \) at all times.

An unknown state to be teleported can be written as \( |\psi_{\text{in}}\rangle = |\alpha\rangle |0\rangle + |\beta\rangle |1\rangle \) with \( |\alpha|^2 + |\beta|^2 = 1 \). It is convenient to rewrite \( |\psi_{\text{in}}\rangle \) as a Bloch vector on a Bloch sphere,

\[ |\psi_{\text{in}}\rangle = \cos \left( \frac{\theta}{2} \right) e^{i \phi_{\text{in}} |0\rangle} + \sin \left( \frac{\theta}{2} \right) e^{-i \phi_{\text{in}} |1\rangle}, \]

(3)

where \( \theta \) and \( \phi \) are the polar and azimuthal angles, respectively. The maximally entangled state of two spin-1/2 particles shared and kept by Alice and Bob is given by

\[ |\beta_{00}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \]

(4)

The input state of the quantum teleportation circuit in Fig. 1 is the product state of \( |\psi_{\text{in}}\rangle \) and \( |\beta_{00}\rangle \). After the implementation of the quantum circuit of Fig. 1 and the measurement of the top two qubits, Bob gets the teleported state \( |\psi_{\text{out}}\rangle \). It is useful to describe the teleportation in terms of density operators

\[ \rho_{\text{out}} = \text{Tr}_{1,2} U_{\text{tel}} \rho_{\text{in}} \otimes \rho_{\text{en}} U^\dagger_{\text{tel}}, \]

(5)

where \( \rho_{\text{in}} = |\psi_{\text{in}}\rangle \langle \psi_{\text{in}}| \), \( \rho_{\text{en}} = |\beta_{00}\rangle \langle \beta_{00}| \), and \( \text{Tr}_{1,2} \) is a partial trace over qubits 1 and 2. The unitary operator \( U_{\text{tel}} \) is implemented by the teleportation circuit as shown in Fig. 1. If the teleportation is ideal, the density-matrix teleported \( \rho_{\text{out}} \) is identical to \( \rho_{\text{in}} \) up to the normalization factor.

As illustrated as dotted boxes in Fig. 1, we consider four different noisy channels, \( A, B, C, \) and \( D \). In case \( A \) an unknown state \( |\psi_{\text{in}}\rangle \) loses its coherence and becomes a mixed state before it is teleported. In case \( B \) an entangled pair, forming a quantum channel, becomes noisy while being shared and kept by Alice and Bob. In cases \( C \) and \( D \), while Alice and Bob perform the Bell measurement and the unitary operation, respectively, noise may set in. For cases \( A \) and \( B \) we obtain both analytic and numerical solutions of Eq. (1), while in cases \( C \) and \( D \) the numerical solutions of Eq. (1) are obtained. For our numerical calculation, \( \tau_{\text{opt}}(t) \) is turned on for the time interval \( \tau \) corresponding to the width of each dotted box in Fig. 1.

The properties of quantum teleportation through noisy quantum channels are quantified by the fidelity that measures the overlap between a state \( |\psi_{\text{in}}\rangle \) to be teleported and the density operator \( \rho_{\text{out}} \) for a teleported state,

\[ F(\theta, \phi) = \langle \psi_{\text{in}} | \rho_{\text{out}} | \psi_{\text{in}} \rangle. \]

(6)

Here the fidelity \( F(\theta, \phi) \) depends on an input state as well as the type of noise acting on qubits. We calculate \( F(\theta, \phi) \) and determine the range of states \( |\psi_{\text{in}}\rangle \) which can be accurately teleported. Since in general a state to be teleported is unknown, it is more useful to calculate the average fidelity given by

\[ F_{av} = \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta F(\theta, \phi), \]

(7)

where \( 4\pi \) is the solid angle.

Case A: States to be teleported are mixed. Alice is not able to know or copy the state to be teleported without disturbing it. So it may be pure or mixed. As Bennett et al. \[\text{[1]}\] noted, the linear property of quantum teleportation enables one to teleport not only a pure state but also a mixed state. The quantum operation \( \mathcal{E} \) transforms a pure state \( \rho_{\text{in}} = |\psi_{\text{in}}\rangle \langle \psi_{\text{in}}| \) to a mixed state \( \mathcal{E}(\rho_{\text{in}}) \). The time evolution of pure states to mixed states is described by Eq. (1) (see Ref. \[\text{[9]}\] for the connection between the two approaches). From Eq. (5), quantum teleportation of mixed states reads

\[ \mathcal{E}(\rho_{\text{out}}) = \text{Tr}_{1,2} U_{\text{tel}} \mathcal{E}(\rho_{\text{in}}) \otimes \rho_{\text{en}} U^\dagger_{\text{tel}}. \]

(8)

The decoherence of the state to be teleported, \( \mathcal{E}(\rho_{\text{in}}) \), is transferred to the state teleportated, \( \mathcal{E}(\rho_{\text{out}}) \). For various types of noises, we obtain both analytic and numerical solutions of Eq. (1) and calculate the fidelity.

Suppose a state to be teleported is subject to the noise \( L_{1,2} \). It is easy to find the analytic solution of Eq. (1) when \( H_S(t) = 0 \). We obtain the mixed state to be teleported, \( \mathcal{E}(\rho_{\text{in}}) \), as \( \rho^{(00)}(t) = \rho_{\text{in}}^{(00)}(0) \), \( \rho^{(11)}(t) = \rho_{\text{in}}^{(11)}(0) \), and \( \rho^{(01)}(t) = \rho_{\text{in}}^{(01)}(0) \exp(-2\kappa t) \). Then from Eqs. (8) and (6), the fidelity can be calculated as

\[ F(\theta, \phi) = 1 - \frac{1}{2} (1 - e^{-2\kappa t})^2 \sin^2 \theta. \]

(9)

If \( 2\kappa \tau \ll 1 \), \( F(\theta, \phi) \approx 1 - \kappa \tau \sin^2 \theta \). On the other hand, if \( 2\kappa \tau \gg 1 \), \( F(\theta, \phi) = \frac{1}{2} (1 + \cos^2 \theta) \). Figure 2(a) is the plot of Eq. (9) for \( 2\kappa \tau = 3.0 \).

Let us consider that the state \( |\psi_{\text{in}}\rangle \) is subject to the noise described by \( L_{1,2} \). After some calculations, we obtain the fidelity

\[ F(\theta, \phi) = \frac{1}{2} [1 + \sin^2 \theta \cos^2 \phi + e^{-2\kappa t} (\cos^2 \theta + \sin^2 \theta \sin^2 \phi)]. \]

(10)

If \( 2\kappa \tau \ll 1 \), \( F(\theta, \phi) \approx 1 - \kappa \tau (\cos^2 \theta + \sin^2 \theta \sin^2 \phi) \). In the limit of \( 2\kappa \tau \gg 1 \), we have \( F(\theta, \phi) = \frac{1}{2} (1 + \sin^2 \theta \cos^2 \phi) \). The plot of Eq. (10) at \( 2\kappa \tau = 3.0 \) is shown in Fig. 2(b).

Substituting Eq. (10) or (9) into Eq. (7), we get the average fidelity.
In Fig. 3, the solid line (case A-1) shows the plot of Eq. (11), the average fidelity as a function of $\kappa \tau$ for the noise modeled by $L_{\kappa}$. Now suppose the isotropic noise ($L_{1/4}, L_{1/3}$, and $L_{1/2}$) is applied to the state $|\psi_{in}\rangle$. The analytic solution of Eq. (1) gives us the fidelity written by

$$F_{av} = F(\theta, \phi) = \frac{1}{2} + \frac{1}{2} e^{-4\kappa \tau}. \quad (12)$$

If $4\kappa \tau \ll 1$, $F(\theta, \phi) = 1 - 2\kappa \tau$. For $4\kappa \tau \gg 1$, we have $F(\theta, \phi) = \frac{1}{4}$, as shown in Fig. 2(c). In Fig. 3, the dotted line (denoted by case A-2) is the plot of Eq. (12).

Case B: Quantum channels are noisy. While being distributed and stored by Alice and Bob, an entangled state of two spin-$\frac{1}{2}$ particles may be subject to noise. The dynamics of an entangled pair subject to quantum noise is described by the quantum operation $\mathcal{E}$ acting on the pure entangled state, $\rho_{en} = \mathcal{E}(\rho_{en})$, or by Eq. (1). From Eq. (5), the quantum teleportation with noisy quantum channels can be written as

$$\mathcal{E}(\rho_{out}) = \text{Tr}_{12}U_{\text{tel}}(y) \otimes \mathcal{E}(\rho_{en}) U_{\text{tel}}^\dagger. \quad (13)$$

We find that the quantum teleportation process transfers the decoherence of the entangled pair $\mathcal{E}(\rho_{en})$ to that of the output state $\mathcal{E}(\rho_{out})$. It should be noted that the quantum operation acting on the entangled pair $\mathcal{E}(\rho_{en})$ is not only a $4 \times 4$ matrix, but effectively a $2 \times 2$ matrix. Thus overall features of case B are similar to case A except the decoherence rates.

Consider the quantum channel subject to the noise acting in one direction, for example, the $z$ direction. This type of noise is modeled by Lindblad operators $L_{2z} = \sqrt{\kappa_{2z}} \sigma_{z}^{(2)}$ and $L_{3z} = \sqrt{\kappa_{3z}} \sigma_{z}^{(3)}$ acting on an entangled pair, qubits 2 and 3, respectively. Here we assume the same strength of decoherence rates, $\kappa_{2z} = \kappa_{3z}$. We obtain the fidelity $F(\theta, \phi)$ with the same form of Eq. (9) except the replacement of $2\kappa \tau$ with $4\kappa \tau$. That is, $F(\theta, \phi) = 1 - \frac{1}{2}[1 - \exp(-4\kappa \tau)] \sin^2 \theta$. For the noise described by $L_{2z}$ and $L_{3z}$, the fidelity $F(\theta, \phi)$ is identical to the form of Eq. (10) with exponent $4\kappa \tau$.

Let us discuss Bennett et al.’s argument: the imperfect quantum channel reduces the range of state $|\psi_{in}\rangle$ that is accurately teleported [1]. Figure 2 shows the fidelity $F(\theta, \phi)$ for various types of noises at $4\kappa \tau = 3.0$. For the noisy channel defined by $L_{2z}$ and $L_{3z}$, the fidelity $F(\theta, \phi)$ is always the maximum value of 1 at $\theta = 0, \pi$, irrespective of $\kappa \tau$ as depicted in Fig. 2(a). These angles indicate states $|0\rangle$ and $|1\rangle$, which are eigenstates of $\sigma_z$. From $F(\theta, \phi) = \frac{1}{4}(1 + \cos^2 \theta)$ in the limit of $4\kappa \tau \gg 1$, the range of states to be teleported with fidelity $F \approx 3/4$ is determined by $0 = \theta \leq \pi/4$ and $3\pi/4 \leq \theta \leq \pi$. The teleported states with fidelity $3/4$ are in the region determined by $\cos \theta = 1/\sqrt{3}$ or $\cos \theta = -1/\sqrt{3}$. When $L_{2z}$ and $L_{3z}$ are applied to qubits 2 and 3, we get $F(\theta, \phi) = 1$ at $\theta = \pi/2$ and $\phi = 0, \pi$ for $4\kappa \tau \gg 1$, which shown in Fig. 2(b). These angles represent states $|\psi\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|\phi\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$, i.e., eigen-
states of $\sigma_z$. The range of states accurately teleported is depicted by contours in Figs. 2(a) and 2(b).

When the quantum channel is subject to noise in one direction, we obtain the average fidelity as depicted in Fig. 3 (denoted by case B-1)

$$F_{av}(\tau) = \frac{2}{3} + \frac{1}{3} e^{-4\tau}. \quad (14)$$

The average fidelity decays exponentially to the limiting value of 2/3. This is the best possible score when Alice and Bob communicate with each other only through the classical channel [3,5].

Consider the case where the quantum channel is affected by isotropic noise, which is described by six Lindblad operators, $L_{2,\alpha}$ and $L_{3,\alpha}$ with $\alpha=x,y,z$. Then the analytic calculation of the fidelity can be written by

$$F_{av}=F(\theta,\phi) = \frac{1}{2} + \frac{1}{2} e^{-8\tau}. \quad (15)$$

As depicted in Fig. 2(c), the fidelity $F(\theta,\phi)$ is independent of angles of input states, $\theta$ and $\phi$, for any value of $\kappa\tau$. For the quantum channel subject to isotropic noise, one could not find the range of states that is accurately teleported. As shown in Fig. 3 (case B-2), the average fidelity decays to the value of 1/2. The number 1/2 can be obtained when Alice and Bob cannot communicate at all and Bob merely selects a state at random.

It should be noted that except decoherence rates $\kappa\tau$, the overall features of cases $A$ and $B$ are identical. This implies that if a state to be teleported is realized by a single particle and not by an ensemble, one may not be able to identify whether the state to be teleported is mixed or the quantum channel is noisy.

**Cases C and D: Noise during Bell’s measurement or the unitary operation.** When Alice performs the Bell’s measurement or Bob does the unitary operation on his particle of an entangled pair, noise may take place as depicted by the boxes $C$ or $D$ in Fig. 1. In contrast to cases $A$ and $B$, it seems to be difficult to find analytic solutions of Eq. (1) for cases $C$ and $D$ because of the time dependence of the qubit Hamiltonian $H_5(t)$. Alice’s Bell measurement on qubits 1 and 2 could be done by a controlled-NOT gate (cNOT) on qubits 1 and 2, and a Hadamard gate $H_1$ on qubit 1 as shown in Fig. 1. With a qubit system modeled by Hamiltonian, Eq. (2), the cNOT gate acting on qubits 1 and 2 could be implemented by the pulse sequence $[12,13]$ $e^{-i\pi/4}H_1R_{2x}(\pi/2)R_{1x}$ $(-\pi/2)U_{22}(\pi/4)R_{1x}(\pi)U_{12}(\pi/4)H_1$. Here $R_{ij}(\theta) = e^{i\theta ij x}$ is a rotation of qubit $j$ by angle $\theta$ about the $x$ axis. A two-qubit operation $U_{12}(\theta)$ on qubits 1 and 2 is implemented by turning on the coupling $J_{12}$ for a time $t$ corresponding to $\theta = J_{12}/\hbar$. During each qubit operation, the noise modeled by Lindblad operators is also switched on. Thus, it does not seem to be simple to obtain an analytic solution and we take a numerical method to solve the problem. Consider the noise modeled by the Lindblad operators $L_{1z}$ and $L_{2z}$ for case $C$ and $L_{3z}$ for case $D$. Here the noise is switched on during the time interval $\tau$ corresponding to the total operation time that it takes to implement Bell’s measurement or controlled $X$ and $Z$ operations. The time interval $\tau$ depends on the operation times of a single gate or a two-qubit gate, proportional to $h/[B^{1/2}]$ and $h/J_{12}$, respectively. Figure 4 shows the fidelity $F(\theta,\phi)$ as a function of angle $\theta$ for various values of $\kappa\tau$. In contrast to the previous cases (cases $A-1, A-2, B-1,$ and $B-2$) whose fidelity is given by Eq. (9) or (10), in cases $C$ and $D$ the degrees of the dependence of fidelity $F(\theta,\phi)$ on angles $\theta$ is maximum at a certain value of $\kappa\tau$. Figure 4(c) shows the differences between the maximum and minimum values of the fidelity, $g(\kappa\tau) = \max\{F(\theta,\phi)\} - \min\{F(\theta,\phi)\}$. It is not clear why $g(\kappa\tau)$ has the maximum at $\kappa\tau \approx 0.98$. As depicted in Fig. 5, the average
In conclusion, we calculated the fidelity and the average fidelity of quantum teleportation subject to various types of noises during different steps of the teleportation. We examined the range of states that can be accurately teleported. Among states to be teleported, the eigenstate of the Lindblad operators is less sensitive to the noise. It was shown that one cannot distinguish whether an unknown state to be teleported, which is realized by a single particle, is mixed or the quantum channel is noisy. We found the dependence of the average fidelity on the type of noise affecting the quantum channel. If the quantum channel is subject to isotropic noise, the average fidelity may decay to 1/2. On the other hand, if the noisy quantum channel is described by a single Lindblad operator, the average fidelity is always greater than the value of 2/3, the best possible value that can be obtained only by the classical communication.

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