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A NEW CLASS OF HIGHER ORDER MIXED FINITE VOLUME METHODS FOR ELLIPTIC PROBLEMS*

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Abstract. We introduce a new class of higher order mixed finite volume methods for elliptic problems. We start from the usual way of changing the given equation into a mixed system using the Darcy's law, $\mathbf{u} = -\mathcal{K}\nabla p$. By integrating the system of equations with some judiciously chosen test spaces on each element, we define new mixed finite volume methods of higher order. We show that these new schemes are equivalent to the nonconforming finite element spaces used to define them. The Darcy velocity can be locally recovered from the solution of nonconforming finite element method. Hence our work opens a way to make higher order mixed method more practicable. Three-dimensional extensions to parallelepiped elements are also presented. This work can be viewed as a generalization of earlier works which were devoted to reducing the (lowest order) mixed finite element method to a corresponding nonconforming finite element method. An optimal error analysis is carried out and numerical results are presented which confirm the theory.

Key words. higher order mixed finite volume methods, nonconforming finite element methods, local velocity recovery, three-dimensional problems

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1. Introduction.

This paper introduces a new class of mixed finite volume methods of higher order for second order elliptic problems and shows that they can be easily implemented by solving the primal problem with some nonconforming finite element methods (FEMs).

Since its introduction in the late 1970s, mixed FEMs (MFEMs) have been the subject of extensive research [1, 2, 3, 4, 5, 6, 8, 17, 19, 21, 24, 25, 26, 27]. The idea of the mixed method is to introduce the Darcy velocity as a new variable and write the equation into a system of differential equations. By discretizing this system, one can compute two variables together and expect a more accurate velocity than solving the scalar equation for the pressure and taking the difference quotients from the discrete pressure. However, the resulting mixed system has many more variables and gives rise to a saddle point problem; thus there were some restrictions to use in industry. For the lowest order mixed methods, it is well known that there exist equivalent positive definite nonconforming FEMs associated with mixed methods. For example, Fraeijis de Veubeke [21] used Lagrangian multipliers to relax the continuity of flux variables and then solve the velocity field equation at an elemental level, where they reduce the saddle point problem into a definite problem involving the multiplier variables only. For an analysis of equivalent nonconforming formulation of mixed finite elements, see Arnold and Brezzi [2], Marini [24], Arbogast and Chen [1], and Chen [8, 9]. However, implementation of the above methods is mostly restricted to the lowest order only. For higher order methods, there exist some equivalent nonconforming finite element formulations [1, 2], but the corresponding nonconforming methods involve projections

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into the vector part of the MFEM spaces, which are not easy to handle. We overcome this difficulty by introducing a finite volume type approach.

Meanwhile, a finite volume type of mixed method [13] and its variants were introduced in [10, 12, 14, 15, 22]. The idea is to integrate the system over each element with certain weight functions. Assuming the scalar variable lies in P_1 or Q_1 nonconforming finite element space, and the velocity variable in the Raviart–Thomas (RT) space, they show the equivalence between the MFEM and the nonconforming FEM. Thus the solutions of the mixed finite volume methods can be obtained through these nonconforming FEMs (with a minor modification). It is extended by Croisille and Greff in [15] to the Brezzi–Douglas–Marini (BDM) space [5] of order 1 enriched by a bubble function. In another direction, a hybrid type of formulation was used in [11] to construct higher order mixed methods.

The equivalence of mixed methods with nonconforming FEMs for higher order has been known for quite some time. Let us briefly review the equivalent formulation with nonconforming FEMs proposed by Arnold and Brezzi in [2] and later extended by Arbogast and Chen in [1]. For example, for even k, Arnold and Brezzi showed that for some nonconforming space N^{k+1}, the solution $\psi_h \in N^{k+1}$ of the hybrid method for triangular grid satisfies

$$\sum_{Q\in\mathcal{Q}_h} \int_Q P_{\mathbf{V}_h}(\nabla\psi_h)\cdot\nabla\chi\,dx = \int_\Omega (P_h f)\chi\,dx, \quad \chi \in N^{k+1}, \tag{1.1}$$

where $P_{\mathbf{V}_h}$ is the projection to the vector part and P_h is the L^2-projection to scalar part of the Raviart–Thomas MFEM space. If this system can be solved, one can compute the solution of the mixed method rather easily, but solving this equation is nontrivial. Arbogast and Chen [1] and Chen [8] extended it to odd k and to the rectangular case by introducing related nonconforming spaces. But again, the equivalent formulation is similar to (1.1), which is nontrivial to solve.

In the lowest order case, Arbogast and Chen showed that the nonconforming space N^1 can be chosen as P_1 plus a P_2 bubble so that the following holds:

$$P_{\mathbf{V}_h}(\nabla v) = \nabla v,\ v \in N^1.$$

Hence (1.1) is equivalent to the P_1 or Q_1 nonconforming FEM with modified right-hand side. For higher order methods, such relation fails; hence (1.1) is not equivalent to the usual nonconforming FEM.

In this paper, we circumvent such difficulty by defining new mixed finite volume type of methods with judiciously chosen test spaces and show that they are equivalent to some nonconforming FEMs. The velocity can be recovered from the corresponding nonconforming solutions cheaply by a local process. This work, on the one hand, is a generalization of the mixed finite volume methods in [13, 10, 14, 15] and can be viewed as a generalization of the work of Arnold and Brezzi [2] and Arbogast and Chen [1] on the other.

The rest of the paper is organized as follows. In the next section, we start with a problem description and brief introduction of various mixed FEMs. Under a series of hypotheses, we define our new class of finite volume mixed methods. Next, we show these methods are equivalent to some nonconforming FEMs and show the velocity can be recovered locally from the FEM solution. In section 3, we prove the error estimate of the nonconforming FEM, and in section 4, we prove the optimal $H(\mathrm{div})$ and L^2-error estimate of our scheme. In sections 5 and 6, we presents examples in rectangular, triangular, and parallelepiped elements. Finally, numerical tests are shown in section 7.

2. Mixed finite volume methods of higher order.

In this section we introduce mixed finite volume methods (MFVM) and relate them to some nonconforming FEMs from which the velocity variables are recovered locally. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the second order elliptic boundary value problem

$$\begin{cases} -\operatorname{div}\mathcal{K}\nabla p + cp = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $0 \le c(x)$, $c \in L^\infty(\Omega)$, and $\mathcal{K} = \mathcal{K}(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, i.e., there exist two positive constants c_1 and c_2 such that

$$c_1\xi^T\xi \le \xi^T\mathcal{K}(\mathbf{x})\xi \le c_2\xi^T\xi \qquad \forall\xi \in \mathbb{R}^2,\ \mathbf{x} \in \overline{\Omega}.$$

For the discussion of higher order methods, we shall require $f \in H^k(\Omega)$ for some integer $k \ge 0$. Let us introduce the vector variable $\mathbf{u} = -\mathcal{K}\nabla p$ and rewrite the problem (2.1) in the mixed form

$$\begin{cases} \mathbf{u} = -\mathcal{K}\nabla p & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} + cp = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Throughout this paper, we assume the following regularity holds. The solution $(\mathbf{u}, p)$ of (2.2) satisfies $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $p \in H^{k+2}(\Omega)$, and there exists some constant $C > 0$ such that

$$\|\mathbf{u}\|_{k+1} + \|p\|_{k+2} \le C\|f\|_k.$$

We introduce some function spaces here. For any domain D, let $H^k(D)$ (or $\mathbf{H}^k(D)$) be the (vector) Sobolev spaces of order $k \ge 0$. We use the standard notation $|\cdot|_{k,D}$, $\|\cdot\|_{k,D}$, and $\|\cdot\|_{\partial D}$ for the seminorm, norm on $H^k(D)$, and norm on $L^2(\partial D)$. When $D = \Omega$, we drop the subscript Ω and write $|\cdot|_k, \|\cdot\|_k$ instead of $|\cdot|_{k,\Omega}, \|\cdot\|_{k,\Omega}$. Let $\mathbf{H}(\text{div};\Omega)$ be the space of functions $\mathbf{u} \in L^2(\Omega)^2$ with $\operatorname{div}\mathbf{u} \in L^2(\Omega)$.

For simplicity of presentation, we assume Ω is a rectangular domain. Let $0 < h$ be a parameter. We consider two triangulations: $\mathcal{Q}_h$ is

- either a uniform triangulation by rectangles
- or a regular triangulation of Ω into triangles: There exists a constant $\alpha > 0$ independent of h such that if h_Q is the size of the largest side and τ_Q is the radius of the smallest circle inscribed in the triangle Q; then

$$\frac{h_Q}{\tau_Q} < \alpha\ \forall Q \in \mathcal{Q}_h.$$

We denote by P_k the set of all polynomials of total degree less than or equal to k and $P_{i,j}$ the set of all polynomials whose degrees are less than or equal to i and j in each variable.

Let $\mathbf{V}_h \times U_h$ be any known mixed finite element space (RT, BDM, Brezzi–Douglas–Fortin–Marini (BDFM), etc.) [26, 5, 3] of order $k \ge 0$ such that for all $\mathbf{u}_h \in \mathbf{V}_h$,

$\mathbf{u}_h \cdot \mathbf{n} \in P_k(e)$ on each edge e of $Q \in \mathcal{Q}_h$ and the local degrees of freedom are

$$\int_e \mathbf{u}_h \cdot \mathbf{n} q\, ds, \quad q \in P_k(e), \quad \text{each edge } e \text{ of } Q, \tag{2.3}$$

$$\int_Q \mathbf{u}_h \cdot \boldsymbol{\phi}\, ds, \quad \boldsymbol{\phi} \in \boldsymbol{\Psi}_h(Q) \quad (k \geq 1) \tag{2.4}$$

for some space $\boldsymbol{\Psi}_h(Q)$ depending on $\mathbf{V}_h(Q)$. Also assume that we have some nonconforming finite element space N_h to be determined later associated with this mixed finite element space. The local spaces $\mathbf{V}_h(Q)$, $N_h(Q)$ of $\mathbf{V}_h$, N_h are obviously defined by restricting to Q.

For an illustration, we review the lowest order finite volume method introduced in [10] for rectangular elements. Let $\mathbf{V}_h$ be the vector part of the lowest order RT space locally spanned by functions of the form $\mathbf{v}_h = (a+bx, c+dy)$ which satisfy the normal flux continuity condition on the interelement boundaries, i.e., $\int_e \mathbf{v}_h \cdot \mathbf{n}_1 + \int_e \mathbf{v}_h \cdot \mathbf{n}_2 = 0$. The corresponding pressure space N_h is the Q_1 nonconforming finite element space locally spanned by element of the form $a + bx + cy + d(x^2 - y^2)$ and having degrees of freedom at the midpoint of interelement boundaries. Let NQ denote the number of elements, NE_i the number of interior edges, and NE_b the number of boundary edges. We consider the following discrete problem. Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times N_h$ satisfying

$$\int_Q (\mathbf{u}_h + \mathcal{K}\nabla p_h) \cdot \nabla \chi\, dx = 0 \ \forall \chi \in N_h(Q), \tag{2.5a}$$

$$\int_Q (\operatorname{div} \mathbf{u}_h + c p_h)\, dx = \int_Q f\, dx. \tag{2.5b}$$

This gives rise to a total of $4NQ(3NQ$ from (2.5a) and NQ from (2.5b)) equations in $2NE_i + NE_b$ unknowns. It is easy to see that this is a square matrix system on rectangular grids (cf. [10]), since we have

$$4NQ = \sum_Q \sum_{\partial Q} 1 = 2 \sum_{e \in E_i} 1 + \sum_{e \in E_b} 1 = 2NE_i + NE_b.$$

Then it is shown that (2.5) is equivalent to the standard Q_1 nonconforming FEMs with modified right-hand side, and the flux can be recovered cheaply. Conventional mixed hybrid schemes using Lagrangian multipliers [1, 2, 24] lead to a similar result for $k = 0$.

Motivated by this, we now propose higher order schemes. Let $\mathbf{V}_h$ be the vector part of any mixed finite element space whose degrees of freedom are given by (2.3) and (2.4). We need some hypotheses.

(H1) There is a nonconforming finite element space N_h of order k associated with the same triangulation such that for $\chi \in N_h$ the moments up to k

$$\int_e \chi q\, ds, q \in P_k(e) \text{ for any edge } e \text{ of } Q \tag{2.6}$$

are continuous across interior edges,

$$\int_e \chi q\, ds = 0, q \in P_k(e) \text{ if edge } e \text{ is part of } \partial\Omega, \tag{2.7}$$

and the interior degrees of freedom are given by

$$\int_Q \chi \phi\, dx,\ \phi \in N_h^i(Q) \tag{2.8}$$

for some space $N_h^i(Q)$ of polynomials.

(H2) The space $N_h(Q)$ has the approximation property

$$\inf_{v_h \in N_h(Q)} (\|v - v_h\|_{0,Q} + h|v - v_h|_{1,Q}) \leq Ch^{k+2}\|v\|_{k+2,Q} \tag{2.9}$$

whenever $v \in H^{k+2}(Q)$.

(H3) Let $U_h(Q) := \operatorname{div} \mathbf{V}_h(Q)$ and $\mathbf{\Psi}_h(Q)$ be the interior degrees of freedom of $\mathbf{V}_h(Q)$. Then the space $U_h(Q)$ satisfies

$$\nabla U_h(Q) \subset \mathbf{\Psi}_h(Q) \text{ and}$$
$$R_s(Q) = U_h(Q) \subset N_h(Q) \text{ for } s = k-1 \text{ or } k,$$

where $R_s(Q) = P_s(Q)$ if Q is a triangle and $R_s(Q) = P_{s,s}(Q)$ if Q is a rectangle.

(H4) There is a space of polynomials $\mathbf{V}_h^S(Q)$ on Q such that

$$\mathbf{Z}_h(Q) := \mathbf{V}_h^S(Q) \oplus \nabla N_h(Q) \supset \mathbf{\Psi}_h(Q) \tag{2.10}$$

and

$$\dim \mathbf{V}_h + \dim N_h = (\dim \nabla N_h(Q) + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q)) \cdot NQ. \tag{2.11}$$

Remark 2.1.

1. By (2.6), the nonconforming space N_h passes the patch test [23] and hence one can expect an optimal order error as long as (H2) holds.
2. Note that the restriction of a function $\chi \in N_h$ to an edge e does not belong to $P_k(e)$ for most of nonconforming examples (see section 5). This makes the error analysis more difficult. See Lemma 4.2 and Theorem 4.3 for how this difficulty is circumvented.
3. Our scheme is easily extended to three dimensions. See section 6.

Now we define a higher order mixed finite volume method. Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times N_h$ which satisfies on every element $Q \in \mathcal{Q}_h$,

$$\int_Q (\mathbf{u}_h + \mathcal{K}\nabla p_h) \cdot \nabla \chi \, dx = 0 \quad \forall \chi \in N_h(Q), \tag{2.12a}$$

$$\int_Q (\mathbf{u}_h + \mathcal{K}\nabla p_h) \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^S(Q), \tag{2.12b}$$

$$\int_Q (\operatorname{div} \mathbf{u}_h + cp_h)\phi \, dx = \int_Q f\phi \, dx \quad \forall \phi \in U_h(Q). \tag{2.12c}$$

One simple but important observation from (2.12 c) is that

$$\operatorname{div} \mathbf{u}_h = P_{U_h}(f - cp_h), \tag{2.13}$$

where P_{U_h} is an L^2-projection onto U_h.

If $\chi \in N_h(Q)$, we have

$$\begin{aligned} \sum_Q \int_Q \mathcal{K}\nabla p_h \cdot \nabla \chi \, dx &= -\sum_Q \int_Q \mathbf{u}_h \nabla \chi \, dx \\ &= -\sum_Q \int_{\partial Q} \mathbf{u}_h \cdot \mathbf{n}\chi ds + \sum_Q \int_Q (\operatorname{div} \mathbf{u}_h)\chi \, dx. \end{aligned} \tag{2.14}$$

Now assume $\chi \in N_h$ (the global nonconforming space). Since χ has continuous moments up to degree k across internal edges and has vanishing moments on $\partial\Omega$, we obtain

$$\sum_Q \int_Q \mathcal{K}\nabla p_h \cdot \nabla\chi \, dx = \int_\Omega P_{U_h}(f - cp_h)\chi dx, \quad \chi \in N_h. \tag{2.15}$$

Observing

$$\int_\Omega P_{U_h}(f - cp_h)\chi dx = \int_\Omega (f - cp_h)P_{U_h}\chi dx,$$

we see (2.15) can be written as

$$\sum_Q \int_Q (\mathcal{K}\nabla p_h \cdot \nabla\chi + cp_h(P_{U_h}\chi)) \, dx = \int_\Omega f(P_{U_h}\chi)dx, \quad \chi \in N_h. \tag{2.16}$$

Thus our new mixed finite volume scheme is equivalent to a standard nonconforming FEM, except that L^2-projections are used on the lower order terms. Thus to implement (2.12), it suffices to solve the equivalent system (2.16) and use the local recovery technique to be described below to find $\mathbf{u}_h$. This is a huge gain over solving the complicated system (2.12). Also, the error analysis follows from that of the nonconforming FEM and the relation (2.12). Temporarily assume the nonconforming FEM (2.16) is well-posed.

2.1. Recovery of flux and velocity. We show how to recover the *velocity variable*. First note that (2.12) is a square system. We see from (2.12) and (2.13) that for any $\chi \in N_h(Q)$

$$\begin{aligned} \int_{\partial Q} \mathbf{u}_h \cdot \mathbf{n}\chi \, ds &= \int_Q \operatorname{div}(\mathbf{u}_h\chi)dx = \int_Q (\operatorname{div}\mathbf{u}_h\chi + \mathbf{u}_h\nabla\chi)dx \\ &= \int_Q (P_{U_h}(f - cp_h))\,\chi dx - \int_Q \mathcal{K}\nabla p_h \cdot \nabla\chi \, dx \\ &= \int_Q (f - cp_h)P_{U_h}\chi \, dx - \int_Q \mathcal{K}\nabla p_h \cdot \nabla\chi \, dx. \end{aligned} \tag{2.17}$$

Now the interior degrees of freedom of $\mathbf{u}_h$

$$\int_Q \mathbf{u}_h \cdot \mathbf{v} \, dx = -\int_Q \mathcal{K}\nabla p_h \cdot \mathbf{v} \, dx, \quad \mathbf{v} \in \mathbf{\Psi}_h(Q) \tag{2.18}$$

are completely given in (2.12a), (2.12b) by (H4). Thus $\mathbf{u}_h$ is determined by (2.17) and (2.18). Incidentally, we have shown the existence and uniqueness of the system (2.12).

In practice, the recovery of *flux* can be separated from the interior variables as follows. Let $\mathbf{u}_h = \sum c_j\boldsymbol{\phi}^j + \sum d_j\boldsymbol{\psi}^j$. Here, $\boldsymbol{\phi}^j$'s are the local basis functions related to the edges and $\boldsymbol{\psi}^j$'s are the ones related to the interior. Then by (H4) we can choose $\mathbf{v} = \boldsymbol{\psi}^\ell \in \mathbf{\Psi}_h(Q)$ in (2.12a), (2.12b), to have

$$\int_Q \mathbf{u}_h \cdot \boldsymbol{\psi}^\ell \, dx = d_\ell = -\int_Q \mathcal{K}\nabla p_h \cdot \boldsymbol{\psi}^\ell \, dx. \tag{2.19}$$

This gives the interior degrees of freedom. By choosing $\chi \in N_h(Q)$, the $k+1$ basis functions associated with the edge e_s, we see by orthogonality of $\boldsymbol{\psi}^\ell \cdot \mathbf{n}$ against any function in $P_k(e)$ along the edge,

$$\begin{aligned}\int_{\partial Q} \mathbf{u}_h \cdot \mathbf{n}\chi\, ds &= \sum_{e\subset\partial Q} \int_e \mathbf{u}_h \cdot \mathbf{n} P^k_{e_s}(\chi)\, ds \\ &= \int_{e_s} \sum_{j=1}^{k+1} c_j \boldsymbol{\phi}^j_s \cdot \mathbf{n} P^k_{e_s}(\chi) ds \\ &= \int_Q (f - cp_h)(P_{U_h}\chi) dx - \int_Q \mathcal{K}\nabla p_h \cdot \nabla\chi\, dx\end{aligned} \tag{2.20}$$

by (2.13) and (2.14), where P^k_e is the L^2-projection onto $P_k(e)$ and $\boldsymbol{\phi}^j_s, j = 1, \ldots, k+1$, is the basis associated with the edge e_s. Thus, to recover $\mathbf{u}_h \cdot \mathbf{n}$ on the edges, it suffices to solve this square system on each edge.

3. Error analysis of a new nonconforming formulation.

In this section we derive optimal error estimates for higher order nonconforming FEMs in (2.16). The analysis is similar to the standard one, but the bilinear form involved is different from the standard. Thus we present some details here. First, we need some notation. For $k \geq 0$ set

$$|\chi|^2_{k,h} = \textstyle\sum_{Q\in\mathcal{Q}_h} |\chi|^2_{k,Q} \qquad \forall\chi \in H^k(\Omega) \oplus N_h$$

and $\|\chi\|^2_{k,h} = \|\chi\|^2_0 + \sum_{j=1}^k |\chi|^2_{j,h}$. We let

$$a_h(p, \chi) := \sum_Q \int_Q (\mathcal{K}\nabla p \cdot \nabla\chi + cp\chi)\, dx, \quad p, \chi \in H^1_0(\Omega) \oplus N_h, \tag{3.1}$$

be the usual bilinear form arising from standard nonconforming FEM. If we define $\|w\|_{a_h} := \sqrt{a_h(w, w)}$, then it is well-known that $\|w\|_{a_h}$ is a norm on $H^1_0(\Omega) \oplus N_h$ equivalent to $\|w\|_{1,h}$. From now on, we assume $\mathcal{K}$ is piecewise H^{k+2}.

Let

$$a^*_h(r, \chi) := \sum_Q \int_Q (\mathcal{K}\nabla r \cdot \nabla\chi + cr(P_{U_h}\chi))\, dx, \quad r,\ \chi \in H^1_0(\Omega) \oplus N_h. \tag{3.2}$$

If p is the solution of (2.1), we easily see that for $\chi \in N_h$

$$a^*_h(p, \chi) - (P_{U_h} f, \chi) = \sum_Q \left\langle \mathcal{K}\frac{\partial p}{\partial n}, \chi \right\rangle - (cp, (I - P_{U_h})\chi)) + ((I - P_{U_h})f, \chi). \tag{3.3}$$

Meanwhile the solution of (2.16) satisfies

$$a^*_h(p_h, \chi) = (f, P_{U_h}\chi), \quad \chi \in N_h. \tag{3.4}$$

Let us discuss the well-posedness of this system. Note that the a^*_h is bounded, i.e.,

$$|a^*_h(u, v)| \leq C\|u\|_{1,h}\|v\|_{1,h},\ u, v \in H^1_0(\Omega) \oplus N_h,$$

and the following Gårding inequality holds:

$$|a^*_h(v, v)| \geq C_1\|v\|^2_{1,h} - C_2\|v\|^2_0,\ v \in H^1_0(\Omega) \oplus N_h.$$

Hence following the procedure in [28], we can show the problem (3.4) has a unique solution provided h is sufficiently small.

Now we turn to the error analysis. We need a variation of the second Strang lemma [7].

LEMMA 3.1. *Let $p \in H_0^1(\Omega)$ and $w \in N_h$ be arbitrary. Then we have*

$$\|p - w\|_{1,h} \le \inf_{v \in N_h} \|p - v\|_{1,h} + \sup_{v \in N_h} \frac{|a_h(p - w, v)|}{\|v\|_{1,h}}.$$

Proof. Let $\tilde{p}_h \in N_h$ satisfy

$$a_h(\tilde{p}_h, \chi) = a_h(p, \chi), \quad \chi \in N_h.$$

Then $a_h(\tilde{p}_h - p, \chi) = 0$ for all $\chi \in N_h$ and hence

$$\|p - \tilde{p}_h\|_{a_h} \le C \inf_{v \in N_h} \|p - v\|_{a_h}. \tag{3.5}$$

Also, we have

$$\|p - w\|_{a_h} \le \|p - \tilde{p}_h\|_{a_h} + \|\tilde{p}_h - w\|_{a_h} = \|p - \tilde{p}_h\|_{a_h} + \sup_{v \in N_h} \frac{|a_h(\tilde{p}_h - w, v)|}{\|v\|_{a_h}}. \tag{3.6}$$

Since

$$a_h(\tilde{p}_h - w, v) = a_h(\tilde{p}_h - p + p - w, v) = a_h(p - w, v),$$

we obtain the result by noting the equivalence of norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{a_h}$. ∎

In particular, this lemma holds when p is the solution of (2.1) and $w = p_h$ is the solution of (3.4).

We need the following estimate [16].

LEMMA 3.2. *Let $\phi \in H^{k+1}(Q)$, $v \in H^1(Q)$ and e be any edge of Q. Then we have*

$$\left| \int_e v(\phi - P_e^k \phi) ds \right| \le C h^{k+1} |\phi|_{k+1,Q} |v|_{1,Q}. \tag{3.7}$$

Proof. See Lemma 3 of [16]. ∎

By definitions of a_h^*, a_h form and (3.3), we have for $v \in N_h$

$$\begin{aligned} a_h(p - p_h, v) &= a_h^*(p - p_h, v) + (c(p - p_h), v - P_{U_h} v) \\ &= a_h^*(p, v) - (P_{U_h} f, v) + (c(p - p_h), v - P_{U_h} v) \\ &= \sum_Q \left\langle \mathcal{K} \frac{\partial p}{\partial n}, v \right\rangle_{\partial Q} + (f - cp, v - P_{U_h} v) + (c(p - p_h), v - P_{U_h} v) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

First note that by (2.6), $\sum_Q \langle P_e^k (\mathcal{K} \frac{\partial p}{\partial n}), v \rangle_{\partial Q} = 0$. Hence by Lemma 3.2

$$\begin{aligned} |\mathrm{I}| &= \left| \sum_Q \sum_{e \subset \partial Q} \left\langle \mathcal{K} \frac{\partial p}{\partial n} - P_e^k \left(\mathcal{K} \frac{\partial p}{\partial n} \right), v \right\rangle_e \right| \\ &\le C h^{k+1} \sum_Q |p|_{k+2,Q} |v|_{1,Q} \le C h^{k+1} |p|_{k+2} |v|_{1,h}. \end{aligned}$$

Next, II can be estimated as

$$|\mathrm{II}| = |((f - cp - P_{U_h}(f - cp)), v - P_{U_h} v)| \leq C h^{k+1} (\|f\|_k + \|p\|_k) \cdot \|v\|_{1,h}.$$

Similarly, we have

$$|\mathrm{III}| \leq C_1 h \|p - p_h\|_{1,h} \cdot \|v\|_{1,h}.$$

Dividing by $\|v\|_{1,h}$ we see, by the equivalence of the norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{a_h}$,

$$\sup_{v \in N_h} \frac{|a_h(p - p_h, v)|}{\|v\|_{1,h}} \leq C h^{k+1} (\|p\|_{k+2} + \|f\|_k) + C_1 h \|p - p_h\|_{1,h}. \tag{3.8}$$

Thus we have the following result.

THEOREM 3.3. *Let p and p_h be the solutions of* (2.1) *and* (3.4). *Suppose $N_h(Q) \supset P_{k+1}(Q)$. Then there is an $h_0 > 0$ such that for all h with $0 < h < h_0$, we have*

$$\|p - p_h\|_0 + h \|p - p_h\|_{1,h} \leq C h^{k+2} \|f\|_k. \tag{3.9}$$

Proof. The estimate follows from Lemma 3.1, the approximation property of $N_h(Q)$, and (3.8) if we choose $h_0 < 1/2C_1$. Now the L^2-estimate can be obtained by duality argument as in [18]. ∎

4. Error estimates for $\mathbf{u}_h$. In order to derive the estimate for $\|\mathbf{u} - \mathbf{u}_h\|_0$, we need a lemma whose proof can be found in [2, 26].

LEMMA 4.1. *Let $\phi \in L^2(Q)$, $\boldsymbol{\phi} \in L^2(Q)^2$, and $\mu \in L^2(\partial Q)$. Then the function $\boldsymbol{\sigma} \in \mathbf{V}_h(Q)$ determined by the degree of freedom of usual mixed finite element*

$$\int_e \boldsymbol{\sigma} \cdot \mathbf{n} q \, ds = \int_e \mu q \, ds, \quad q \in P_k(e) \text{ for all edges } e \text{ of } Q, \tag{4.1a}$$

$$\int_Q \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_Q \boldsymbol{\phi} \cdot \mathbf{v} \, dx, \quad \mathbf{v} \in \boldsymbol{\Psi}_h(Q), \tag{4.1b}$$

satisfies

$$\|\boldsymbol{\sigma}\|_{0,Q} \leq C(\|\boldsymbol{\phi}\|_{0,Q} + h^{1/2} \|\mu\|_{\partial Q}).$$

Similarly, the function $\chi \in N_h(Q)$ determined by

$$\int_e \chi q \, dx = \int_e \mu q \, ds, \quad q \in P_k(e) \text{ for all edges } e \text{ of } Q, \tag{4.2a}$$

$$\int_Q \chi v \, dx = \int_Q \phi v \, dx, \quad v \in N_h^i(Q), \tag{4.2b}$$

satisfies

$$\|\chi\|_{0,Q} \leq C(\|\phi\|_{0,Q} + h^{1/2} \|\mu\|_{\partial Q}).$$

Let

$$N_h^{\partial}(Q) = \left\{ v \in N_h(Q) : \int_Q v \phi \, dx = 0, \ \phi \in N_h^i(Q) \right\} \tag{4.3}$$

and define a projection $\tilde{\Pi}_h : \mathbf{H}^1(Q) \to \mathbf{V}_h(Q)$ by

$$\int_e \tilde{\Pi}_h \mathbf{u} \cdot \mathbf{n} P_e^k \chi \, ds = \int_e \mathbf{u} \cdot \mathbf{n} \chi \, ds, \quad \chi \in N_h^{\partial}(Q) \text{ for all edges } e \text{ of } Q, \tag{4.4a}$$

$$\int_Q \tilde{\Pi}_h \mathbf{u} \cdot \mathbf{v} \, dx = \int_Q \mathbf{u} \cdot \mathbf{v} \, dx, \quad \mathbf{v} \in \mathbf{\Psi}_h(Q). \tag{4.4b}$$

This operator is clearly well-defined. Let $P_{\partial Q}^k := \oplus_e P_e^k : N_h^{\partial}(Q) \to \cup_e P_k(e)$ be the extension of P_e^k. Then one can check that

$$\int_{\partial Q} \tilde{\Pi}_h \mathbf{u} \cdot \mathbf{n} P_{\partial Q}^k \chi \, ds = \int_{\partial Q} \mathbf{u} \cdot \mathbf{n} \chi \, ds, \quad \chi \in N_h^{\partial}(Q). \tag{4.5}$$

Now the following approximation property holds.

LEMMA 4.2. *Let* $\mathbf{u}$ *be any function in* $\mathbf{H}^{k+1}(\Omega)$ *such that* $\operatorname{div} \mathbf{u}$ *belongs to* $H^{s+1}(\Omega)$. *Then*

$$\|\mathbf{u} - \tilde{\Pi}_h \mathbf{u}\|_0 \le C h^{k+1} \|\mathbf{u}\|_{k+1}, \tag{4.6}$$

$$\|\operatorname{div}(\mathbf{u} - \tilde{\Pi}_h \mathbf{u})\|_0 \le C h^{s+1} \|\operatorname{div} \mathbf{u}\|_{s+1}, \tag{4.7}$$

where $s = k-1$ *or* k *depending on the choice of the mixed finite element* $\mathbf{V}_h$. *Here, we interpret the divergence as a piecewise operator when applied to* $\mathbf{V}_h$. *The same remark applies to Theorem* 4.3.

Proof. These are standard as long as $\tilde{\Pi}_h$ preserves $P_k^2(Q)$ and the following commuting diagram holds:

$$\begin{array}{ccc} \mathbf{H}^1(Q) & \xrightarrow{\ \operatorname{div}\ } & L^2(Q) \\ \downarrow \tilde{\Pi}_h & & \downarrow P_h \\ \mathbf{V}_h(Q) & \xrightarrow{\ \operatorname{div}\ } & U_h(Q) \end{array}$$

In fact, for $q \in \operatorname{div} \mathbf{V}_h(Q) \subset N_h(Q)$ we have $P_{\partial Q}^k q = q$ by (H3). Hence by (4.5)

$$\begin{aligned} \int_Q \operatorname{div} \tilde{\Pi}_h \mathbf{u} \, q \, dx &= \int_{\partial Q} \tilde{\Pi}_h \mathbf{u} \cdot \mathbf{n} \, q \, ds - \int_Q \tilde{\Pi}_h \mathbf{u} \cdot \nabla q \, dx \\ &= \int_{\partial Q} \mathbf{u} \cdot \mathbf{n} \, q \, ds - \int_Q \mathbf{u} \cdot \nabla q \, dx \\ &= \int_Q \operatorname{div} \mathbf{u} \, q \, dx. \qquad \square \end{aligned} \tag{4.8}$$

THEOREM 4.3. *Let* $(\mathbf{u}_h, p_h)$ *be the solution of the system* (2.12). *Then there exists a constant* C *independent of* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C h^{k+1} (\|\mathbf{u}\|_{k+1} + \|f\|_k), \tag{4.9a}$$

$$\|\operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_h\|_0 \le C h^{s+1} (\|\operatorname{div} \mathbf{u}\|_{s+1} + \|f\|_k), \tag{4.9b}$$

where $s = k-1$ *or* k, *provided that* $\mathbf{u}$ *is in* $\mathbf{H}^{k+1}(\Omega)$, $\operatorname{div} \mathbf{u} \in H^{s+1}(\Omega)$ *and* f *is in* $H^k(Q)$ *for all* $Q \in \mathcal{Q}_h$.

Proof. From (2.2), (2.12), and (2.13), we have the error equations

$$\int_{\partial Q}(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\chi\,ds = \int_Q (f - cp - P_{U_h}(f-cp_h))\chi dx - \int_Q \mathcal{K}\nabla(p-p_h)\cdot\nabla\chi\,dx, \quad \chi\in N_h(Q), \tag{4.10a}$$

$$\int_Q(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{v}\,dx = -\int_Q \mathcal{K}\nabla(p-p_h)\cdot\mathbf{v}\,dx, \quad \mathbf{v}\in\mathbf{\Psi}_h(Q). \tag{4.10b}$$

Now from (4.10), (4.5), and the definition of $\tilde\Pi_h$, the following holds:

$$\int_{\partial Q}(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}P^k_{\partial Q}\chi\,ds = \int_Q (-cp - P_{U_h}(f-cp_h))\chi dx - \int_Q \mathcal{K}\nabla(p-p_h)\cdot\nabla\chi\,dx, \quad \chi\in N_h^{\partial}(Q), \tag{4.11a}$$

$$\int_Q(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{v}\,dx = -\int_Q \mathcal{K}\nabla(p-p_h)\cdot\mathbf{v}\,dx, \quad \mathbf{v}\in\mathbf{\Psi}_h(Q). \tag{4.11b}$$

Let $\chi\in N_h$ be the solution of (4.2) with $\mu = (\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}|_e$ on each edge e of Q, and $\phi = 0$. Then $\chi\in N_h^{\partial}(Q)$ and $P_e^k(\chi) = (\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}|_e$ on e, and we have

$$\|\chi\|_{0,Q}\le Ch^{1/2}\|(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\|_{\partial Q}. \tag{4.12}$$

Substituting this χ into (4.11a), we see by the approximation property of P_{U_h}, (3.9), (4.12), and inverse inequality,

$$\begin{aligned}
&\|(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\|^2_{\partial Q}\\
&\quad\le C(\|f-P_{U_h}f\|_{0,Q}+\|cp-P_{U_h}(cp_h)\|_{0,Q})\|\chi\|_{0,Q}+C|p-p_h|_{1,Q}\cdot|\chi|_{1,Q}\\
&\quad\le C(h^{k+1/2}\|f\|_{k,Q}+h^{-1/2}|p-p_h|_{1,Q})\|(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\|_{\partial Q}\\
&\quad\le Ch^{k+1/2}(\|f\|_{k,Q}+|p|_{k+2,Q})\|(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\|_{\partial Q}.
\end{aligned}$$

Now we apply the first part of Lemma 4.1 to (4.11b) with $\boldsymbol{\sigma} = \tilde\Pi_h\mathbf{u}-\mathbf{u}_h$, $\mu = (\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}$, $\boldsymbol{\phi} = \mathcal{K}\nabla(p-p_h)$. We have

$$\begin{aligned}
\|\tilde\Pi_h\mathbf{u}-\mathbf{u}_h\|^2_{0,Q} &\le C|p-p_h|^2_{1,Q}+Ch\|(\tilde\Pi_h\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}\|^2_{\partial Q}\\
&\le Ch^{2k+2}\|f\|^2_{k,Q}.
\end{aligned}$$

Summing over all elements we have

$$\|\tilde\Pi_h\mathbf{u}-\mathbf{u}_h\|^2_0\le Ch^{2k+2}\|f\|^2_k.$$

Now the triangle inequality $\|\mathbf{u}-\mathbf{u}_h\|_0\le\|\mathbf{u}-\tilde\Pi_h\mathbf{u}\|_0+\|\tilde\Pi_h\mathbf{u}-\mathbf{u}_h\|_0$ and Lemma 4.2 completes the proof. The estimate in divergence norm follows from (2.13). ∎

5. Examples.

We first show a lemma regarding the cardinality.

LEMMA 5.1. *The condition* (2.11) *is equivalent to*

$$\dim\mathbf{V}_h^S(Q) = \dim\mathbf{\Psi}_h(Q) - \dim U_h(Q) + 1. \tag{5.1}$$

Proof. We start from the Euler formula. Let E_i be the set of all interior edges and E_b be the set of all boundary edges of the triangulation. Then

$$2\cdot NE_i + NE_b = \begin{cases}3NQ & \text{for a triangular grid,}\\ 4NQ & \text{for a rectangular grid.}\end{cases} \tag{5.2}$$

From (2.11), the cardinality of $\mathbf{V}_h^S(Q)$ satisfies

$$\dim \mathbf{V}_h + \dim N_h = (\dim N_h(Q) - 1 + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q)) \cdot NQ. \tag{5.3}$$

Hence we see

$$\begin{aligned}(k+1)\cdot(NE_i + NE_b) &+ \dim \mathbf{\Psi}_h(Q)\cdot NQ + (k+1)\cdot NE_i + \dim N_h^i(Q)\cdot NQ \\ &= (t(k+1) + \dim N_h^i(Q) - 1 + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q))\cdot NQ,\end{aligned} \tag{5.4}$$

where $t = 3$ for a triangular element and $t = 4$ for a rectangular grid. Hence by (5.2), this can be written as

$$\begin{aligned}&(\dim \mathbf{\Psi}_h(Q) + \dim N_h^i(Q))\cdot NQ \\ &\quad = (\dim N_h^i(Q) - 1 + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q))\cdot NQ,\end{aligned} \tag{5.5}$$

which is exactly (5.1). □

5.1. RT-like MFVM for rectangular element. For simplicity, we let $\hat{Q} = Q = [-1,1]\times[-1,1]$. Let $\mathbf{V}_h$ be the RT element of order k:

$$\mathbf{V}_h(Q) = P_{k+1,k} \times P_{k,k+1}.$$

Here the interior degrees of freedom in (2.4) are determined by $\mathbf{\Psi}_h = P_{k-1,k}\times P_{k,k-1}$. Now consider a nonconforming finite element space introduced by Arbogast and Chen [1]:

$$N_h(Q) = P_{k,k} \oplus X^k(Q) \oplus Y^k(Q), \tag{5.6}$$

where with ℓ_i, the Legendre polynomial on $[-1,1]$,

$$X^k(Q) = \left\{ \sum_{i=0}^{k} [a_{i,0}\ell_{k+1}(x_1) + a_{i,1}\ell_{k+2}(x_1)]\ell_i(x_2) \right\}, \tag{5.7}$$

$$Y^k(Q) = \left\{ \sum_{i=0}^{k} [b_{i,0}\ell_{k+1}(x_2) + b_{i,1}\ell_{k+2}(x_2)]\ell_i(x_1) \right\}. \tag{5.8}$$

The function in $N_h(Q)$ is completely determined by (2.6) and (2.8) with the choice of $N_h^i(Q) = P_{k,k}$. Hence (H1) holds. Since $N_h(Q) \supset P_{k+1}(Q)$, (H2) holds and (H3) is trivial.

We choose

$$\mathbf{V}_h^S(Q) = P_{k-1,k}\times P_{k,k-1} \cap \nabla P_{k,k}^{\perp}.$$

We count the dimension of $\mathbf{V}_h^S(Q)$. Since $\nabla P_{k,k} \subset P_{k-1,k}\times P_{k,k-1}$,

$$\dim \mathbf{V}_h^S(Q) = \dim P_{k-1,k}\times P_{k,k-1} - \dim \nabla P_{k,k} = 2(k+1)k - (k+1)^2 + 1 = k^2.$$

To check (H4), we first see that $\mathbf{Z}_h(Q) \supset \mathbf{\Psi}_h(Q)$ and

$$\dim \mathbf{V}_h^S = k^2, \quad \dim \mathbf{\Psi}_h = 2k(k+1), \quad \dim U_h = (k+1)^2.$$

So (5.1) is satisfied. This completes the verification of (H4).

5.2. BDFM-like MFVM for rectangular element. Let $\mathbf{V}_h$ be the BDFM space[3] of order $k+1$ given by

$$\mathbf{V}_h(Q) = (P_{k+1} \setminus \{y^{k+1}\}) \times (P_{k+1} \setminus \{x^{k+1}\}) \text{ with } \mathbf{\Psi}_h = P^2_{k-1}.$$

To define an MFVM based on BDFM spaces, we need a lemma.

LEMMA 5.2. *We have*

$$P^2_k = \nabla P_{k+1} \oplus \mathbf{S}_k, \tag{5.9}$$

where $\mathbf{S}_0 = \{0\}$ *and* $\mathbf{S}_k(k \geq 1) \equiv \oplus_{\ell=1}^k \tilde{\mathbf{S}}_\ell$. *Here* $\tilde{\mathbf{S}}_\ell$ *is the space of polynomials of the form*

$$\sum_{i,j\neq 0, i+j=\ell+1} \begin{pmatrix} ia_{i,j}x^{i-1}y^j \\ -ja_{i,j}x^i y^{j-1} \end{pmatrix}. \tag{5.10}$$

Proof. Let $\tilde{P}_{\ell+1}$ be the homogenous polynomials of degree $\ell+1$. Then

$$\nabla P_{k+1} = \oplus_{\ell=0}^k \nabla \tilde{P}_{\ell+1},$$

where $\nabla \tilde{P}_{\ell+1}$ is the space of polynomials of the form

$$\sum_{0\leq i+j=\ell+1} \begin{pmatrix} ia_{i,j}x^{i-1}y^j \\ ja_{i,j}x^i y^{j-1} \end{pmatrix}. \tag{5.11}$$

Then it is easy to see that

$$\dim \nabla \tilde{P}_{\ell+1} = \ell + 2 \text{ and } \dim \tilde{\mathbf{S}}_\ell = \ell. \tag{5.12}$$

Since $\nabla \tilde{P}_{k+1} \cap \tilde{\mathbf{S}}_k = \{0\}$ by construction of $\tilde{\mathbf{S}}_k$, we have $\nabla P_{k+1} \cap \mathbf{S}_k = \{0\}$ and

$$\nabla P_{k+1} \oplus \mathbf{S}_k \subset P^2_k. \tag{5.13}$$

Now counting the dimensions, we see that

$$\dim P^2_k = \dim \nabla P_{k+1} + \dim \mathbf{S}_k \tag{5.14}$$

and the proof is complete. ∎

Now we use the same nonconforming finite element space as before,

$$N_h(Q) = P_{k,k} \oplus X^k(Q) \oplus Y^k(Q).$$

Then $U_h(Q) = P_k$ and (H1)–(H3) clearly hold. Since $\mathbf{\Psi}_h(Q) = P^2_{k-1}$ we choose $\mathbf{V}^S_h(Q) = \mathbf{S}_{k-1}(Q)$; then we see by Lemma 5.2 that $\mathbf{Z}_h(Q)$ contains $\mathbf{\Psi}_h(Q) = P^2_{k-1}$. Finally, (5.1) holds by (5.14). Hence (H4) holds.

5.3. BDM-like MFVM for rectangular element. The BDM [5] spaces of order k is

$$\mathbf{V}_h(Q) = P^2_k \oplus Span\{\text{curl}\,(x^{k+1}y), \text{curl}\,(xy^{k+1})\}, \quad \mathbf{\Psi}_k(Q) = P^2_{k-2}.$$

Let us choose the space given in [5]:

$$N_h(Q) = P_{k+1} \oplus \left\{x^{k+1}y, xy^{k+1}, q^{k+1}\right\},$$

where

$$q^{k+1} = \begin{cases} x^{k+2}y - xy^{k+2}, k \text{ odd}, \\ x^{k+2} - y^{k+2}, k \text{ even } \geq 2. \end{cases}$$

A function $v \in N_h$ is uniquely determined by the conditions

$$\int_e vq\,ds, \quad q \in P_k(e),$$
$$\int_Q vw\,dx, \quad w \in N_h^i(Q) := P_{k-3} + Span(l_{k-1}(x)l_{k-1}(y)),$$

where l_{k-1} is the Laguerre polynomial of degree $k-1$. Take $\mathbf{V}_h^S(Q) = \mathbf{S}_{k-2}(Q)$ in Lemma 5.2. All the hypotheses (H1)–(H3) hold (with $s = k-1$ in (H3)). By Lemma 5.2, $\mathbf{Z}_h(Q) \supset \mathbf{\Psi}_k(Q)$ holds and we have

$$\dim\, \mathbf{S}_{k-2}(Q) = \dim\, P_{k-2}^2 - \dim\, P_{k-1} + 1.$$

Hence (5.1) holds.

5.4. RT-like MFVM for triangular element. In this subsection, we let Q denote a typical triangular element. Let $\tilde{P}_k$ be the homogenous polynomial of degree k. Then the RT space

$$\mathbf{P}_k^2 \oplus \mathbf{x}\tilde{P}_k$$

is determined by

$$\int_e \mathbf{u}_h \cdot \mathbf{n} q\,ds, \quad q \in P_k(e) \text{ for all edges } e \text{ of } Q, \tag{5.15}$$
$$\int_Q \mathbf{u}_h \cdot \boldsymbol{\psi}\,dx, \quad \boldsymbol{\psi} \in \mathbf{\Psi}_h(Q) := P_{k-1}^2. \tag{5.16}$$

When k is even, we use the following nonconforming spaces [2]. Let $N_h(Q) = P_{k+1}$ with degrees of freedom:

$$\int_e vq\,ds \text{ for } q \in P_k(e), \tag{5.17}$$
$$\int_Q vp\,dx \text{ for } p \in P_{k-2}. \tag{5.18}$$

Take $\mathbf{V}_h^S(Q) = \mathbf{S}_{k-1}(Q)$ in Lemma 5.2. Then we see by Lemmas 5.1 and 5.2 that the hypotheses (H1) through (H4) are satisfied.

We do not find a general formula for k odd. However, for $k = 1$ we may use the Fortin–Soulie [20] element given by $N_h(Q) = P_2 \oplus \{\beta(Q)\}$, where $\beta(Q)$ is the nonconforming bubble function of the form $2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$. Take $\mathbf{V}_h^S(Q) = 0$; then all the hypotheses (H1)–(H4) hold.

5.5. BDFM-like MFVM for triangular element. The BDFM space order $k+1$ is

$$\mathbf{V}_{k+1}(Q) = (P_{k+1})^2,$$

where the functions are determined by

$$\int_e \mathbf{u}_h \cdot \mathbf{n} q\, ds, \quad q \in P_k(e) \text{ for all edges } e \text{ of } Q, \tag{5.19}$$

$$\int_Q \mathbf{u}_h \cdot \nabla q\, dx, \quad q \in P_k, \tag{5.20}$$

$$\int_Q \mathbf{u}_h \cdot \mathbf{curl}\,(b_Q p)\, dx, \quad q \in P_{k-1},\ k \geq 1. \tag{5.21}$$

For k even, $N_h(Q) = P_{k+1}$ and $\mathbf{V}_h^S(Q) = \mathbf{S}_k(Q)$ in Lemma 5.2 works. For $k = 1$, the Fortin–Soulie space with $\mathbf{V}_h^S(Q) = \{\beta(Q)\}$ works.

5.6. BDM-like MFVM for triangular element. The BDM space order k is

$$\mathbf{V}_k(Q) = (P_k)^2,$$

where the degrees of freedom are

$$\int_e \mathbf{u}_h \cdot \mathbf{n} q\, ds, \quad q \in P_k(e) \text{ for all edges } e \text{ of } Q, \tag{5.22}$$

$$\int_Q \mathbf{u}_h \cdot \nabla q\, dx, \quad q \in P_{k-1}, \tag{5.23}$$

$$\int_Q \mathbf{u}_h \cdot \mathbf{curl}\,(b_Q p)\, dx, \quad q \in P_{k-2},\ k \geq 2. \tag{5.24}$$

For k even, the choice $N_h(Q) = P_{k+1}$ and $\mathbf{V}_h^S(Q) = \mathbf{S}_{k-1}(Q)$ in Lemma 5.2 again works. For $k = 1$, the Fortin–Soulie space with $\mathbf{V}_h^S(Q) = 0$ gives the proper space.

6. Three-dimensional examples. Now we consider three-dimensional rectangular paralellepipeds. Assume $Q = [-1, 1]^3$ and observe the relation

$$6NQ = \sum_Q \sum_{F \subset \partial Q} = 2NF_i + NF_b, \tag{6.1}$$

where F denotes a face of Q, NF_i is the number of interior faces, and NF_b is the number of faces that meet with $\partial\Omega$.

Given a mixed finite element space $\mathbf{V}_h$, we would like to find a nonconforming finite element space N_h and a supplementary test space $\mathbf{V}_h^S(Q)$ such that the hypotheses (H1) through (H4) hold (where "face" replaces "edge"). We see

$$\begin{aligned}&(k+1)^2 \cdot (2NF_i + NF_b) + \dim \mathbf{\Psi}_h(Q) \cdot NQ + \dim N_h^i(Q) \cdot NQ \\ &\quad = (6(k+1)^2 + \dim N_h^i(Q) - 1 + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q)) \cdot NQ.\end{aligned} \tag{6.2}$$

Hence by (6.1) this can be written as

$$\begin{aligned}&(\dim \mathbf{\Psi}_h(Q) + \dim N_h^i(Q)) \cdot NQ \\ &\quad = (\dim N_h^i(Q) - 1 + \dim \mathbf{V}_h^S(Q) + \dim U_h(Q)) \cdot NQ\end{aligned} \tag{6.3}$$

from which we again see (5.1) holds. Now choose Raviart–Thomas–Nedelec spaces [25]

$$\mathbf{V}_h(Q) = P_{k+1,k,k} \times P_{k,k+1,k} \times P_{k,k,k+1}.$$

Here we assume $Q = [-1,1]\times[-1,1]\times[-1,1]$ and the interior degrees of freedom in (2.4) are determined by $\mathbf{\Psi}_h = P_{k-1,k,k} \times P_{k,k-1.k} \times P_{k,k,k-1}$. Now consider a nonconforming finite element space introduced in [8, 1]:

$$N_h(Q) = P_{k,k,k} \oplus X^k(Q) \oplus Y^k(Q) \oplus Z^k(Q), \tag{6.4}$$

where

$$X^k(Q) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}[a_{i,0}\ell_{k+1}(x_1) + a_{i,1}\ell_{k+2}(x_1)]\ell_i(x_2)\ell_j(x_3) \right\}, \tag{6.5}$$

$$Y^k(Q) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}[b_{i,0}\ell_{k+1}(x_2) + b_{i,1}\ell_{k+2}(x_2)]\ell_i(x_1)\ell_j(x_3) \right\}, \tag{6.6}$$

$$Z^k(Q) = \left\{ \sum_{i=0}^{k}\sum_{j=0}^{k}[c_{i,0}\ell_{k+1}(x_3) + c_{i,1}\ell_{k+2}(x_3)]\ell_i(x_1)\ell_j(x_2) \right\}. \tag{6.7}$$

Note that $N_h(Q) \supset P_{k+1}$, so (H2) holds. We choose $U_h(Q) = P_{k,k,k}$ and

$$\mathbf{V}_h^S(Q) = P_{k-1,k,k} \times P_{k,k-1,k} \times P_{k,k,k-1} \cap \nabla P_{k,k,k}^{\perp}.$$

Then the dimension of $\mathbf{V}_h^S(Q)$ is $(k+1)^2(2k-1)+1$ and we check

$$\dim \mathbf{\Psi}_h(Q) - \dim U_h(Q) + 1 = (k+1)^2(2k-1)+1.$$

So (5.1) is satisfied. Other hypotheses are easy to check.

Extensions to other elements such as BDM and BDDF [4, 5] are also possible.

7. Numerical experiments. We solved the problem when $\Omega = [0,1]^2$ using RT-like MFVM with rectangular element (section 5.1) for $k = 1$. The error was measured at four Gauss points. Table 6.1 is the result when $\mathcal{K} = 1, c = 1$ with the exact solution $p = \sin(2\pi x)\sin(2\pi y)$.

In Table 6.2, we tested with another exact solution, $p = x^2(1-x)y(1-y)^2$. In Table 6.3, we tested a problem with discontinuous coefficients. Here the exact solution is $\frac{1}{\mathcal{K}}\sin(2\pi x)\sin(2\pi y)$ and the coefficients are

$$\mathcal{K} = \begin{cases} 1 & \text{for } (x-\frac{1}{2})(y-\frac{1}{2}) > 0, \\ 100 & \text{for } (x-\frac{1}{2})(y-\frac{1}{2}) \le 0. \end{cases}$$

TABLE 6.1
Errors and orders of convergence with $\mathcal{K} = 1, p = \sin(2\pi x)\sin(2\pi y)$.

	$\Vert p-p_h\Vert_{L^2(\Omega)}$		$\Vert \mathbf{u}-\mathbf{u}_h\Vert_{L^2(\Omega)}$		$\Vert \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\Vert_{L^2(\Omega)}$	
$1/h$	Error	Order	Error	Order	Error	Order
4	0.006493	*	0.063869	*	0.712968	*
8	0.000439	3.8875	0.008507	2.9084	0.087496	3.0265
16	2.787e-05	3.9767	0.001079	2.9789	0.010869	3.0089
32	1.748e-06	3.9945	0.000135	2.9949	0.001356	3.0024
64	1.094e-07	3.9988	1.693e-05	2.9987	0.000169	3.0006
128	6.834e-09	4.0003	2.117e-06	2.9997	2.118e-05	3.0001

TABLE 6.2
Errors and orders of convergence with $\mathcal{K} = 1, p = x^2(1-x)y(1-y)^2$.

	$\|p - p_h\|_{L^2(\Omega)}$		$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$		$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)}$	
$1/h$	Error	Order	Error	Order	Error	Order
4	2.112e-05	2.9828	0.000197	1.6284	0.000870	3.0902
8	1.813e-06	3.5418	4.425e-05	2.1575	0.000107	3.0247
16	1.462e-07	3.6326	8.535e-06	2.3743	1.331e-05	3.0064
32	1.200e-08	3.6067	1.559e-06	2.4525	1.662e-06	3.0016
64	1.014e-09	3.5658	2.793e-07	2.4808	2.076e-07	3.0004
128	8.735e-11	3.5365	4.966e-08	2.4916	2.596e-08	2.9999

TABLE 6.3
Errors and orders of convergence for discontinuous coefficient.

	$\|p - p_h\|_{L^2(\Omega)}$		$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$		$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)}$	
$1/h$	Error	Order	Error	Order	Error	Order
4	0.004403	*	0.063987	*	0.707043	*
8	0.000295	3.8971	0.008511	2.9104	0.087309	3.0176
16	1.875e-05	3.9785	0.001079	2.9794	0.010863	3.0067
32	1.176e-06	3.9949	0.000135	2.9950	0.001356	3.0018
64	7.356e-08	3.9988	1.693e-05	2.9988	0.000169	3.0005
128	4.601e-09	3.9989	2.117e-06	2.9997	2.118e-05	3.0001

TABLE 6.4
Errors and orders of convergence with $\mathcal{K} = 1 + 10x + y$.

	$\|p - p_h\|_{L^2(\Omega)}$		$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$		$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)}$	
$1/h$	Error	Order	Error	Order	Error	Order
4	4.062e-05	*	0.004127	*	0.006044	*
8	4.161e-06	3.28697	0.000784	2.39652	0.000755	2.9999
16	3.961e-07	3.39308	0.000142	2.46585	9.444e-05	2.9999
32	3.626e-08	3.44973	2.529e-05	2.48799	1.180e-05	3.0000
64	3.255e-09	3.47762	4.485e-06	2.49545	1.476e-06	2.9999
128	2.896e-10	3.49041	7.939e-07	2.49813	1.883e-07	2.9701

In Table 6.4, we tested a variable coefficient, $\mathcal{K} = 1 + 10x + y$ with the exact solution $p = x^2(1-x)y(1-y)^2$. In every experiment, the result shows more than optimal order of convergence. Tables 6.1 and 6.3 show one higher order than the theory predicts. This phenomenon seems due to the choice of special exact solution $\sin(2\pi x)\sin(2\pi y)$. For all other test problems, the result show one half order higher than the usual MFEM for velocity and pressure. This phenomenon may be some kind of superconvergence and will be left to future investigation.

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REFERENCES

[1] T. ARBOGAST AND Z. CHEN, *On the implementation of mixed methods as nonconforming methods for second order elliptic problems*, Math. Comp., 64 (1995), pp. 943–973.

[2] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates*, RAIRO Model. Math. Anal. Numer., 19 (1985), pp. 7–32.

[3] F. BREZZI, J. DOUGLAS, M. FORTIN, AND L. MARINI, *Efficient rectangular mixed finite elements in two and three variables*, RAIRO Model. Math. Numer. Anal., 21 (1987), pp. 581–604.

[4] F. Brezzi, R. Duràn, and M. Fortin, *Mixed finite elements for seconder order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.

[5] F. Brezzi, J. Douglas, and L. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[6] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[7] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Dordrecht, The Netherlands, 1978.

[8] Z. Chen, *Analysis of mixed methods using conforming and nonconforming finite element methods*, RAIRO Model. Math. Anal. Numer., 27 (1993), pp. 9–34.

[9] Z. Chen, *Equivalence between and multigrid algorithms for nonconforming and mixed methods for second-order elliptic problems*, East-West J. Numer. Math., 4 (1996), pp. 1–33.

[10] S. H. Chou, D. Y. Kwak, and K. Y. Kim, *Mixed finite volume methods on nonstaggered quadrilateral grids for elliptic problems*, Math. Comp., 72 (2002), pp. 525–539.

[11] S. H. Chou, D. Y. Kwak, and K. Y. Kim, *Flux recovery from primal hybrid finite element methods*, SIAM J. Numer. Anal., 40 (2002), pp. 403–415.

[12] S. H. Chou and S. Tang, *Conservative $P1$ conforming and nonconforming Galerkin FEMs: Effective flux evaluation via a nonmixed method approach*, SIAM J. Numer. Anal., 38 (2000), pp. 660–680.

[13] B. Courbet and J. P. Croisille, *Finite volume box schemes on triangular meshes*, RAIRO Math. Model. Numer. Anal., 32 (1998), pp. 631–649.

[14] J. P. Croisille, *Finite volume box schemes and mixed methods*, RAIRO Math. Model. Numer. Anal., 34 (2000), pp. 1087–1106.

[15] J. P. Croisille and I. Greff, *Some nonconforming mixed box schemes for elliptic problems*, Numer. Methods Partial Differential Equations, 18 (2002), pp. 355–373.

[16] M. Crouzeix and P. A. Rarviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, RAIRO Math. Model. Numer. Anal., 7 (1973), pp. 33–76.

[17] J. Douglas, Jr., and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.

[18] J. Douglas, Jr., J. E. Santos, D. Sheen, and X. Ye, *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 747–770.

[19] M. Fortin, *An analysis of the convergence of mixed finite element methods*, RAIRO Anal. Numér., 11 (1977), pp. 341–354.

[20] M. Fortin and M. Soulie, *A non-conforming piecewise quadratic finite element on triangles*, Internat. J. Numer. Methods Engrg., 19 (1983), pp. 505–520.

[21] B. M. Fraeijis de Veubeke, *Displacement and equilibrium models in the finite element method*, in Stress Analysis, O. C. Zienkiewicz and G. Holister, eds., John Wiley and Sons, New York, 1965.

[22] I. Greff, *Nonconforming box-schemes for elliptic problems on rectangular grids*, SIAM J. Numer. Anal., 45 (2007), pp. 946–968.

[23] B. M. Irons and A. Razzaque, *Experience with the patch test for the convergence of finite element*, in the Mathematical Foundations of the Finite Element Methods with Application, Part II, A. K. Aziz, ed., Academic Press, New York, 1972, pp. 557–587.

[24] L. D. Marini, *An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method*, SIAM J. Numer. Anal., 22 (1995), pp. 493–496.

[25] J. D. Nedelec, *Mixed finite elements in $\mathbb{R}^3$*, Numer. Math., 35 (1980), pp. 315–341.

[26] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Proceedings of the Conference on Mathematical Aspects of Finite Element Methods, Lecture Notes in Math. 606, Springer-Verlag, Berlin, 1977, pp. 292–315.

[27] P. A. Raviart and J. M. Thomas, *Primal hybrid finite element methods for 2nd order elliptic equations*, Math. Comp., 31 (1977), pp. 391–413.

[28] A. Schatz, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.